Mechanics I

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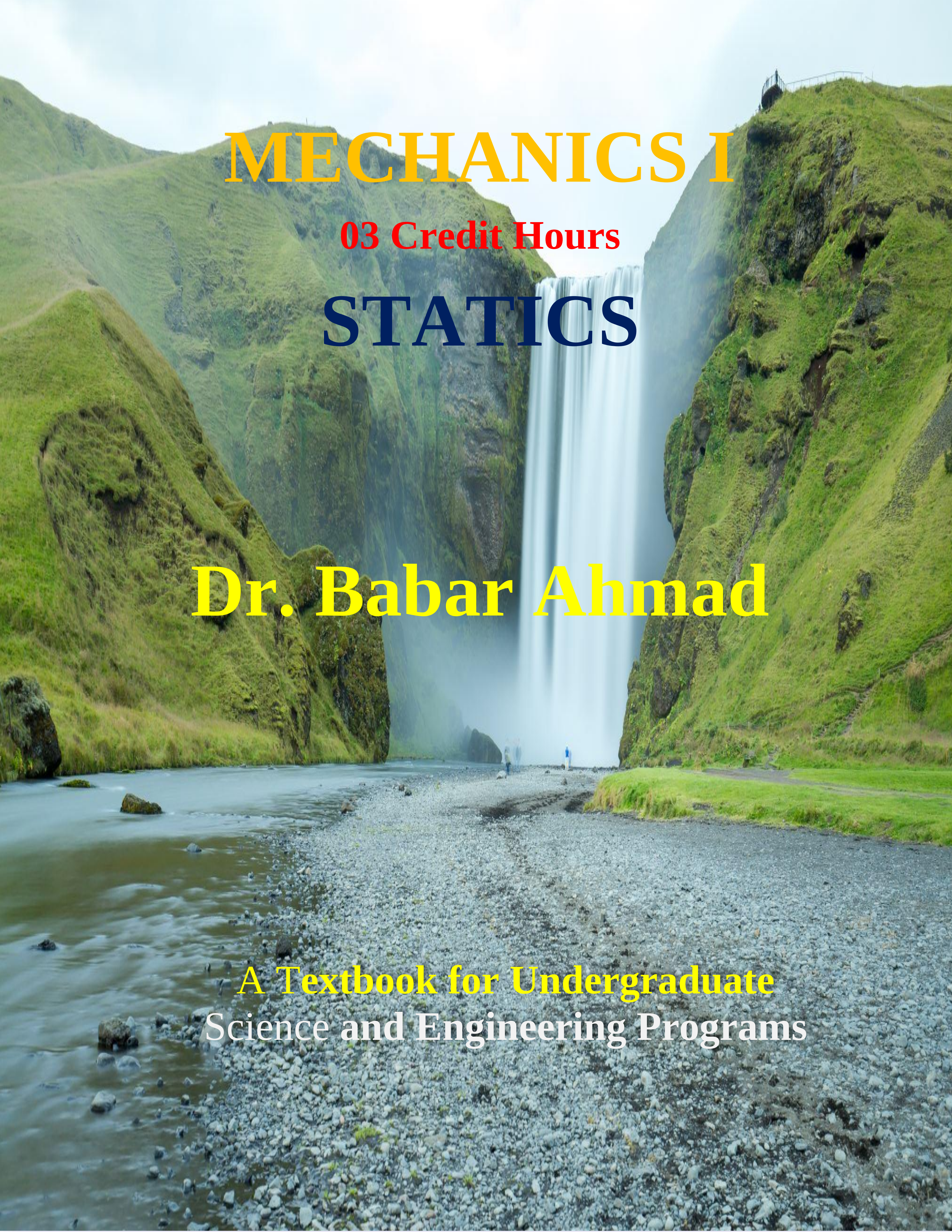

MECHANICS I

03 Credit Hours

STATICS

Dr. Babar Ahmad

A Textbook for Undergraduate
Science and Engineering Programs

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To My Parents

Mr. & Mrs. Rana Muhammad Hanif (*late*)

(*May ALLAH bless them*)

PREFACE

Mechanics is one of the most important course in maximum disciplines of science and engineering. No matter what your interest in science or engineering, mechanics will be important for you.

Mechanics is a branch of physics which deals with the bodies at rest and in motion. During the early modern period, scientists such as Galileo, Kepler, and Newton laid the foundation for what is now known as classical mechanics. Hence there is an extensive use of mathematics in its foundation.

Mechanics is core course for undergraduate Mathematics, Physics and many engineering disciplines. It appears under different names as Analytical/Classical Mechanics, Theoretical Mechanics, Mechanics I, Mechanics II, Mechanics III, Analytical Dynamics.

This textbook is designed to support teaching activities in Theoretical Mechanics specially Statics. It covers the contents of "Mechanics" for many undergraduate science and engineering programs. It presents simply and clearly the main theoretical aspects of mechanics.

It is assumed that the students have completed their courses in Calculus, Linear Algebra and Differential Equations. This book also lay the foundations for further studies in physics, physical sciences, and engineering.

For each concept a number books, documents and lecture notes are consulted. I wish to express my gratitude to the authors of such works.

In Chapter 1, the units and dimension of different physical quantities are given. They are referred from The International System of Units (SI), NIST Special Publication 330 2008 Edition, Barry N. Taylor and Ambler Thompson, Editors. Chapter 2 is about vector and scalar quantities. In this chapter we will learn to represent a vector in one, two and three dimensional coordinate systems. We will also learn techniques to add or decompose vectors. Also we will learn to take their products, which may be scalar or vector.

In Chapter 3, we will learn how forces can be composed into a single force and how a force can be resolved into different components. Also couple of forces and moment of a force are presented in this chapter. Chapter 4 contains concepts of equilibrium. Friction, its types and laws are presented in chapter 5.

Chapter 6 is about Linear Momentum, Impulse and Collision . It also contains law of conservation of linear momentum. In Chapter 7, we discuss Angular Momentum and its law of conservation. Chapter 8 contains theoretical aspects of Work, Energy, Power and Conservative Force. Also the law of conservation of energy is presented in this chapter. In these three chapters, three laws of conservation are presented.

In chapter 9, the concepts of virtual displacement and virtual work are presented. Center of mass and center of gravity of homogeneous objects are discussed in chapter 10. In this chapter, we consider discrete and continuous distribution of mass of homogeneous objects only. Non-homogeneous objects will be considered in next edition. In chapter 11, the concepts of Moments of Inertia and Products of Inertia with examples are presented. Some properties of rigid bodies like, Radius of Gyration, angular momentum, rotational kinetic energy, Euler dynamical equations are also presented in this chapter.

In a book of this concept, level and size, there may be a possibility that some misprint might have remained uncorrected. If you find such misprints or want to give some suggestions for its improvement, please write me at: babar.sms@gmail.com

Dr. Babar Ahmad

Islamabad, Pakistan
June, 2020

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Chapter 1

Units and Dimensions of Physical Quantities

In science problems, it is important to know the numerical value of the quantities but it is also important to understand the units and physical dimensions of the variable(s) involving in that problem. For units the mostly used system is INTERNATIONAL SYSTEM OF UNITS (SI).

1.1 International System of Units (SI)

The International System of Units (SI), NIST Special Publication 330, 2008 Edition, B.N. Taylor, editor. United States Department of Commerce, National Institute of Standards and Technology Gaithersburg, MD 20899.
There are two classes of SI units.

- base units;
- derived units.

1.1.1 Base Units

From the scientific point of view, the division of SI units into these two classes is to a certain extent arbitrary, because it is not essential to the physics of the subject. Nevertheless, the CGPM (General Conference on Weights and Measures), considering the advantages of a single, practical, world-wide system of units for international relations, for teaching, and for scientific work, decided to base the International System on a choice of seven well-defined units which by convention are regarded as dimensionally independent: the meter, the kilogram, the second, the ampere, the kelvin, the mole, and the candela. These SI units are called base units and are given in table 1.1.

Table 1.1: SI base units

Physical quantity	Name of unit	Symbol
length	meter	m
mass	kilogram	kg
time	second	s
electric current	ampere	A
thermodynamic temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

1.1.2 Derived Units with Special Names

The second class of SI units is that of derived units . These are units that are formed as products of powers of the base units according to the algebraic relations linking the quantities concerned. The names and symbols of some units thus formed in terms of base units may be replaced by special names and symbols which can themselves be used to form expressions and symbols for other derived units.

a) **Units expressed in terms of base units**

In table 1.2 lists some examples of derived units expressed directly in terms of base units. The derived units are obtained by multiplication and division of base units.

Table 1.2: Examples of SI derived units expressed in terms of base units

Derived quantity	Name of unit	Symbol
area	square meter	m^2
volume	cubic meter	m^3
speed, velocity	meter per second	m/s
acceleration	meter per second squared	m/s^2
wave number	reciprocal meter	m^{-1}
density, mass density	kilogram per cubic meter	kg/m^3
specific volume	cubic meter per kilogram	m^3/kg
current density	ampere per square meter	A/m^2
magnetic field strength	ampere per meter	A/m
concentration (of amount of substance)	mole per cubic meter	mol/m^3
luminance	candela per square meter	cd/m^2
refractive index	(the number) one	$1^{(a)}$

b) **More derived units with special names and symbols**

For convenience, certain derived units, which are listed in table 1.3, have been given special names and symbols. These names and symbols may themselves be used to express other derived units

Table 1.3: SI derived units with special names and symbols

Physical quantity	Name of unit	Symbol
plane angle	radian	rad
solid angle	steradian	sr
frequency	hertz	Hz
energy	joule	J
force	newton	N
pressure	pascal	Pa
power	watt	W
electric charge	coulomb	C
electric potential	volt	V
electric resistance	ohm	Ω
electric conductance	siemens	S
electric capacitance	farad	F
magnetic flux	weber	Wb
inductance	henry	H
magnetic flux density	tesla	T
luminous flux	lumen	lm
illuminance	lux	lx
celsius temperature	degree celsius	$^{\circ}C$
activity (of a radioactive source)	becquerel	Bq
absorbed dose (of ionizing radiation)	gray	Gy
dose equivalent	sievert	Sv

1.1.3 Decimal Multiples and Submultiples of SI Units

The CGPM adopted a series of prefixes for use in forming the decimal multiples and submultiples of SI units. Following CIPM (International Committee for Weights and Measures) Recommendation 1 (1969) mentioned above, these are designated by the name SI prefixes.

1.2 Units outside the SI

Some non-SI unit systems like British Engineering System, still appear widely in the scientific, technical and commercial literature, and some will probably continue to be used for many years. Other non-SI units, such as the units of time, are so widely used in everyday life, and are so deeply embedded in the history and culture of the human race, that they

Table 1.4: SI prefixes

Factor	Name	Symbol	Factor	Name	Symbol
10^{24}	yotta	Y	10^{-1}	deci	d
10^{21}	zetta	Z	10^{-2}	centi	c
10^{18}	exa	E	10^{-3}	milli	m
10^{15}	peta	P	10^{-6}	micro	μ
10^{12}	tera	T	10^{-9}	nano	n
10^{9}	giga	G	10^{-12}	pico	p
10^{6}	mega	M	10^{-15}	femto	f
10^{3}	kilo	k	10^{-18}	atto	a
10^{2}	hecto	h	10^{-21}	zepto	z
10	deca	da	10^{-24}	yocto	y

will continue to be used for the foreseeable future. For these reasons some of the more important non-SI units are listed in the tables below.

Table 1.5 lists non-SI units which are accepted for use with the SI. These units are in continuous everyday use.

Note: The neper and bel are used to express values of such logarithmic quantities as field

Table 1.5: Non-SI units accepted for use with the International System

Name	Symbol	Value in SI units
minute	min	1 min=60 s
hour	h	1 h = 60 min = 3600 s
day	d	1 d= 24 h= 86 400 s
degree	$^\circ$	$1^\circ = (\pi/180)$ rad
minute	$'$	$1' = (1/60)^\circ = (\pi/10800)$ rad
second	”	1” $= (1/60)' = (\pi/648000)$ rad
liter	l, L	1 L $= 1dm^3 = 10^{-3}\ m^3$
metric ton	t	1t $= 10^3$ kg
neper	N_p	$1N_p = 1$
bel	B	1 B $= (1/2)\ln 10(N_p)^{(i)}$

level, power level, sound pressure level, and logarithmic decrement. Natural logarithms are used to obtain the numerical values of quantities expressed in nepers and logarithms to base ten are used to obtain the numerical values of quantities expressed in bels.

Table 1.6 lists three non-SI units which are also accepted for use with the SI, whose values expressed in SI units must be obtained by experiment and are therefore not known exactly. These units are in common use in certain specialized fields. Table 1.7 lists some other non-SI units which are currently accepted for use with the SI to satisfy the needs of commercial, legal, and specialized scientific interests. These units should be defined in relation to the SI in every document in which they are used. Their use is not encouraged. The barn is a special unit employed in nuclear physics to express effective cross-sections.

Table 1.6: Non-SI units accepted for use with the International System, whose values in SI units are obtained experimentally

Name	Symbol	Value in SI units
electronvolt	eV	1 eV = $1.60217733(49) \times 10^{19}$J
unified atomic mass unit	u	1 u = $1.6605402(10) \times 10^{27}$ kg
astronomical unit	ua	1 ua = $1.49597870(30) \times 10^{11}$ m

Table 1.7: Other non-SI units currently accepted for use with the International System

Name	Symbol	Value in SI units
nautical mile		1 nautical mile = 1852 m
knot		1 nautical mile per hour = (1852/3600) m/s
are	a	1 a = 1 dam^2 = $10^2 m^2$
hectare	ha	1 ha = 1 hm^2 = $10^4 m^2$
bar	bar	1 bar=0.1 MPa=100 kPa=1000 hPa=10^5 Pa
barn	b	1 b = 100 fm^2 = $10^{-28} m^2$
curie	Ci	
roentgen	R	
rad	rad	
rem	rem	

1.3 Dimensions

A quantity can be measured in size, number, weight or amount of something. All physical quantities on Earth have dimensions that can be expressed in terms and combinations of 5 basic dimensions: mass (M), length (L), time (T), electrical current (I), and temperature (θ). These 5 dimensions are considered basic because they are easy to measure in experiments.

Dimensions are not the same as units. Rather, units express the system of measurement for the various dimensions. For example, speed can measured in units of metres per second (m/s) or kilometers per hour (km/hr) but the dimensions of speed are always a length (L) divided by time (T), or simply LT^{-1}. Similarly, the dimensions of area are $L \times L$ or L^2 and the units can be expressed in m^2. This is a useful means of working with physical quantities as it enables to determine the "dimensions" involved and the appropriate units of the quantity, especially in equations involving many variables and parameters. The dimensions of some useful quantities are listed in table . Note that the angle and solid angle are included in this list but are actually dimensionless quantities.

Some examples

Table 1.8: Different quantities with units and dimensional formula,

Derived quantity	Symbol of unit	Dimensioin
angle	rad	1
area	m^2	L^2
volume	m^3	L^3
speed, velocity	m/s	LT^{-1}
acceleration	m/s^2	LT^{-2}
Frequency	hertz s^{-1}	T^{-1}
density, mass density	kg/m^3	ML^{-3}
specific volume	m^3/kg	L^3T^{-1}
Force	Newton ($N = kgms^{-2}$)	MLT^{-2}
Impulse	$N.s$	MLT^1
Work	$N.m$	MLT^2
Power	Joule or J	MLT^2

Unit of Force: Force is measured in **Newtons** in the SI system:

$$1N = (1kg)\left(m/s^2\right)$$

1 Newton is the force required to give a mass of $1kg$ an acceleration of $1m/s^2$. and in the Non SI or British Engineering system:

$$1lb = 1slugft/s2 = 4.448N$$

The value of g in SI system is

$$g = 9.81m/s^2$$

and in non SI system is

$$g = 32.2ft/s^2$$

Weight The weight of 1 kg mass in SI system is:

$$\begin{aligned} W &= mg \\ &= (1kg)(9.81m/s^2) \\ &= 9.81N \end{aligned}$$

The mass of an object that weighs 1 pound:

$$\begin{aligned} F &= ma \\ 1lb &= m(1ft/s2) \\ m &= 1lbs^2/ft = 1slug \end{aligned}$$

Exercises

1. A wrist watch gains time at the rate of 5.5 second per day. Calculate the error after (i) an hour (ii) a month (iii) an year
2. The acceleration of a particle due to gravity is 9.80 m/s^2. What is its value in ft/s^2?
3. Convert the value of G from SI system to C.G.S.
4. Calculate the dimension of $F = \frac{mv^2}{r}$

Chapter 2

Scalars and Vectors

In mechanics, we come across various quantities such as mass, length, time, speed, velocity, area, volume, acceleration and force etc. These quantities are of two types namely scalars and vectors.

2.1 Scalar Quantities or Scalars

The quantities which possess only magnitude are called scalars. For example, the length of a bar is 2 m. Other examples of scalars are distance, speed, volume, density, temperature etc.

2.2 Vector Quantities or Vectors

The quantities which are specified by magnitude as well as direction are called vectors. For example, winds are usually described by giving their speed and direction, say 20 km/h northeast. The wind speed and wind direction together form a vector quantity called the wind velocity. Other examples of vectors are displacement, acceleration, force, weight, momentum etc.

2.2.1 Geometric Representation of a Vector

Geometrically a vectors can be represented by an arrow in $2 - space$ or $3 - space$; the direction of the arrow specifies the direction of the vector and the length of the arrow describes its magnitude. The tail of the arrow is called the *initial point* and the tip of the arrow is called the *terminal point* of the vector.
Physical quantities are also represented by symbols, lowercase or uppercase alphabet. We denote vectors with overline arrow such as $\vec{a}$, $\vec{p}$, $\vec{v}$, $\vec{F}$, and $\vec{W}$. When discussing vectors, we will refer to real numbers as scalars. Scalars will be denoted by a, m, t, v, and E.

From above we can say that vector quantities are completely specified by the following four characteristics:

- Magnitude
- Point of application
- Line of action, and
- Direction

First consider vectors whose initial point or reference point is origin.

2.2.2 Position Vector with reference to the Origin

In mechanics, a vector may be one dimensional, two dimensional or three dimensional.

(a) Vector in $1-space$

In $1-space$, the position vector of P relative to origin O is a vector $\vec{r}=\vec{OP}$ (from point O to point P), having magnitude of the length of line $\overline{OP}$. A plus or minus sign is enough to specify its direction. If point P is on right from O, we assign plus sign, and if it is on left from O, we assign minus sign. The vector $\vec{r}$ is shown in Fig. 2.1, as an arrow from O to P.

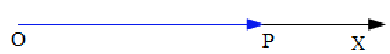

Figure 2.1: Position vector w.r.t origin in 1-space

(b) Vector in $2-space$

In $2-space$, the origin has coordinates $O(0,0)$. Take a point $P(x,y)$, then its position vector relative to O is a vector $\vec{r}=\vec{OP}$ (from point O to point P), having magnitude of the length of line $\overline{OP}$ and direction parallel to line $\overline{OP}$. The vector $\vec{r}$ is shown in Fig. 2.2, as an arrow from O to P, and is written as

$$\vec{r}=\vec{OP} \quad = \quad \langle x,y\rangle \tag{2.2.1}$$

It means that if the initial point of a vector is at the origin then the components of a vector are the coordinates of its terminal point. The position vector of P relative to P is a zero vector.

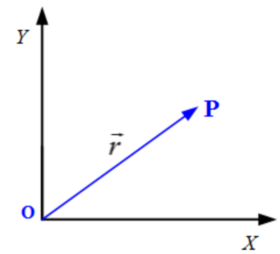

Figure 2.2: Position vector w.r.t origin in 2-space

Example 2.2.1. *The position vector of $P(2,1)$ relative to O is*

$$\vec{r} = \langle 2,1\rangle$$

(c) Vector in $3-space$

In $3-space$, the origin has coordinates $O(0,0,0)$. Take a point $P(x,y,z)$, then its position vector relative to O is a vector $\vec{r} = \vec{OP}$ having magnitude of the length of line $\overline{OP}$ and direction parallel to line $\overline{OP}$. The vector $\vec{r}$ is shown in Fig. 2.3, as an arrow from O to P, and is written as

$$\vec{r} = \vec{OP} = \langle x,y,z\rangle \tag{2.2.2}$$

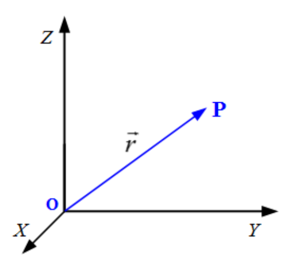

Figure 2.3: Position vector w.r.t origin in 3-space

Example 2.2.2. *The position vector of $P(2,3,2)$ relative to O is*

$$\vec{r} = \langle 2,3,2\rangle$$

2.2.3 Position Vector with reference to a point other than the Origin

(a) Vector in $2-space$

The components of a vector whose initial point is not at the origin can be calculated by two points $A(x_1, y_1)$ and $B(x_2, y_2)$ whose position vectors are

$$\begin{aligned}\vec{a} = \vec{OA} &= \langle x_1, y_1\rangle \\ \vec{b} = \vec{OB} &= \langle x_2, y_2\rangle\end{aligned}$$

Then $\vec{AB}$ is

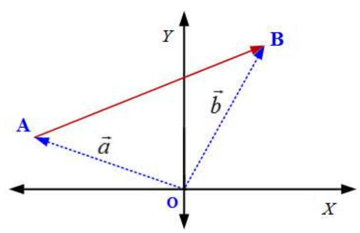

Figure 2.4: Position vector w.r.t a point other than origin in $2-space$

$$\begin{aligned}\vec{AB} &= \vec{OB} - \vec{OA} \\ &= \langle x_2 - x_1, y_2 - y_1\rangle \qquad (2.2.3)\end{aligned}$$

(b) Vector in $3-space$

For $3-space$, consider two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ whose position vectors are

$$\begin{aligned}\vec{a} = \vec{OA} &= \langle x_1, y_1, z_1\rangle \\ \vec{b} = \vec{OB} &= \langle x_2, y_2, z_2\rangle\end{aligned}$$

Then $\vec{AB}$ is

$$\begin{aligned}\vec{AB} &= \vec{OB} - \vec{OA} \\ &= \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle \qquad (2.2.4)\end{aligned}$$

Example 2.2.3. *In 2-space the vector from* $A(1, 2)$ *to* $B(3, -2)$ *is*

$$\begin{aligned}\vec{AB} &= \langle 3-1, -2-2\rangle \\ &= \langle 2, -4\rangle\end{aligned}$$

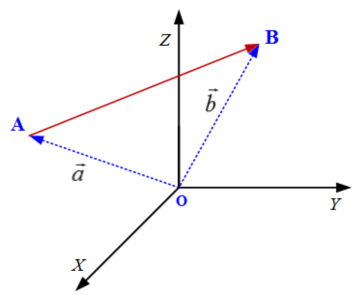

Figure 2.5: Position vector w.r.t a point other than origin in $3-space$

and in 3-space the vector from $P(1,-2,-1)$ *to* $Q(3,3,-1)$ *is*

$$\begin{aligned}\vec{PQ} &= \langle 3-1, 3+2, -1+1\rangle \\ &= \langle 2,5,0\rangle\end{aligned}$$

2.2.4 Unit Vectors

A vector of magnitude 1 is called a unit vector. In an $xy-$coordinate $(2-space)$ system the unit vectors along the $x-$ and $y-axes$ are denoted by $\hat{i}$ and $\hat{j}$ respectively as shown in left of Fig. 2.6; and in an $xyz-$coordinate $(3-space)$ system the unit vectors along the $x-$, $y-$, and $z-$axes are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively as shown in right of Fig. 2.6. The unit vectors in $2-space$ are

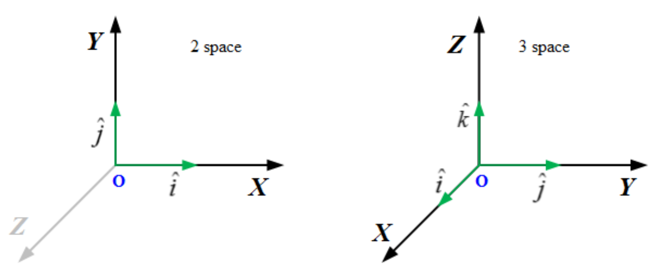

Figure 2.6: Unit vectors along coordinate axes

$$\begin{aligned}\hat{i} &= \langle 1,0\rangle \\ \hat{j} &= \langle 0,1\rangle\end{aligned}$$

The $\vec{r}$ with the combination of unit vectors $\hat{i}, \hat{j}$ can also be written as

$$\vec{r} = \vec{OP} = x\hat{i} + y\hat{j} \tag{2.2.5}$$

and in $3 - space$, the unit vectors are

$$\begin{aligned} \hat{i} &= \langle 1, 0, 0 \rangle \\ \hat{j} &= \langle 0, 1, 0 \rangle \\ \hat{k} &= \langle 0, 0, 1 \rangle \end{aligned}$$

The $\vec{r}$ with the combination of unit vectors $\hat{i}, \hat{j}, \hat{k}$ can also be written as

$$\vec{r} = \vec{OP} = x\hat{i} + y\hat{j} + z\hat{k} \tag{2.2.6}$$

In example 2.2.2, $\vec{r}$ can also be written as

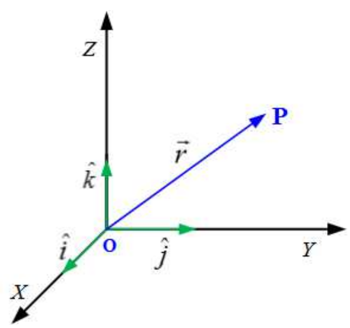

Figure 2.7: Position vector w.r.t origin.

$$\vec{r} = 2\hat{i} + 3\hat{j} + 2\hat{k}$$

2.2.5 Magnitude of a Vector

The magnitude of $\vec{r} = x\hat{i} + y\hat{j}$ in $2 - space$ is

$$\|\vec{r}\| = r = \sqrt{x^2 + y^2} \tag{2.2.7}$$

The magnitude of $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in $3 - space$ is

$$\|\vec{r}\| = r = \sqrt{x^2 + y^2 + z^2} \tag{2.2.8}$$

The magnitude of $\vec{r} = \langle 2, 1, 2 \rangle$ is

$$\begin{aligned} r &= \sqrt{2^2 + 1^2 + 2^2} \\ &= \sqrt{4 + 1 + 4} = \sqrt{9} \\ &= 3 \end{aligned}$$

2.2.6 Normalizing a Vector

Normalizing a vector is to find a unit vector $\hat{r}$ that has the same direction as some given nonzero vector $\vec{r}$. This unit vector is obtained by multiplying $\vec{r}$ by the reciprocal of its magnitude r; If $\vec{r}$ is vector in $2-space$, the unit vector is

$$\begin{aligned}\hat{r} &= \frac{\vec{r}}{r} \\ &= \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} \qquad (2.2.9)\end{aligned}$$

If $\vec{r}$ is vector in $3-space$, the unit vector is

$$\begin{aligned}\hat{r} &= \frac{\vec{r}}{r} \\ &= \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k} \qquad (2.2.10)\end{aligned}$$

The unit vector $\hat{r}$ for the $\vec{r} = \langle 2, 1, 2\rangle$ is

$$\begin{aligned}\hat{r} &= \frac{\langle 2,1,2\rangle}{3} \\ &= \left\langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle\end{aligned}$$

2.2.7 Equal Vectors

Two vectors are equal if and only if their corresponding components are equal. Thus two vectors $\vec{a} = \langle x_1, y_1, z_1\rangle$ and $\vec{b} = \langle x_2, y_2, z_2\rangle$ are equal if and only if $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$

2.2.8 Parallel Vectors

Two vectors $\vec{a}$ and $\vec{b}$ are parallel if there exist a scalar $k \in \mathbb{R}$ such that

$$\vec{a} = k\vec{b} \qquad (2.2.11)$$

If $k > 0$, vectors $\vec{a}$ and $\vec{b}$ have same direction and if $k < 0$, vectors $\vec{a}$ and $\vec{b}$ have opposite direction. In Fig. 2.8, on left side, two parallel vectors $\vec{a}$ and $\vec{b}$ are acting in same direction and on right side, two parallel vectors $\vec{a}$ and $\vec{b}$ are acting in opposite directions.

Norm of (2.2.11) is

$$\|\vec{a}\| = |k|\|\vec{b}\|$$

Example Vectors $\vec{a} = \langle 1, -2, 1\rangle$, $\vec{b} = \langle 3, -6, 3\rangle$ and $\vec{c} = \langle -0.5, 1, -0.5\rangle$ are parallel vectors. As we can write:

$$\begin{aligned}\vec{a} &= \frac{1}{3}\vec{b} \quad \text{or} \quad \vec{b} = 3\vec{a} \\ \vec{a} &= -2\vec{c} \quad \text{or} \quad \vec{c} = -0.5\vec{a} \\ \vec{b} &= -6\vec{c} \quad \text{or} \quad \vec{c} = -\frac{1}{6}\vec{b}\end{aligned}$$

Figure 2.8: Parallel vectors

Vectors $\vec{a}$ and $\vec{b}$ are acting in the same direction while vector $\vec{c}$ acts in opposite direction.

2.3 Free-body diagram

The free-body diagram is a very helpful to find the solution of problems involving vectors. It is a simplified line sketch of the body, showing position, direction and point of application of all vectors described in problem. To draw it we can follow the following steps.

1. First fix an appropriate coordinate system.
2. Define the particular body from the statement of the problem.
3. Label each vector with an appropriate name.
4. Mention all vectors described in the problem. We may also consider their rectangular components. Separate horizontal and vertical vectorss.

2.4 System of Vectors

When several vectors act simultaneously on a body, they constitute a system of vectors. These system are named, depending on the position of line of action of the vectors as follows:

- **Concurrent vectors** If the line of action of all the vectors in a system pass through a single point, the vectors are termed as concurrent vectors as shown in Fig. 2.9.
- **Collinear vectors** If the line of action of all the vectors lie along a single line, the vectors are called collinear vector as shown in Fig. 2.9. Example is, forces on a rope in a tug of war.
- **Coplanar vectors** If all the vectors in a system lie in a single plane, they are called coplanar vectors and are shown in Fig. 2.10. In Fig. 2.11 (a) coplanar and concurrent vectors are shown and in Fig. 2.11 (b) coplanar and non-concurrent vectors are shown.

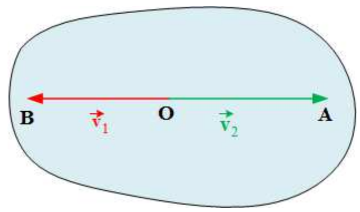

Figure 2.9: Concurrent and collinear vectors

- **Non-Coplanar vectors** If all the vectors in a system do not lie in a single plane they are called non-coplanar vectors or vectors in space.In Fig. 2.12 (a) non-coplanar and concurrent vectors are shown and in Fig. 2.12 (b) non-coplanar and non-concurrent vectors are shown.

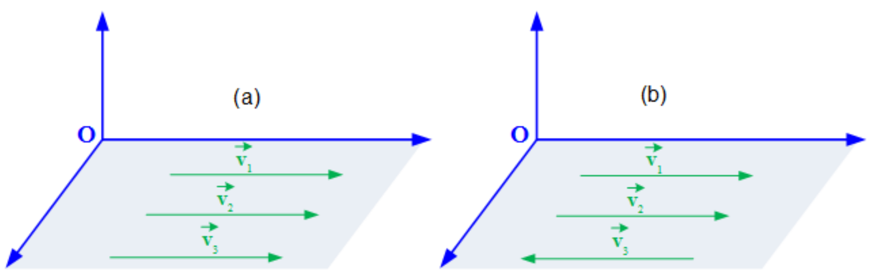

Figure 2.10: (a) Coplanar like parallel (b) Coplanar parallel vectors

2.5 Scalar and Vector Products of Two Vectors

In this section we will define a new kind of multiplication in which two vectors are multiplied to produce a scalar or a vector.

2.5.1 Scalar or Dot Products of Two Vectors

If $\vec{a} = \langle x_1, y_1 \rangle$ and $\vec{b} = \langle x_2, y_2 \rangle$ are vectors in $2 - space$, then the dot product of $\vec{a}$ and $\vec{b}$ is written as $\vec{a} \cdot \vec{b}$ and is defined as

$$\vec{a} \cdot \vec{b} = x_1 x_2 + y_1 y_2 \tag{2.5.1}$$

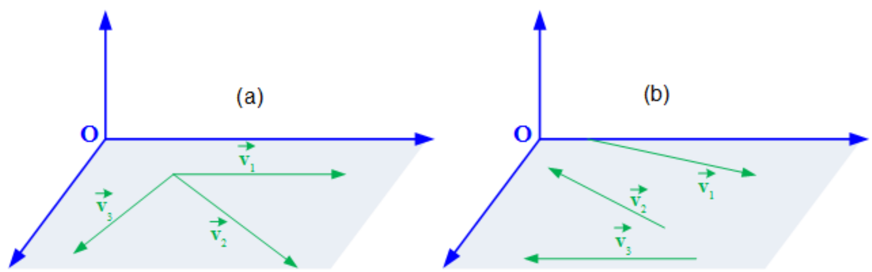

Figure 2.11: (a) Coplanar and Concurrent (b) Coplanar and Non-Concurrent vectors

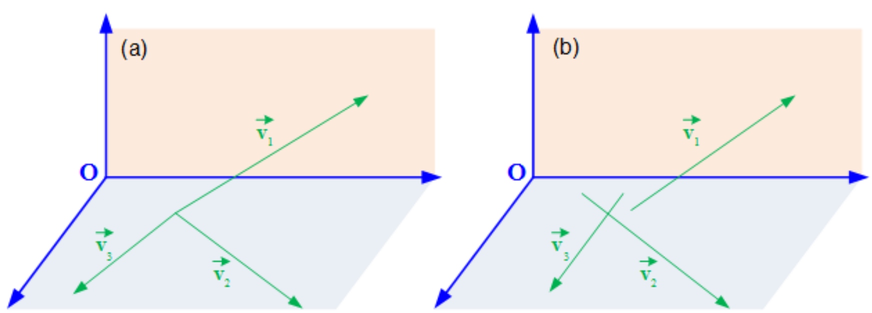

Figure 2.12: (a) Non-Coplanar and Concurrent (b) Non-Concurrent and Non-Coplanar vectors

Similarly if $\vec{a} = \langle x_1, y_1, z_1 \rangle$ and $\vec{b} = \langle x_2, y_2, z_2 \rangle$ are vectors in $3 - space$, then the dot product of $\vec{a}$ and $\vec{b}$ is written as $\vec{a}.\vec{b}$ and is defined as

$$\vec{a} \cdot \vec{b} = x_1 x_2 + y_1 y_2 + z_1 z_2 \tag{2.5.2}$$

2.5.2 Angle Between Two Vectors

If $\vec{a}$ and $\vec{b}$ are nonzero vectors in $2 - space$ or $3 - space$, and let θ satisfies the condition $0 \leq \theta \leq \pi$, is the angle between them as shown in Fig. 2.13, then

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{ab} \tag{2.5.3}$$

Hence $\vec{a} \cdot \vec{b}$ can also be written as

$$\vec{a} \cdot \vec{b} = ab \cos\theta \tag{2.5.4}$$

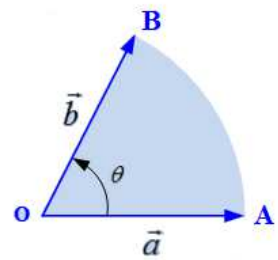

Figure 2.13: angle between two vectors.

Example 2.5.1. *Find dot product and angle between $\vec{a}$ and $\vec{b}$ for*

(a) 2-space vectors $\vec{a} = \langle 2, -4\rangle$ and $\vec{b} = \langle 2, 2\rangle$.

(b) 3-space vectors $\vec{a} = \hat{\imath} - 7\hat{\jmath} + 6\hat{k}$ and $\vec{b} = 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$.

Solution

(a) The dot product of $\vec{a}$ and $\vec{b}$ is given by using (2.5.1).

$$\begin{aligned} \vec{a}\cdot\vec{b} &= (2)(2) + (-4)(2) \\ &= -4 \end{aligned}$$

For angle, we first calculate the magnitudes of $\vec{a}$ and $\vec{b}$

$$\begin{aligned} a &= \sqrt{2^2 + (-4)^2} = 2\sqrt{5} \\ b &= \sqrt{2^2 + (2)^2} = 2\sqrt{2} \end{aligned}$$

Using (2.5.3), the angle between $\vec{a}$ and $\vec{b}$ is

$$\begin{aligned} \cos\theta &= \frac{4}{(2\sqrt{5})(2\sqrt{2})} \\ &= \frac{1}{\sqrt{10}} \end{aligned}$$

Hence the angle is

$$\begin{aligned} \theta &= \arccos\left(\frac{1}{\sqrt{10}}\right) \\ &= 71.56^\circ \end{aligned}$$

(b) The dot product of $\vec{a}$ and $\vec{b}$ is given by using (2.5.1).

$$\begin{aligned}\vec{a}\cdot\vec{b} &= (1)(2)+(-7)(2)+(6)(2)\\ &= 0\end{aligned}$$

For angle, we first calculate magnitudes of $\vec{a}$ and $\vec{b}$

$$\begin{aligned}a &= \sqrt{1^2+(-7)^2+(6)^2}=\sqrt{86}\\ b &= \sqrt{2^2+(2)^2+(2)^2}=2\sqrt{3}\end{aligned}$$

Using (2.5.3), the angle between $\vec{a}$ and $\vec{b}$ is

$$\begin{aligned}\cos\theta &= \frac{0}{(\sqrt{86})(2\sqrt{3})}\\ &= 0\end{aligned}$$

Hence the angle is

$$\begin{aligned}\theta &= \arccos(0)\\ &= \frac{\pi}{2}\end{aligned}$$

Note If the dot product of two nonzero vectors is zero, then the vectors are orthogonal.

2.5.3 Direction Angles

In an xy-coordinate system ($2-space$), the direction of a nonzero vector $\vec{r}$ is completely determined by the angles α and β between $\vec{r}$ and the unit vectors $\hat{i}$ and $\hat{j}$ respectively (see Fig. 2.14 (a)), and in an xyz-coordinate system ($3-space$) the direction is completely determined by the angles α, β, and γ between $\vec{r}$ and the unit vectors $\hat{i}, \hat{j}$, and $\hat{k}$ respectively (see Fig. 2.14 (b)). In both $2-space$ and $3-space$ the angles between a nonzero vector $\vec{r}$ and the vectors $\hat{i}, \hat{j}$, and $\hat{k}$ are called the direction angles of $\vec{r}$, and the cosines of those angles are called the direction cosines of $\vec{r}$. Formulas for the direction cosines of a vector can be obtained from (2.5.3). The $\vec{r}$ in $2-space$ is

$$\vec{r} = x\hat{i}+y\hat{j}$$

Since α is the angle between $\hat{i}$ and $\vec{r}$, then

$$\begin{aligned}\cos\alpha &= \frac{\vec{r}\cdot\hat{i}}{\|\vec{r}\|\|\hat{i}\|}\\ &= \frac{x}{r}\end{aligned}$$

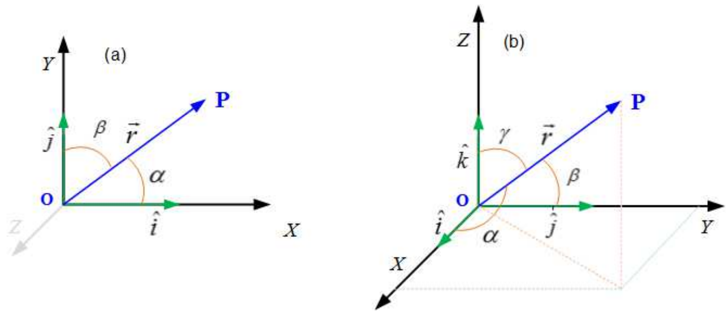

Figure 2.14: Direction angles of $\vec{r}$.

Similarly

$$\cos\beta = \frac{\vec{r}\cdot\hat{j}}{\|\vec{r}\|\|\hat{j}\|} = \frac{y}{r}$$

Then (2.2.9) can be written as

$$\hat{r} = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} = \cos\alpha\hat{i} + \cos\beta\hat{j} \quad (2.5.5)$$

The direction cosines of a vector satisfy the equation

$$\cos^2\alpha + \cos^2\beta = 1 \quad (2.5.6)$$

The $\vec{r}$ in $3 - space$ is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Since α is the angle between $\hat{i}$ and $\vec{r}$, then

$$\cos\alpha = \frac{\vec{r}\cdot\hat{i}}{\|\vec{r}\|\|\hat{i}\|} = \frac{x}{r}$$

Similarly

$$\cos\beta = \frac{\vec{r}\cdot\hat{j}}{\|\vec{r}\|\|\hat{j}\|} = \frac{y}{r}$$

and

$$\cos\gamma = \frac{\vec{r}\cdot\hat{k}}{\|\vec{r}\|\|\hat{k}\|} = \frac{z}{r}$$

Then (2.2.10) can be written as

$$\hat{r} = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k} \quad (2.5.7)$$

The direction cosines of a vector satisfy the equation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \quad (2.5.8)$$

Example 2.5.2. *Find the direction cosines of the vector $\vec{a} = 2\hat{i} - 4\hat{j} + 4\hat{k}$, and approximate the direction angles to the nearest degree.*

Solution

For direction cosines, we have to normalize $\vec{a}$, the components of $\hat{a}$ will give the direction cosines. First its magnitude is

$$a = \sqrt{(2)^2 + (-4)^2 + (4)^2} = \sqrt{4+16+16} = 6$$

and $\hat{a}$ is

$$\hat{a} = \frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

Thus

$$\cos\alpha = \frac{1}{3}$$
$$\cos\beta = -\frac{2}{3}$$
$$\cos\gamma = \frac{2}{3}$$

Then the direction angles are

$$\alpha = \arccos\left(\frac{1}{3}\right) \approx 71^\circ$$
$$\beta = \arccos\left(-\frac{2}{3}\right) \approx 132^\circ$$
$$\gamma = \arccos\left(\frac{2}{3}\right) \approx 48^\circ$$

2.5.4 Decomposing Vector into Orthogonal Components

In many applications it is desirable to decompose a vector into a sum of two orthogonal vectors with convenient specified directions. These components are also known as rectangular components of a vector. For example, Figure 2.15 shows a block on an inclined plane. The weight of the block $\vec{W}$ (the downward force) can be decomposed into the sum

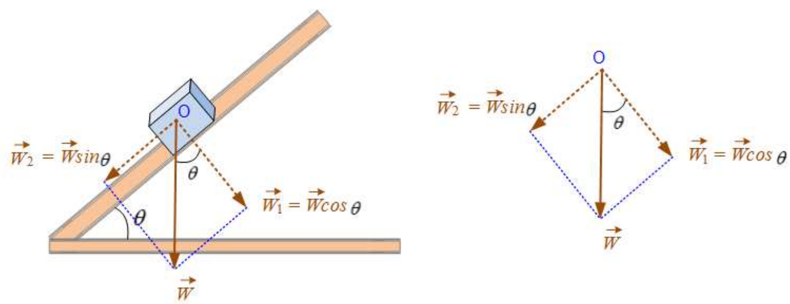

Figure 2.15: Orthogonal components of a vector.

$$\vec{W} = \vec{W}_1 + \vec{W}_2$$

Considering O as the origin of a 2-space system. Let $\hat{e_1}$ and $\hat{e_2}$ be two orthogonal unit vectors considering $\hat{e_1}$ along $\vec{W}_1$ and $\hat{e_2}$ along $\vec{W}_2$. Then $\vec{W}$ is

$$\vec{W} = W_1\hat{e_1} + W_2\hat{e_2} \tag{2.5.9}$$

W_1 and W_2 can be calculated by taking the dot product of $\vec{W}$ with $\hat{e_1}$ and $\hat{e_2}$ respectively.

$$\begin{aligned} \vec{W}\cdot\hat{e_1} &= (W_1\hat{e_1} + W_2\hat{e_2})\cdot\hat{e_1} \\ &= W_1(\hat{e_1}\cdot\hat{e_1}) + W_2(\hat{e_2}\cdot\hat{e_1}) \\ &= W_1(1) + W_2(0) \\ &= W_1 \end{aligned}$$

Similarly

$$\vec{W}\cdot\hat{e_2} = W_2$$

Thus (2.5.9) can be written in another form as

$$\vec{W} = \left(\vec{W}\cdot\hat{e_1}\right)\hat{e_1} + \left(\vec{W}\cdot\hat{e_2}\right)\hat{e_2} \tag{2.5.10}$$

The vector component $\left(\vec{W}\cdot\hat{e_1}\right)\hat{e_1}$ of $\vec{W}$ is along $\hat{e_1}$ direction and the vector component $\left(\vec{W}\cdot\hat{e_2}\right)\hat{e_2}$ of $\vec{W}$ is along $\hat{e_2}$ direction. Let θ be the angle between $\vec{W}$ and $\hat{e_1}$. Then the orthogonal components of $\vec{W}$ are

$$\begin{aligned} W_1 &= \vec{W}\cdot\hat{e_1} \\ &= \|\vec{W}\|\|\hat{e_1}\|\cos\theta \\ &= W(1)\cos\theta \\ &= W\cos\theta \end{aligned}$$

The scalar W is the magnitude of $\vec{W}$ and is obtained as

$$W = \sqrt{W_1^2+W_2^2} \qquad (2.5.11)$$

And

$$\begin{aligned} W_2 &= \vec{W}\cdot\hat{e_2} \\ &= \|\vec{W}\|\|\hat{e_2}\|\cos(90-\theta) \\ &= W(1)\sin\theta \\ &= W\sin\theta \end{aligned}$$

Then (2.5.9) can also be written in another form as

$$\vec{W} = W\cos\theta\hat{\imath}+W\sin\theta\hat{\jmath}$$

Example 2.5.3. *Consider the vectors* $\vec{a}=2\hat{\imath}+3\hat{\jmath}$, $\hat{e}_1=\frac{1}{\sqrt{2}}\hat{\imath}+\frac{1}{\sqrt{2}}\hat{\jmath}$ *and* $\hat{e}_2=\frac{1}{\sqrt{2}}\hat{\imath}-\frac{1}{\sqrt{2}}\hat{\jmath}$. *Find*

a) *The scalar components of* $\vec{a}$ *along* $\hat{e}_1$ *and* $\hat{e}_2$ *and*

b) *The vector components of* $\vec{a}$ *along* $\hat{e}_1$ *and* $\hat{e}_2$.

Solution

a) The scalar component of $\vec{a}$ along $\hat{e}_1$ is

$$\begin{aligned} a_1 &= \vec{a}\cdot\hat{e_1} \\ &= 2\left(\frac{1}{\sqrt{2}}\right)+3\left(\frac{1}{\sqrt{2}}\right) \\ &= \frac{5}{\sqrt{2}} \end{aligned}$$

And the scalar component of $\vec{a}$ along $\hat{e}_2$ is

$$\begin{aligned} a_2 &= \vec{a}\cdot\hat{e_2} \\ &= 2\left(\frac{1}{\sqrt{2}}\right)+3\left(-\frac{1}{\sqrt{2}}\right) \\ &= -\frac{1}{\sqrt{2}} \end{aligned}$$

b) The vector components of $\vec{a}$ along $\hat{e}_1$ is

$$\begin{aligned} a_1\hat{e_1} &= \frac{5}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}\right) \\ &= \frac{5}{2}\hat{i}+\frac{5}{2}\hat{j} \end{aligned}$$

And the vector components of $\vec{a}$ along $\hat{e}_2$ is

$$\begin{aligned} a_2\hat{e_2} &= -\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\hat{i}-\frac{1}{\sqrt{2}}\hat{j}\right) \\ &= -\frac{1}{2}\hat{i}+\frac{1}{2}\hat{j} \end{aligned}$$

Example 2.5.4. *Let the vector $\vec{a}=\langle 2,2\rangle$,makes an angle $\theta=\frac{\pi}{4}$ with $\hat{e}_1$. Find the scalar components of $\vec{a}$ along $\hat{e}_1$ and $\hat{e}_2$.*

Solution
We first need the magnitude of $\vec{a}$, that is

$$a = \sqrt{(2)^2+(2)^2}=2\sqrt{2}$$

The scalar component of $\vec{a}$ along $\hat{e}_1$ is

$$\begin{aligned} a_1 &= a\cos\theta \\ &= 2\sqrt{2}\cos\left(\frac{\pi}{4}\right) \\ &= 2\sqrt{2}(\frac{1}{\sqrt{2}})=2 \end{aligned}$$

The scalar component of $\vec{a}$ along $\hat{e}_2$ is

$$\begin{aligned} a_2 &= a\sin\theta \\ &= 2\sqrt{2}\sin\left(\frac{\pi}{4}\right) \\ &= 2\sqrt{2}(\frac{1}{\sqrt{2}})=2 \end{aligned}$$

2.5.5 Orthogonal Components of a Vector with Reference to the Origin

This can be discussed in two steps, a vector in $2 - space$ and a vector in $3 - space$. First consider a vector in $2 - space$.

(a) Vector in $2 - space$

If the initial point of a vector $\vec{r}$ is at origin, in $2 - space$ the orthogonal/rectangular components will be along coordinate axes. The component along $x\ axis$ is called horizontal component of $\vec{r}$ and the component along $y\ axis$ is called vertical component of $\vec{r}$. If $\vec{r}$ makes an angle θ with $x\ axis$ then its orthogonal components are

$$\begin{aligned} r_X &= r\cos\theta \\ r_Y &= r\sin\theta \end{aligned}$$

And $\vec{r}$ can be written as

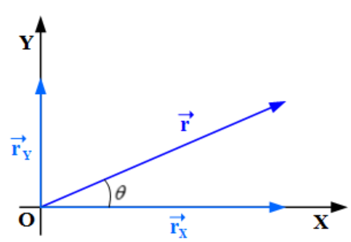

Figure 2.16: Orthogonal components of a vector in 2 *space*.

$$\begin{aligned} \vec{r} &= r_X\hat{i} + r_Y\hat{j} \\ &= r\cos\theta\hat{i} + r\sin\theta\hat{j} \end{aligned}$$

Example 2.5.5. *Let a vector $\vec{a}$ in $2 - space$ of magnitude 3 makes an angle $\theta = \frac{\pi}{4}$ with $x\ axis$. Find $\vec{a}$ from its rectangular components.*

Solution

The given data is

$$\begin{aligned} a &= 3 \\ \theta &= \frac{\pi}{4} \end{aligned}$$

The x component of $\vec{a}$ is

$$\begin{aligned} a_1 &= a\cos\theta \\ &= 3\cos\left(\frac{\pi}{4}\right) \\ &= 3(0.707) = 2.12 \end{aligned}$$

The y component of $\vec{a}$ is

$$\begin{aligned} a_2 &= a\sin\theta \\ &= 3\sin\left(\frac{\pi}{4}\right) \\ &= 3(0.707) = 2.12 \end{aligned}$$

$\vec{a}$ in the combination of its rectangular component is

$$\vec{a} = 2.12\hat{i} + 2.12\hat{j}$$

(b) Vector in $3-space$

If the initial point of a vector is at origin, in $3-space$ the orthogonal/rectangular components will be along coordinate axes. Let $\vec{r}$ makes an angle ϕ with $z\ axis$ then its orthogonal components are

$$\begin{aligned} r_Z &= r\cos\phi \\ r_V &= r\sin\phi \end{aligned}$$

$\vec{r_V}$ is the projection of $\vec{r}$ in xy plane. Let it makes an angle θ with $x\ axis$. It will further

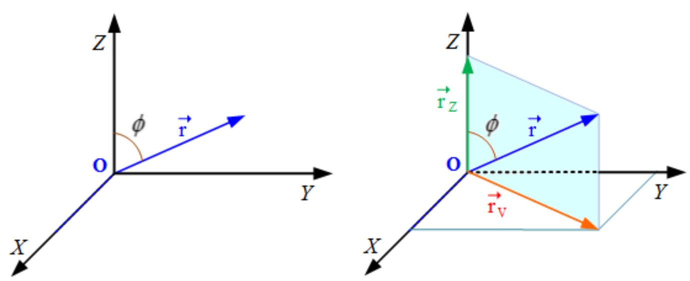

Figure 2.17: Vector in 3 *space*.

decompose into orthogonal components as

$$\begin{aligned} r_X &= r\sin\phi\cos\theta \\ r_Y &= r\sin\phi\sin\theta \end{aligned}$$

It provides the idea of spherical coordinates. The rectangular components of $\vec{r}$ are

$$\begin{aligned} r_X &= r\sin\phi\cos\theta \\ r_Y &= r\sin\phi\sin\theta \\ r_Z &= r\cos\phi \end{aligned}$$

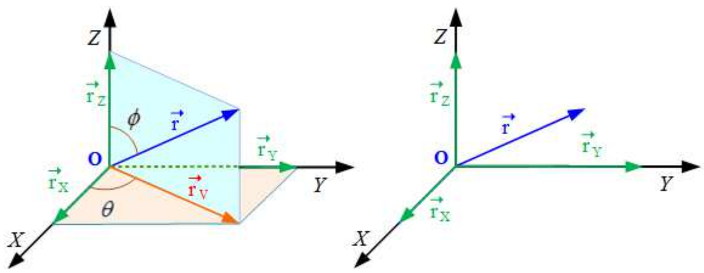

Figure 2.18: Orthogonal components of a vector in 3 *space*.

And $\vec{r}$ can be written as

$$\begin{aligned}\vec{r} &= r_X\hat{i} + r_Y\hat{j} + r_Z\hat{k} \\ &= r\sin\phi\cos\theta\hat{i} + r\sin\phi\sin\theta\hat{j} + r\cos\phi\hat{k}\end{aligned}$$

2.5.6 Vector or Cross Product of Two Vectors

If $\vec{a} = \langle x_1, y_1, z_1\rangle$ and $\vec{b} = \langle x_2, y_2, z_2\rangle$ are vectors in $3 - space$, then the cross product of $\vec{a}$ and $\vec{b}$ is written as $\vec{a} \times \vec{b}$ and is defined as

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \qquad (2.5.12) \\ &= (y_1z_2 - y_2z_1)\,\hat{i} - (x_1z_2 - x_2z_1)\,\hat{j} + (x_1y_2 - x_2y_1)\,\hat{k}\end{aligned}$$

If θ is the angle between $\vec{a}$ and $\vec{b}$ as shown in Fig. 2.19, then the magnitude of $\vec{a} \times \vec{b}$ is

$$\left\|\vec{a} \times \vec{b}\right\| = ab\sin\theta$$

Also

$$A = \left\|\vec{a} \times \vec{b}\right\|$$

is the area of the parallelogram that has $\vec{a}$ and $\vec{b}$ as adjacent sides.

Note

- $\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$
- The cross product is defined only for vectors in 3-space, whereas the dot product is defined for vectors in 2-space and 3-space.

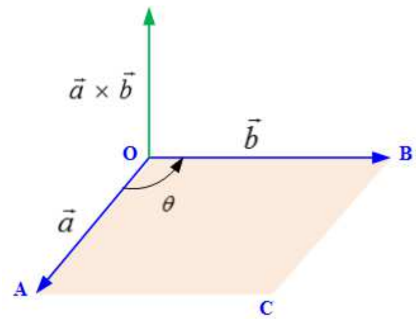

Figure 2.19: Cross product of two vectors

- The cross product of two vectors is a vector, whereas the dot product of two vectors is a scalar.
- $\vec{r} \times \vec{r} = 0$ for any vector $\vec{r}$ in 3-space.
- $\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j},$
  $\hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i},\ \hat{i} \times \hat{k} = -\hat{j}$

Example 2.5.6. *Let the vector* $\vec{a} = \langle 2, -2, 1 \rangle$, *and* $\vec{b} = \langle 3, 0, 1 \rangle$ *Find*

a) $\vec{a} \times \vec{a}$

b) $\vec{a} \times \vec{b}$

c) $\vec{b} \times \vec{a}$

d) Find the area of the parallelogram whose adjacent sides are $\vec{a}$, *and* $\vec{b}$

Solution
The vector products can be calculated by using (2.5.12), first

a) $\vec{a} \times \vec{a}$ is

$$\begin{aligned} \vec{a} \times \vec{a} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{vmatrix} \\ &= (-2+2)\,\hat{i} - (2-2)\,\hat{j} + (-4+4)\,\hat{k} \\ &= 0 \end{aligned}$$

Hence the vector product of a vector with itself is zero. Next

b) $\vec{a} \times \vec{b}$ is

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 3 & 0 & 1 \end{vmatrix} \\ &= (-2+0)\,\hat{i} - (2-3)\,\hat{j} + (0+6)\,\hat{k} \\ &= -2\hat{i} + \hat{j} + 6\hat{k} \end{aligned}$$

c) For $\vec{b} \times \vec{a}$, we use the concept $\vec{b} \times \vec{a} = -\left(\vec{a} \times \vec{b}\right)$. Hence

$$\begin{aligned} \vec{b} \times \vec{a} &= -\left(\vec{a} \times \vec{b}\right) \\ &= -\left(-2\hat{i} + \hat{j} + 6\hat{k}\right) \\ &= 2\hat{i} - \hat{j} - 6\hat{k} \end{aligned}$$

d) The area of the parallelogram is the magnitude of $\vec{a} \times \vec{b}$, that is

$$\begin{aligned} \|\vec{a} \times \vec{b}\| &= \sqrt{(-2)^2 + (1)^2 + (6)^2} \\ &= \sqrt{4+1+36} = \sqrt{41} \\ &\approx 6.4 \; units^2 \end{aligned}$$

Hence the area of the parallelogram whose sides are $\vec{a}$ and $\vec{b}$, is 6.4 $units^2$

2.6 Scalar or Dot Product of Three Vectors

If $\vec{a} = \langle x_1, y_1, z_1 \rangle$, $\vec{b} = \langle x_2, y_2, z_2 \rangle$ and $\vec{c} = \langle x_3, y_3, z_3 \rangle$ are vectors in $3 - space$, then the scalar product of three vectors is a number written as $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)$ and is defined as

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \tag{2.6.1}$$

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are adjacent edges of the parallelepiped, as shown in Fig. 2.20, then the magnitude of $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)$ is is volume V of the parallelepiped.

$$V = \left|\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)\right|$$

Note

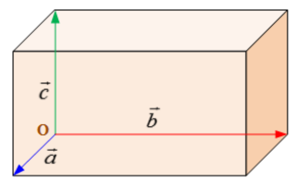

Figure 2.20: scalar triple product is volume of parallelepiped

- If $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = 0$ then the three vectors are coplanar.
- If $\vec{a} \cdot \left(\vec{a} \times \vec{b}\right) = 0$ then $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$.
- $\vec{b} \cdot \left(\vec{a} \times \vec{b}\right) = 0$ then $\vec{a} \times \vec{b}$ is orthogonal to $\vec{b}$.

Example 2.6.1. *Find the triple dot product of the vectors* $\vec{a} = \langle 3, 2, 1\rangle$, $\vec{b} = \langle 2, 1, 1\rangle$ *and* $\vec{c} = \langle 1, 2, 4\rangle$. *Also find the volume of the parallelepiped whose edges are* $\vec{a}$, $\vec{b}$ *and* $\vec{c}$.

Solution
The triple dot product of vectors is given by using (2.6.1)

$$\begin{aligned}
\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) &= \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix} \\
&= 3(4-2) - 2(8-1) + 1(4-1) \\
&= 6 - 14 + 3 = -5
\end{aligned}$$

The volume of the parallelepiped with edges $\vec{a}$, $\vec{b}$ and $\vec{c}$ is

$$\begin{aligned}
V &= \left|\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)\right| \\
&= 5 \ units^3
\end{aligned}$$

2.7 Addition of Vectors

Two or more vectors can be added geometrically (graphically) and analytically. Following methods can be used to add vectors.

2.7.1 Head to Tail Rule of Vector Addition

Given two vectors $\vec{P}$ and $\vec{Q}$. Their resultant $\vec{P}+\vec{Q}$ is obtained by joining the tail of $\vec{Q}$ with the head of $\vec{P}$ without making any alteration in the direction of vectors. Draw a vector from the tail of $\vec{P}$ to the head of $\vec{Q}$. This vector is $\vec{P}+\vec{Q}$. Also we can obtain $\vec{Q}+\vec{P}$, the same vector, let it be $\vec{R}$. Hence we can say

$$\vec{R} = \vec{P} + \vec{Q} \quad = \quad \vec{Q} + \vec{P}$$

This means vector addition is commutative. Geometrically this sum is illustrated in Fig.

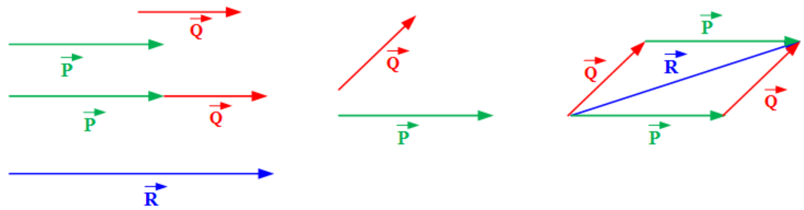

Figure 2.21: Vector addition by head to tail rule

3.1, on left side two vectors are acting in same directions and on right side two vectors are not acting in same directions. The length of $\vec{P}+\vec{Q}$ is the magnitude and its inclination with $\vec{P}$ is its direction.
Any number of vectors can be added by this method. For two adjacent vectors we can apply triangular law and for more than two vectors we can apply polygon law.

2.7.2 Triangle Law of Vector Addition

If two vectors are represented in magnitude and direction by two sides of a triangle taken in order, then their resultant is the closing side of the triangle taken in the opposite order.
Proof Two vectors $\vec{P}$ and $\vec{Q}$ are represented completely by two sides OA and AB of a triangle OAB. Then by vector addition (head to tail rule) the resultant vector $\vec{R}$ of two vectors $\vec{P}$ and $\vec{Q}$ is

$$\vec{R} \quad = \quad \vec{P} + \vec{Q}$$

Geometrically this sum is illustrated in Fig. 3.4. The length of $\vec{R}$ is the magnitude and its inclination with $\vec{P}$ is its direction.

2.7.3 Polygon Law of Vector Addition

The triangle rule can be made more general to apply to any geometrical shape - or polygon. This then becomes the polygon law. It can be stated as:

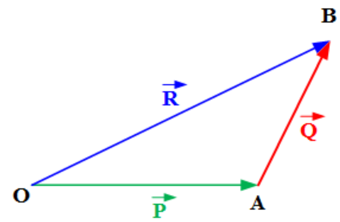

Figure 2.22: Vector addition by triangular law

If a number of vectors are represented both in magnitude and direction by the sides of a polygon taken in the same order, then their resultant is represented both in magnitude and direction by the closing side of the polygon taken in the opposite order.
Proof Four vectors $\vec{F_1}, \vec{F_2}, \vec{F_3}$ and $\vec{F_4}$ are represented by four sides $OA.AB, BC$ and CD of a polygon $OABCD$. Then by vector addition (head to tail rule) the resultant vector $\vec{R}$ of four vectors is

$$\vec{R} \ = \ \vec{F_1} + \vec{F_2} + \vec{F_3} + \vec{F_4}$$

Geometrically this sum is illustrated in Fig. 3.7. The length of $\vec{R}$ is the magnitude and its

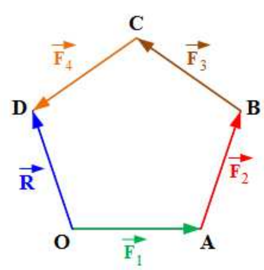

Figure 2.23: Vector addition by polygon law

inclination with $\vec{F_1}$ is its direction.
Since vector addition is associative, the resultant vector obtained by the polygon rule is independent of the order of composition of vectors.

2.7.4 Subtraction of vectors

The subtraction of a vector from another is a vector obtained by adding one vector to the negative of the other. It is also called difference of vectors.

Given two vectors $\vec{P}$ and $\vec{Q}$. Their difference $\vec{P}-\vec{Q}$ is obtained by joining the tail of $-\vec{Q}$ with the head of $\vec{P}$ without making any alteration in the direction of vectors. Draw a vector from the tail of $\vec{P}$ to the head of $-\vec{Q}$. This vector is $\vec{R}=\vec{P}-\vec{Q}$.

$$\vec{R}=\vec{P}-\vec{Q} \quad = \quad \vec{P}+\left(-\vec{Q}\right)$$

Geometrically this difference vector is illustrated in Fig. 3.8, on left side the two vectors are parallel and on right side the two vectors are not parallel. The length of $\vec{R}$ is the magnitude

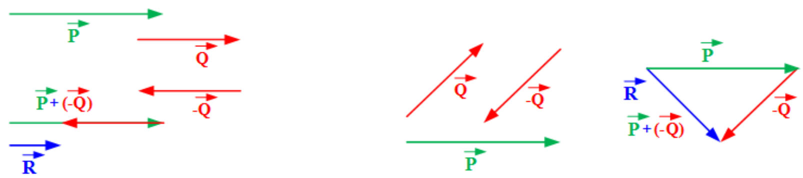

Figure 2.24: Subtraction of two vectors.

and its inclination with $\vec{P}$ is its direction.

2.7.5 Parallelogram Law of Vector Addition

If two vectors are represented in magnitude and direction by two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram, passing through the point of intersection of the vectors as shown in Fig. 3.13. and is given by (3.4.1)

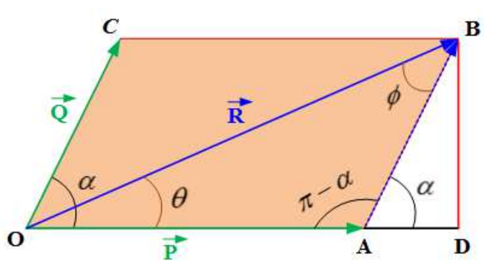

Figure 2.25: Parallelogram of vectors

$$\vec{R} \quad = \quad \vec{P}+\vec{Q} \tag{2.7.1}$$

And by law of sines, the magnitude of the resultant is

$$\frac{R}{\sin(\pi-\alpha)} = \frac{P}{\sin\phi} = \frac{Q}{\sin\theta} \tag{2.7.2}$$

And by law of cosines, the magnitude of the resultant is

$$R = \sqrt{P^2+Q^2+2PQ\cos\alpha} \tag{2.7.3}$$

where α is the angle between these two vectors. The resultant makes an angle θ with the horizontal vector.

$$\tan\theta = \frac{Q\sin\alpha}{P+Q\cos\alpha} \tag{2.7.4}$$

Note: We usually refer (3.4.2) for magnitude of the resultant.

Proof The two vectors $\vec{P}$ and $\vec{Q}$ are represented completely by two adjacent sides OA and OC of a parallelogram $OABC$. The vector $\vec{Q}$ can be represented by AB side, then by vector addition (head to tail rule) the resultant vector $\vec{R}$ of two vectors $\vec{P}$ and $\vec{Q}$ is

$$\vec{R} = \vec{P}+\vec{Q}$$

In Fig. 3.12, a triangle OAB is formed by three vectors $\vec{P}$, $\vec{Q}$, $\vec{R}$. In this triangle,

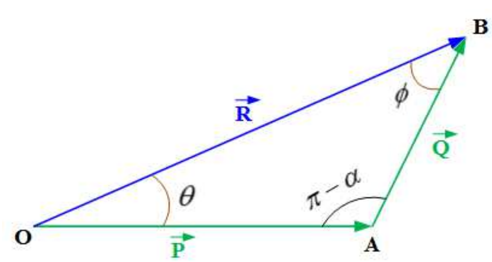

Figure 2.26: Triangle of vectors

$\angle A = \pi-\alpha$, $\angle O = \theta$, and $\angle B = \phi$, then by law of sines we can write

$$\frac{R}{\sin(\pi-\alpha)} = \frac{P}{\sin\phi} = \frac{Q}{\sin\theta}$$

Again consider triangle OAB, by law of cosines we can write

$$\begin{aligned}
\cos(\angle OAB) &= \frac{|\overline{OA}|^2+|\overline{AB}|^2-|\overline{OB}|^2}{2|\overline{OA}||\overline{AB}|}\\
\cos(\pi-\alpha) &= \frac{P^2+Q^2-R^2}{2PQ}\\
-\cos\alpha &= \frac{P^2+Q^2-R^2}{2PQ}\\
-2PQ\cos\alpha &= P^2+Q^2-R^2
\end{aligned}$$

Then R can be written as

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

Draw a perpendicular BD from B on OA, which meets OA at D produced. In right angle triangle ADB

$$\cos\alpha = \frac{|\overline{AD}|}{|\overline{AB}|}$$

then

$$\begin{aligned} |\overline{AD}| &= |\overline{AB}|\cos\alpha \\ &= Q\cos\alpha \end{aligned} \tag{2.7.5}$$

Similarly

$$|\overline{BD}| = Q\sin\alpha \tag{2.7.6}$$

Let the resultant $\vec{R}$ makes an angle θ with the vector $\vec{P}$ as shown in Fig. 3.13. In right angle triangle ODB, $\angle BOD = \theta$, then

$$\tan\theta = \frac{|\overline{BD}|}{|\overline{OB}|} \tag{2.7.7}$$

Since

$$|\overline{OD}| = |\overline{OA}| + |\overline{AD}|$$

then (3.4.6) becomes

$$\tan\theta = \frac{|\overline{BD}|}{|\overline{OA}| + |\overline{AD}|} \tag{2.7.8}$$

Using (3.4.4) and (3.4.5), (3.4.7) becomes

$$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha}$$

or

$$\theta = \arctan\left(\frac{Q\sin\alpha}{P + Q\cos\alpha}\right) \tag{2.7.9}$$

If the resultant $\vec{R}$ makes an angle θ with the vector $\vec{Q}$, the resultant will remain the same, but in (3.4.8), P and Q will interchange.
Particular Cases Here some particular cases can be discussed for different values of the angle between the vectors.

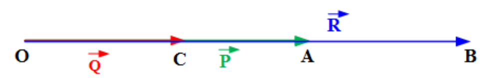

Figure 2.27: Vectors acting in the same direction

Case 1: If the two vectors are acting in the same direction, then the angle between them is $\alpha = 0\ rad$, as shown in Fig. 3.14 and by (3.4.2), their resultant is

$$\begin{aligned} R &= \sqrt{P^2 + Q^2 + 2PQ\cos(0)} \\ &= \sqrt{P^2 + Q^2 + 2PQ} \\ &= \sqrt{(P+Q)^2} \\ &= P + Q \end{aligned}$$

(3.4.9) gives the magnitude of the resultant. For direction, consider (3.4.3)

$$\begin{aligned} \tan\theta &= \frac{Q\sin(0)}{P + Q\cos(0)} \\ &= 0 \end{aligned} \tag{2.7.10}$$

(3.4.10) gives the direction of the resultant. In this case the magnitude of the resultant of the vectors is the sum of the individual magnitudes of the vectors and it acts in the direction of the vectors resultant is known as resultant of greatest magnitude.

Case 2: If the two vectors are perpendicular, then the angle between them is $\alpha = \frac{\pi}{2}\ rad$, as shown in Fig. 3.15 and by (3.4.2), their resultant is

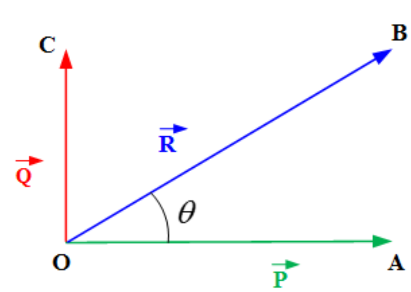

Figure 2.28: Vectors are orthogonal

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos(\frac{\pi}{2})}$$
$$= \sqrt{P^2 + Q^2} \qquad (2.7.11)$$

(3.4.11) gives the magnitude of the resultant, when the two vectors acting on a body are perpendicular. For direction, consider (3.4.3)

$$\tan\theta = \frac{Q\sin(\frac{\pi}{2})}{P + Q\cos(\frac{\pi}{2})}$$
$$= \frac{Q}{P} \qquad (2.7.12)$$

(3.4.12) gives the direction of the resultant.

Case 3: If the two vectors are acting in the opposite direction, then the angle between them is $\alpha = \pi\ rad$, and by (3.4.2), their resultant is

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos(\pi)}$$
$$= \sqrt{P^2 + Q^2 - 2PQ}$$
$$= \sqrt{(P - Q)^2}$$
$$= |P - Q| \qquad (2.7.13)$$

(3.4.13) gives the magnitude of the resultant, when the vectors are acting in opposite direction, as shown in Fig. 3.16. For direction, consider (3.4.3)

Figure 2.29: Vectors acting in opposite direction

$$\tan\theta = \frac{Q\sin(\pi)}{P + Q\cos(\pi)}$$
$$= 0$$

and

$$\theta = 0 \text{ or } \pi \qquad (2.7.14)$$

(3.4.14) gives the direction of the resultant. In this case the magnitude of the resultant of the vectors is the difference of the individual magnitudes of the vectors and it acts in the direction of a forces with greater magnitude. This resultant is known as resultant of least magnitude.

2.7.6 Resolved Components of a Vector in Two given Directions

Let $\vec{R}$ be a given vector and $\vec{P}$ and $\vec{Q}$ be its resolved parts making angles θ and ϕ with it. Completing parallelogram $OACB$ as shown in Fig. 3.28. In this figure, consider the triangle OAB, in which the sides OA, AB and OB represents the forces P, Q and R in magnitude respectively. Then by law of sine's we have

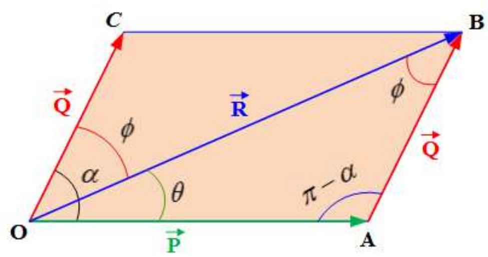

Figure 2.30: Resolved components of a vector in two given directions

$$\frac{R}{\sin(\pi-\alpha)} = \frac{P}{\sin\phi} = \frac{Q}{\sin\theta}$$

and we can write

$$P = \frac{R\sin\phi}{\sin(\pi-\alpha)} \tag{2.7.15}$$

and

$$Q = \frac{R\sin\theta}{\sin(\pi-\alpha)} \tag{2.7.16}$$

Then P and Q given by (3.5.1) and (3.5.3) respectively gives the magnitudes of resolved parts of a vector.
If the angle between the forces $\alpha = \frac{\pi}{2}$, then $\phi = \frac{\pi}{2} - \theta$,

$$\sin(\pi-\alpha) = \sin\left(\frac{\pi}{2}\right)$$

and

$$\sin\phi = \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

Then the resolved components of a vector are

$$P = R\cos\theta \tag{2.7.17}$$

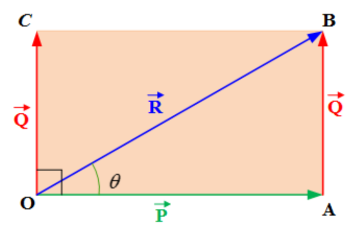

Figure 2.31: Resolved components of a vector in orthogonal directions

and

$$Q = R\sin\theta \tag{2.7.18}$$

are known as rectangular components of a vector and are shown in Fig. 3.31.

Example 2.7.1. *Consider two vectors with magnitudes* $3N$ *and* $4N$ *and angle between them is* 60 *degrees. Find their resultant vector using parallelogram law of vectors.*

Solution: Let

$$\begin{aligned} P &= 3\ N \\ Q &= 4\ N \end{aligned}$$

and the angle between them is

$$\alpha = 60^\circ$$

Using (3.4.2), the resultant is

$$\begin{aligned} R &= \sqrt{3^2 + 4^2 + 2\cos 60} \\ &= \sqrt{9 + 16 + 1} \\ &= \sqrt{26} \approx 5.099N \end{aligned}$$

Let P is the initial vector and Q is the terminal vector, then using (3.4.3), the angle of resultant with P is

$$\begin{aligned} \theta &= \arctan\left(\frac{4\sin(60)}{3 + 4\cos(60)}\right) \\ &= \arctan\left(\frac{3.4641}{5}\right) \\ &= \arctan(0.6928) = 34.72^\circ \approx 35^\circ \end{aligned}$$

The resultant makes an angle $\theta = 35^\circ$ with force P.

Theorem 2.7.1. *If A and B are two points having position vectors $\vec{a}$ and $\vec{b}$ in a system with O as origin. Next C is a point which divides AB in ratio $\lambda : \mu$, then show that position vector of C relative to O is*

$$\vec{c} = \frac{\mu\vec{a} + \lambda\vec{b}}{\lambda + \mu} \tag{2.7.19}$$

Proof This theorem is also known as λ, μ **theorem** of vector addition.
Two points A and B having position vectors $\vec{a}$ and $\vec{b}$ relative to O are shown in Fig. 3.27. If C divides AB in ratio $\lambda : \mu$, then from Fig. 3.27, we can write

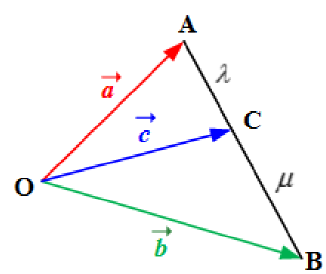

Figure 2.32: λ μ Theorem.

$$AC = \frac{\lambda}{\lambda + \mu} AB$$

Hence

$$\vec{AC} = \frac{\lambda}{\lambda + \mu} \vec{AB} \tag{2.7.20}$$

Following *head to tail rule* of vector addition, $\vec{OB}$ can be written as a sum of $\vec{OA}$ and $\vec{AB}$ (see Fig. 3.27). Then

$$\vec{AB} = \vec{OB} - \vec{OA} \tag{2.7.21}$$

Using (3.4.23), (3.4.22) can be written as

$$\vec{AC} = \frac{\lambda}{\lambda + \mu} \left(\vec{OB} - \vec{OA}\right) \tag{2.7.22}$$

Again following *head to tail rule* of vector addition, $\vec{OC}$ can be written as (see Fig. 3.27)

$$\vec{OC} = \vec{OA} + \vec{AC} \tag{2.7.23}$$

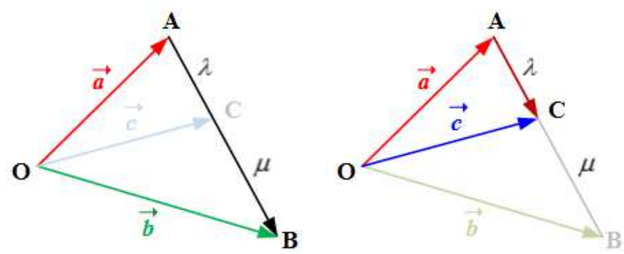

Figure 2.33: Vector addition by head to tail rule.

Since $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$ and $\vec{OC} = \vec{c}$ is given by (3.4.24). Then (3.4.25) becomes

$$\begin{aligned}\vec{c} &= \vec{a} + \frac{\lambda}{\lambda+\mu}\left(\vec{b} - \vec{a}\right) \\ &= \frac{(\lambda+\mu-\lambda)\,\vec{a} + \lambda\vec{b}}{\lambda+\mu} \\ &= \frac{\mu\vec{a} + \lambda\vec{b}}{\lambda+\mu}\end{aligned}$$

Hence the result.
Special case: When $\lambda = \mu$, then C is the mid point of AB and $\vec{c}$ is

$$\vec{c} = \frac{\vec{a} + \vec{b}}{2}$$

Example 2.7.2. *In triangle OAB, $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$ as shown in Fig. 3.25. If C*

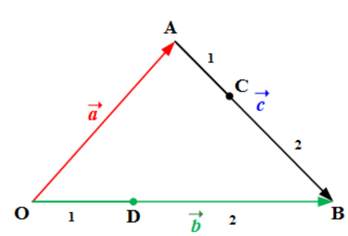

Figure 2.34: triangle of vectors

divides the line AB in ratio $1:2$ and D divides the line OB in ratio $1:2$. Find $\vec{DC}$ and *hence show that $\vec{DC}$ is parallel to $\vec{OA}$.*

Solution Following $\lambda,\ \mu$ theorem, the sum of $\vec{OA}$ and $\vec{OB}$ is $\vec{OC}$ as shown in Fig. 3.26 (a) with $\lambda = 1,\ \mu = 2$. Hence $\vec{OC}$ is

$$\vec{OC} = \frac{2\vec{a} + \vec{b}}{3}$$

In Fig. 3.26 (b) $\vec{DO}$ is

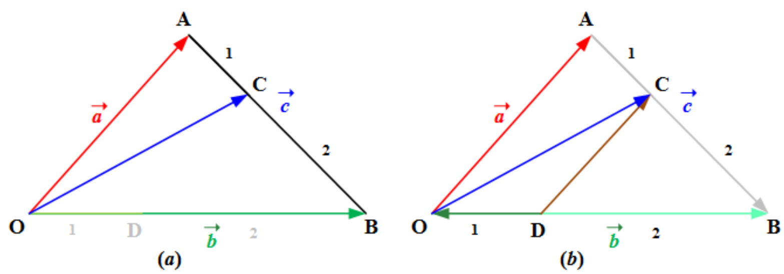

Figure 2.35: Vector addition by head to tail rule.

$$\vec{DO} = -\frac{1}{3}\vec{b}$$

following *head to tail rule* of vector addition, in Fig. 3.26 (b) $\vec{DC}$ can be written as

$$\begin{aligned}\vec{DC} &= \vec{DO} + \vec{OC} \\ &= -\frac{1}{3}\vec{b} + \frac{2}{3}\vec{a} + \frac{1}{3}\vec{b} \\ &= \frac{2}{3}\vec{a}\end{aligned}$$

Since $\vec{DC}$ is a scalar multiple of $\vec{a}$ and is acting in the direction of $\vec{a}$; that is $\vec{DC}$ is parallel to $\vec{OA}$.

2.8 Resultant of Coplanar Vectors

Following subsection 2.5.4, the rectangular components of a vector can be obtained. The rectangular components of resultant vector of two or more coplanar vectors can be obtained by summing rectangular components of individual vectors. For two dimensional system, the horizontal components of resultant vector of two or more vectors is the sum of horizontal components of individual vectors and the vertical components of resultant vector of two or more vectors is the sum of vertical components of individual vectors. First consider a system of two vectors in $2-space$.

2.8.1 Resultant of Two Coplanar Vectors

Consider two coplanar vectors $\vec{P}$ and $\vec{Q}$ as shown in Fig. 3.34. Their representations in

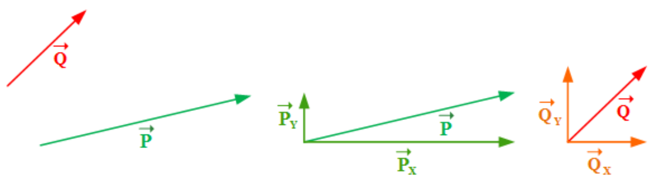

Figure 2.36: Two coplanar vectors.

rectangular components are

$$\vec{P} = P_X\hat{\imath} + P_Y\hat{\jmath}$$

and

$$\vec{Q} = Q_X\hat{\imath} + Q_Y\hat{\jmath}$$

Then by vector addition (head to tail rule) the resultant vector $\vec{R}$ of two vectors $\vec{P}$ and $\vec{Q}$ is

$$\vec{R} = \vec{P} + \vec{Q} \tag{2.8.1}$$

$\vec{R}$ in rectangular components is

$$\vec{R} = R_X\hat{\imath} + R_Y\hat{\jmath}$$

In rectangular components, (3.6.1) can be written as

$$\begin{aligned} R_X\hat{\imath} + R_Y\hat{\jmath} &= \left(P_X\hat{\imath} + P_Y\hat{\jmath}\right) + \left(Q_X\hat{\imath} + Q_Y\hat{\jmath}\right) \\ &= (P_X + Q_X)\,\hat{\imath} + (P_Y + Q_Y)\,\hat{\jmath} \end{aligned}$$

Comparing components, we have

$$R_X = P_X + Q_X$$

The horizontal component of resultant is sum of horizontal components of individual vectors. And

$$R_Y = P_Y + Q_Y$$

the vertical component of resultant is sum of vertical components of individual vectors. The magnitude of resultant is

$$R = \sqrt{R_X^2 + R_Y^2} \tag{2.8.2}$$

And the direction is

$$\theta = \arctan\left(\frac{R_Y}{R_X}\right) \tag{2.8.3}$$

Geometrically this sum is illustrated in Fig. 3.35. The length of $\vec{R}$ is the magnitude and

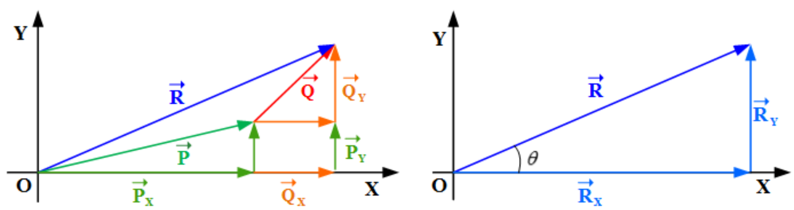

Figure 2.37: Addition of two coplanar vectors.

its inclination with horizontal is its direction.

2.8.2 Resultant of n Coplanar Vectors

For n coplanar vectors $\vec{F_1}, \vec{F_2}, ...\vec{F_n}$, the rectangular components of resultant vector are

$$R_X = F_{1X} + F_{2X} + ... + F_{nX} = \sum_{i=1}^{n} F_{iX}$$

And

$$R_Y = F_{1Y} + F_{2Y} + ... + F_{nY} = \sum_{i=1}^{n} F_{iY}$$

The magnitude and direction of resultant vector can be calculated by using (3.6.2) and (3.6.3) respectively.

2.8.3 Resultant of n Non-coplanar Vectors

Next consider a system of n vectors $\vec{F_1}, \vec{F_2}, ...\vec{F_n}$in $3 - space$.
The rectangular components of resultant vector are

$$\begin{aligned} R_X &= F_{1X} + F_{2X} + ... + F_{nX} = \sum_{i=1}^{n} F_{iX} \\ R_Y &= F_{1Y} + F_{2Y} + ... + F_{nY} = \sum_{i=1}^{n} F_{iY} \end{aligned}$$

And

$$R_Z = F_{1Z} + F_{2Z} + ... + F_{nZ} = \sum_{i=1}^{n} F_{iZ}$$

The magnitude of resultant is

$$R = \sqrt{R_X^2 + R_Y^2 + R_Z^2} \tag{2.8.4}$$

If the resultant R makes angles α with x $axis$, β with y $axis$ and γ with z $axis$, then direction cosines are

$$\begin{aligned} \cos\alpha &= \frac{R_X}{R} \\ \cos\beta &= \frac{R_Y}{R} \end{aligned}$$

and

$$\cos\gamma = \frac{R_Z}{R}$$

which will help to determine the direction of the resultant.

2.9 Resultant of Concurrent and Coplanar Vectors

Following subsection 2.5.4, the rectangular components of a vector can be obtained. The rectangular components of resultant vector of two or more concurrent and coplanar vectors can be obtained by summing rectangular components of individual vectors. For two dimensional system, the horizontal components of resultant vector of two or more vectors is the sum of horizontal components of individual vectors and the vertical components of resultant vector of two or more vectors is the sum of vertical components of individual vectors. First consider a system of two vectors in $2 - space$.

2.9.1 Resultant of Two Concurrent and Coplanar Vectors

Consider two concurrent and coplanar vectors $\vec{P}$ and $\vec{Q}$. Let their point of application is origin O of cartesian coordinate system. Vector $\vec{P}$ makes an angle α_1 and vector $\vec{Q}$ makes an angle α_2 with x *axis* as shown in Fig. 3.36. Their representations in rectangular

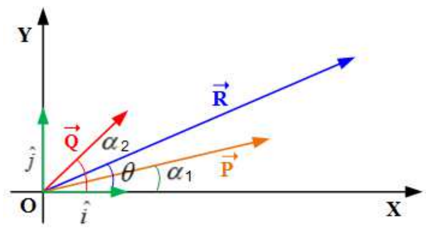

Figure 2.38: Resultant of two concurrent and coplanar vectors

components are

$$\begin{aligned}\vec{P} &= P_X\hat{i} + P_Y\hat{j}\\ &= P\cos\alpha_1\hat{i} + P\sin\alpha_1\hat{j}\end{aligned}$$

and

$$\begin{aligned}\vec{Q} &= Q_X\hat{i} + Q_Y\hat{j}\\ &= Q\cos\alpha_2\hat{i} + Q\sin\alpha_2\hat{j}\end{aligned}$$

If $\vec{R}$ is their resultant making an angle θ with x *axis* as shown in Fig. 3.36 and is given by (3.6.1). Its representation in rectangular components is

$$\begin{aligned}\vec{R} &= R_X\hat{i} + R_Y\hat{j}\\ &= R\cos\theta\hat{i} + R\sin\theta\hat{j}\end{aligned}$$

In rectangular components, (3.6.1) can be written as

$$\begin{aligned}R_X\hat{i} + R_Y\hat{j} &= \left(P_X\hat{i} + P_Y\hat{j}\right) + \left(Q_X\hat{i} + Q_Y\hat{j}\right)\\ &= \left(P_X + Q_X\right)\hat{i} + \left(P_Y + Q_Y\right)\hat{j}\end{aligned}$$

Comparing components, we have

$$\begin{aligned}R_X &= P_X + Q_X\\ &= P\cos\alpha_1 + Q\cos\alpha_2\end{aligned}$$

The horizontal component of resultant is sum of horizontal components of individual vectors. And

$$
\begin{aligned}
R_Y &= P_Y + Q_Y \\
&= P \sin \alpha_1 + Q \sin \alpha_2
\end{aligned}
$$

the vertical component of resultant is sum of vertical components of individual vectors. The length of $\vec{R}$ is the magnitude and its inclination with x *axis* is its direction can be calculated by using (3.6.2) and (3.6.3) respectively.

2.9.2 Resultant of n Concurrent and Coplanar Vectors

For n vectors $\vec{F_1}, \vec{F_2}, ...\vec{F_n}$ making angles $\alpha_1, \alpha_2, ...\alpha_n$ with x *axis* as shown in Fig. 3.37. If $\vec{R}$ is their resultant making an angle θ with x *axis* as shown in Fig. 3.37. Rectangular

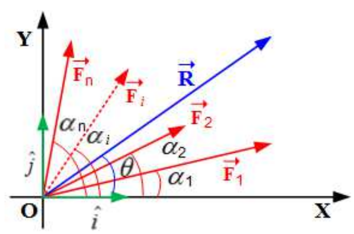

Figure 2.39: Resultant of n concurrent and coplanar vectors

components of resultant vector are

$$
\begin{aligned}
R_X &= F_{1X} + F_{2X} + ... + F_{nX} = \sum_{i=1}^{n} F_{iX} \\
R \cos \theta &= F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + ... + F_{nX} \cos \alpha_n = \sum_{i=1}^{n} F_i \cos \alpha_i
\end{aligned}
$$

And

$$
\begin{aligned}
R_Y &= F_{1Y} + F_{2Y} + ... + F_{nY} = \sum_{i=1}^{n} F_{iY} \\
R \sin \theta &= F_1 \sin \alpha_1 + F_2 \sin \alpha_2 + ... + F_{nX} \sin \alpha_n = \sum_{i=1}^{n} F_i \sin \alpha_i
\end{aligned}
$$

The magnitude and direction of resultant vector can be calculated by using (3.6.2) and (3.6.3) respectively.

Example 2.9.1. *A boat is rowed at a velocity of* 12 km/h *across a river. The velocity of stream is* 8 km/h. *Determine the resultant velocity of the boat.*

Solution: Fix the starting point of boat as origin O. Mark the downstream as horizontal axis and across the river as vertical axis. Accordingly the boat will move vertically, but as the velocity of the stream joins, as a result it will move making an angle θ with x $axis$ as shown in the Fig. 2.40. It means velocity of the boat will be vertical component and

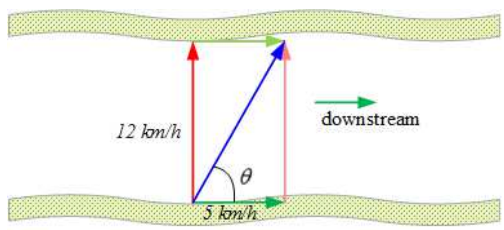

Figure 2.40: Resultant of two concurrent and coplanar velocities

velocity of the river will be the horizontal component of the resultant velocity.

$$v_X = 5\ km/h$$
$$v_Y = 12\ km/h$$

The resultant can be obtained by using addition of vectors by rectangular components. Using (3.6.2) magnitude of resultant velocity is

$$\begin{aligned} v &= \sqrt{(v_X)^2 + (v_Y)^2} \\ &= \sqrt{(5)^2 + (12)^2} \\ &= \sqrt{169} \\ &= 13\ km/h \end{aligned}$$

Let the resultant velocity makes an angle θ with x $axis$. Using (3.6.3) direction of resultant velocity is

$$\begin{aligned} \theta &= \arctan\left(\frac{v_Y}{v_X}\right) \\ &= \arctan\left(\frac{12}{5}\right) = \arctan(2.4) \\ &\simeq 1.17\ rad \simeq 71^\circ \end{aligned}$$

2.10 Resultant of Three Concurrent and Non-coplanar Vectors

Consider three concurrent and non-coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. Let their point of application is origin O. Complete the parallelepiped $OABCDEFG$ whose edges OA, OB and OC represents the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively as shown in Fig. 2.41. Consider parallelogram

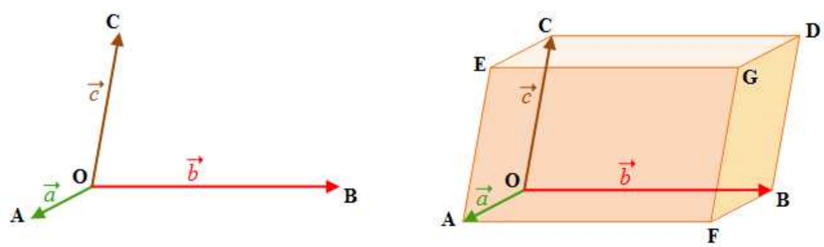

Figure 2.41: Three concurrent and non-coplanar vectors

$OAFB$, whose two adjacent sides OA and OB represents the vectors $\vec{a}$ and $\vec{b}$. Then by law of parallelogram of vector addition, OF represents the sum of $\vec{a}$ and $\vec{b}$. Let it be $\vec{u}$. Next consider parallelogram $OFGC$, whose two adjacent sides OF and OC represents the

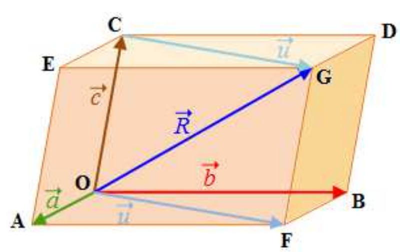

Figure 2.42: Addition of three concurrent and non-coplanar vectors

vectors $\vec{u}$ and $\vec{c}$. Then by law of parallelogram of vector addition, OG represents the sum of $\vec{u}$ and $\vec{c}$. Let it be $\vec{R}$.
Hence the resultant of three concurrent and non-coplanar vectors, acting at O, is represented by the diagonal, drawn through O, of a parallelepiped with the given vectors for its edges.

Theorem 2.10.1. *For any vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ and any scalars k and l, the following relationships hold:*

(a) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

(b) $\left(\vec{a}+\vec{b}\right)+\vec{c}=\vec{a}+\left(\vec{b}+\vec{c}\right)$

(c) $\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$

(d) $\vec{a}+(-\vec{a})=\vec{0}$

(e) $k\,(l\vec{a})=(kl)\,\vec{a}$

(f) $k\left(\vec{a}+\vec{b}\right)=k\vec{a}+k\vec{b}$

(g) $(k+l)\,\vec{a}=k\vec{a}+l\vec{a}$

(h) $1\vec{a}=\vec{a}$

Proof The results in this theorem can be proved algebraically by using components. Let $\vec{a}=\langle x_1,y_1\rangle$, $\vec{b}=\langle x_2,y_2\rangle$, $\vec{c}=\langle x_3,y_3\rangle$ and $\vec{0}=\langle 0,0\rangle$ be vectors in $2-space$.

(a) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

Consider left hand side

$$\begin{aligned}\vec{a}+\vec{b} &= \langle x_1,y_1\rangle+\langle x_2,y_2\rangle\\ &= \langle x_1+x_2,y_1+y_2\rangle\end{aligned}$$

Since commutative law holds in $\mathbb{R}$ (set of real numbers), so we can write

$$\begin{aligned}\vec{a}+\vec{b} &= \langle x_2+x_1,y_2+y_1\rangle\\ &= \langle x_2,y_2\rangle+\langle x_1,y_1\rangle\\ &= \vec{b}+\vec{a}\end{aligned}$$

Geometrically this result is illustrated in Fig. 2.43

(b) $\left(\vec{a}+\vec{b}\right)+\vec{c}=\vec{a}+\left(\vec{b}+\vec{c}\right)$

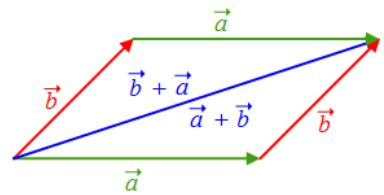

Figure 2.43: Vector addition is commutative

Consider left hand side

$$\begin{aligned}\left(\vec{a}+\vec{b}\right)+\vec{c} &= (\langle x_1, y_1\rangle + \langle x_2, y_2\rangle) + \langle x_3, y_3\rangle \\ &= \langle x_1+x_2, y_1+y_2\rangle + \langle x_3, y_3\rangle \\ &= \langle (x_1+x_2)+x_3, (y_1+y_2)+y_3\rangle\end{aligned}$$

Since associative law holds in $\mathbb{R}$, so we can write

$$\begin{aligned} &= \langle x_1+(x_2+x_3), y_1+(y_2+y_3)\rangle \\ &= \langle x_1, y_1\rangle + \langle x_2+x_3, y_2+y_3\rangle \\ &= \vec{a}+\left(\vec{b}+\vec{c}\right)\end{aligned}$$

(c) $\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$

Consider left hand side

$$\begin{aligned}\vec{a}+\vec{0} &= \langle x_1, y_1\rangle + \langle 0, 0\rangle \\ &= \langle x_1+0, y_1+0\rangle\end{aligned}$$

Since 0 is identity with respect to addition in $\mathbb{R}$, so we can write

$$\begin{aligned} &= \langle x_1, y_1\rangle \\ &= \vec{a}\end{aligned}$$

Similarly

$$\vec{0}+\vec{a} = \vec{a}$$

(d) $\vec{a}+(-\vec{a})=\vec{0}$

The vector $-\vec{a}=\langle -x_1, -y_1\rangle$. Then

$$\begin{aligned}\vec{a}+(-\vec{a}) &= \langle x_1, y_1\rangle + \langle -x_1, -y_1\rangle \\ &= \langle x_1-x_1, y_1-y_1\rangle\end{aligned}$$

Since $-x_1$ is additive inverse of x_1 in $\mathbb{R}$, so we can write

$$\langle x_1 - x_1, y_1 - y_1\rangle = \langle 0, 0\rangle = \vec{0}$$

Hence $\vec{a} + (-\vec{a}) = \vec{0}$

(e) $k\,(l\vec{a}) = (kl)\,\vec{a}$

Multiplication of a scalar l with a vector $\vec{a}$ is

$$\begin{aligned} l\vec{a} &= l\langle x_1, y_1\rangle \\ &= \langle lx_1, ly_1\rangle \end{aligned}$$

Again multiplication of a scalar k with a vector $l\vec{a}$ will give

$$\begin{aligned} k\,(l\vec{a}) &= k\langle lx_1, ly_1\rangle \\ &= \langle klx_1, kly_1\rangle \\ &= (kl)\langle x_1, y_1\rangle \\ &= (kl)\,\vec{a} \end{aligned}$$

(f) $k\left(\vec{a} + \vec{b}\right) = k\vec{a} + k\vec{b}$

Consider left hand side

$$\begin{aligned} k\left(\vec{a} + \vec{b}\right) &= k\left(\langle x_1, y_1\rangle + \langle x_2, y_2\rangle\right) \\ &= k\left(\langle x_1 + x_2, y_1 + y_2\rangle\right) \\ &= \langle k\left(x_1 + x_2\right), k\left(y_1 + y_2\right)\rangle \\ &= \langle kx_1 + kx_2, ky_1 + ky_2\rangle \\ &= \langle kx_1, ky_1\rangle + \langle kx_2, ky_2\rangle \\ &= \left(\langle kx_1, ky_1\rangle\right) + \left(\langle kx_2, ky_2\rangle\right) \\ &= k\vec{a} + k\vec{b} \end{aligned}$$

(g) $(k + l)\,\vec{a} = k\vec{a} + l\vec{a}$

Multiplication of a scalar $(k + l)$ with a vector $\vec{a}$ is

$$\begin{aligned} (k + l)\,\vec{a} &= (k + l)\,\langle x_1, y_1\rangle \\ &= \langle (k + l)\,x_1, (k + l)\,y_1\rangle \end{aligned}$$

Since distributive law holds in $\mathbb{R}$, so we can write

$$\begin{aligned} &= \langle kx_1 + lx_1, ky_1 + ly_1\rangle \\ &= \langle kx_1, ky_1\rangle + \langle lx_1, ly_1\rangle \\ &= k\langle x_1, y_1\rangle + l\langle x_1, y_1\rangle \\ &= k\vec{a} + l\vec{a} \end{aligned}$$

(h) $1\vec{a} = \vec{a}$

Multiplication of a scalar 1 with a vector $\vec{a}$ is

$$\begin{aligned} 1\vec{a} &= 1\langle x_1, y_1\rangle \\ &= \langle 1x_1, 1y_1\rangle \end{aligned}$$

Since 1 is identity with respect to multiplication in $\mathbb{R}$, so we can write

$$\langle 1x_1, 1y_1\rangle = \langle x_1, y_1\rangle = \vec{a}$$

2.11 Vector Field

A vector field in a plane is a function that associates with each point P in the plane a unique vector $\vec{F}(P)$ parallel to the plane.
If $\vec{F}(P)$ is a vector field in an $xy-$coordinate system, and $P(x,y)$ is a point in xy plane, then the associated vector with P will have components that are functions of x and y. Thus, the vector field $\vec{F}(P)$ can be expressed as

$$\vec{F}(x,y) = f(x,y)\hat{i} + g(x,y)\hat{j} \tag{2.11.1}$$

Similarly, a vector field in $3-$space is a function that associates with each point P in $3-$space a unique vector $\vec{F}(P)$ in $3-$space.
If $\vec{F}(P)$ is a vector field in an $xyz-$coordinate system, and $P(x,y,z)$ then the vector field $\vec{F}(P)$ can be expressed as

$$\vec{F}(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k} \tag{2.11.2}$$

If $\vec{r}$ is the position vector of P, the vector field of P can also be represented as $\vec{F}(P) = \vec{F}(r)$. For example the gravitational field can be written as

$$\vec{F}(\vec{r}) = -\frac{GmM}{r^3}\vec{r}$$

2.11.1 Gradient of a Function

The gradient of a function (at a point) is a vector that points in the direction in which the function increases most rapidly. In $2-$space the gradient of a function $f(x,y)$ is denoted by ∇f and is defined as

$$\nabla f(x,y) = f_x(x,y)\hat{i} + f_y(x,y)\hat{j} \tag{2.11.3}$$

The gradient of a $3-$space function $f(x,y,z)$ is

$$\nabla f(x,y,z) = f_x(x,y,z)\hat{i} + f_y(x,y,z)\hat{j} + f_z(x,y,z)\hat{k} \tag{2.11.4}$$

Note

- At (x, y) the surface $z = f(x, y)$ has its maximum slope in the direction of the gradient, and the maximum slope is $\|f(x, y)\|$.
- At (x, y) the surface $z = f(x, y)$ has its minimum slope in the direction that is opposite to the gradient, and the minimum slope is $-\|f(x, y)\|$.

Flux
The flux of a vector field is the volume of fluid flowing through an element of surface area per unit time.
Conservative Fields and Potential functions A vector field $\vec{F}(r)$ in $2-$space or $3-$space is said to be conservative in a region if it is the gradient field for some function U in that region, that is, if

$$\vec{F} = -\nabla U \tag{2.11.5}$$

The function U is called a potential function for $\vec{F}$ in the region. This concept will be discussed in detail in chapter **??**.

2.11.2 Divergence and Curl

Divergence and curl are two important operations on vector fields in 3-space. These names originate in the study of fluid flow, in which case the divergence relates to the way in which fluid flows toward or away from a point and the curl relates to the rotational properties of the fluid at a point.
The divergence of a vector field is defined as the flux divergence per unit volume. More clearly the divergence of a vector field is a number that can be thought of as a measure of the rate of change of the density of the fluid at a point. Consider the vector field

$$\vec{F}(x, y, z) = f(x, y, z)\hat{i} + g(x, y, z)\hat{j} + h(x, y, z)\hat{k} \tag{2.11.6}$$

the divergence of $\vec{F}$, written $div\vec{F}$, to be the function given by

$$div\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \tag{2.11.7}$$

The curl of a vector field measures the tendency of the vector field to rotate about a point. The curl of a vector field at a point is a vector that points in the direction of the axis of rotation and has magnitude represents the speed of the rotation. Mathematically the curl of $\vec{F}$, written $curl\vec{F}$, to be the vector field given by

$$\begin{aligned} curl\vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \qquad (2.11.8) \\ &= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{i} - \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{k} \end{aligned}$$

The ∇ Operator
The symbol ∇ appearing in the gradient expression ∇U can be viewed as an operator known as del operator

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \tag{2.11.9}$$

which when applied to $U(x, y, z)$ produces the gradient

$$\nabla U = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial h}{\partial z}\hat{k} \tag{2.11.10}$$

The del operator allows us to express the divergence of a vector field

$$\vec{F}(x, y, z) = f(x, y, z)\hat{i} + g(x, y, z)\hat{j} + h(x, y, z)\hat{k} \tag{2.11.11}$$

in dot product notation as

$$div\vec{F} = \nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \tag{2.11.12}$$

and the curl of this field in cross-product notation as

$$curl\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \tag{2.11.13}$$

Example 2.11.1. *Find divergence and curl for a vector field*

$$\vec{F} = x^2e^{-z}\hat{i} + 2yz^2\hat{j} + 3ye^z\hat{k}$$

Solution

$$\begin{aligned} div\vec{F} &= \nabla \cdot \vec{F} \\ &= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle x^2e^{-z}, 2yz^2, 3ye^z \rangle \\ &= \frac{\partial x^2e^{-z}}{\partial x} + \frac{\partial 2yz^2}{\partial y} + \frac{\partial 3ye^z}{\partial z} \\ &= 2xe^{-z} + 2z^2 + 3ye^z \end{aligned}$$

and the curl of this field is

$$\begin{aligned} curl\vec{F} &= \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2e^{-z} & 2yz^2 & 3ye^z \end{vmatrix} \\ &= (3e^z - 4yz)\,\hat{i} - x^2e^{-z}\hat{j} \end{aligned}$$

Exercises

1. Find the vector $\overrightarrow{AB}$ and its unit vector from the following pair of points. Also find a vector with magnitude 4 having same direction of $\overrightarrow{AB}$ vector.

   a) $A(1,5),\ B(2,5)$
   b) $A(5,2),\ B(0,0)$
   c) $A(4,2),\ B(4,4)$
   d) $A(4,2,0),\ B(2,2,2)$
   e) $A(0,0,0),\ B(1,4,1)$
   f) $A(3,1,3),\ B(9,4,3)$

2. Find the terminal point of $\vec{a} = 3\hat{\imath} + 2\hat{\jmath}$ if the initial point is (2,2).

3. Find the initial point of $\vec{a} = \langle 3, 1, -2 \rangle$ if the terminal point is (3, 0,3).

4. Consider the vectors $\vec{a} = \langle 2, -1, 3 \rangle$ and $\vec{b} = \langle -7, 2, -1 \rangle$. Find

   a) $\vec{a} \cdot \vec{a}$
   b) $\vec{a} \cdot \vec{b}$
   c) $\hat{a}$ and $\hat{b}$
   d) the angle between $\vec{a}$ and $\vec{b}$.
   e) $\vec{a} \times \vec{a}$
   f) $\vec{a} \times \vec{b}$ and check that it is orthogonal to both $\vec{a}$ and $\vec{b}$.
   g) $\vec{b} \times \vec{a}$
   h) the orthogonal components of $\vec{a}$, $\vec{b}$ and $\vec{a} \times \vec{b}$
   i) the direction cosines of $\vec{a}$, $\vec{b}$ and $\vec{a} \times \vec{b}$ and approximate the direction angles to the nearest degree.
   j) the area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$

5. Whether the points $A(2,2,0),\ B(1,0,2)$, and $C(0,4,3)$ are collinear or non-collinear. If non-collinear find the area of the triangle determined by these points.

6. Find $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)$. Also find the volume V of the parallelepiped whose adjacent edges are $\vec{a}$, $\vec{b}$ and $\vec{c}$ given as follow:

   a) $\vec{a} = \langle 2, -3, 1 \rangle,\ \vec{b} = \langle 4, 1, -3 \rangle$ and $\vec{c} = \langle 0, 1, 5 \rangle$
   b) $\vec{a} = \langle 1, -2, 2 \rangle,\ \vec{b} = \langle 0, 3, 2 \rangle$ and $\vec{c} = \langle -4, 1, -3 \rangle$
   c) $\vec{a} = \hat{\imath},\ \vec{b} = \hat{\imath} + \hat{\jmath}$ and $\vec{c} = \hat{\imath} + \hat{\jmath} + \hat{k}$
   d) $\vec{a} = 2\hat{\imath} + \hat{\jmath},\ \vec{b} = \hat{\imath} - 3\hat{\jmath} + \hat{k}$ and $\vec{c} = 4\hat{\imath} + \hat{k}$

7. Check whether the vectors $\vec{a} = \langle 3, -2, 5 \rangle$, $\vec{b} = \langle 0, 3, 2 \rangle$ and $\vec{c} = \langle 1, 4, -4 \rangle$ are co-planer, if not find the volume V of the parallelepiped that has $\vec{a}$, $\vec{b}$ and $\vec{c}$ as adjacent edges.

8. Find the resultant of two vectors with magnitudes $2N$ and $4N$ if the angle between them is (a) $0°$, (b) $30°$ (c) $45°$ (d) $60°$ (e) $90°$, (f) $120°$ and (g) $180°$.

9. A block of weight 400 N rests on a smooth ramp that is inclined at an angle of $30°$ with the ground (Figure 8.12). Let there are no frictional forces, then how much force does the block exert against the ramp, and how much force must be applied to the rope in a direction parallel to the ramp to prevent the block from sliding down the ramp?

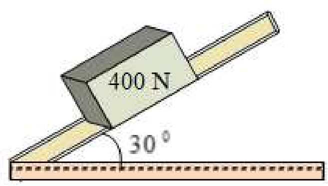

Figure 2.44: Block on inclined plane

10. A wagon is pulled horizontally by exerting a constant force of $50N$ on the handle at an angle of $60°$ with the horizontal. How much work is done in moving the wagon 10 m?

11. A force of $\vec{F} = \langle 3, -1, 2 \rangle$ N is applied to a point that moves on a line from $A(0, 1, -1)$ to $B(4, 1, 2)$. If distance is measured in feet, how much work is done?

12. If $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$ be two vectors and $\vec{OC} = \vec{c}$ be their resultant as shown in Fig. 2.45. If any transversal cuts their lines of actions in the points D, E and F as shown in Fig. 2.45, prove that

$$\frac{a}{OD} + \frac{b}{OE} = \frac{c}{OF}$$

13. Find divergence and curl for the following vector fields
(a) $\vec{F} = x^2yz^3\hat{\imath} + 2xy^2 \sin z\hat{\jmath} + 3ze^y\hat{k}$
(b) $\vec{F} = x^2e^y\hat{\imath} + 2xe^y\hat{\jmath} + 3\cos ye^x\hat{k}$
(c) $\vec{F} = x^2y\hat{\imath} + 2xy^2z^3\hat{\jmath} + 3ze^x\hat{k}$

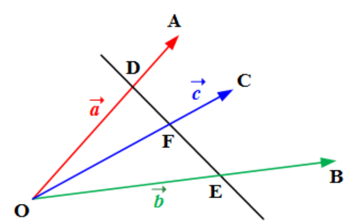

Figure 2.45: triangle of vectors

Chapter 3

Composition and Resolution of Forces

In mechanics, the term force is very common as it is an important factor of this subject. We define it as:
an agent which produces or tends to produce motion, equivalently destroys or tends to destroy motion.
For example, a street hawker push the cart to set it in motion and pull it to make it at rest. Hence force is applied to move or to stop the cart.

3.1 Force

Newton's fundamental laws and gravitational law are about force. Newton defined force as:
Force is the external agency applied on a body to change its state of rest and motion.
It means a force can cause the acceleration of the body. It is a vector quantity and is usually denoted by $\vec{F}$. Newton's second law of motion provides mathematical concept of force. If a force F acts on a body of mass m and produces an acceleration a, then the magnitude of the force F is

$$F = ma$$

In SI, its unit of measure is N ($Newton$). A force in action may be constant force of variable force.
Since force is a vector quantity, so it is completely specified by the following four characteristics:

- Magnitude

- Point of application
- Line of action and
- Direction

3.1.1 Effect of Force

When a force acts on a body, it may have the following effects:

(a) It may bring a moving body at rest.

(b) It may bring a body at rest in motion.

(c) It may change the size or shape of the body.

3.1.2 Fundamental Natural Forces

The following four forces exist naturally may be regarded as fundamental forces.

1. Gravitational force
2. Electromagnetic force
3. Strong nuclear force and
4. Weak nuclear force

The above forces are defined as:

1. **Gravitational Force**

A force with which the earth attracts every body towards its center is called gravitational force. It is an external force and always acts downward. Remember that gravity is a non-contact force. For a body of mass m, the magnitude of this force is

$$W = mg$$

Gravitational force is the weakest force among the fundamental forces of nature but has the greatest largescale impact on the universe. Unlike the other forces, gravity works universally on all matter and energy, and is universally attractive.

2. **Electromagnetic Force**

This force exist between charged particles such as the force between two electrons, or the force between two current carrying wires. It is attractive for unlike charges and repulsive for like charges. The electromagnetic force obeys inverse square law. It is very strong compared to the gravitational force. It is the combination of electrostatic and magnetic forces.

3. **Strong Nuclear Force**

It is the strongest of all the basic forces of nature. It, however, has the shortest range, of the order of $10^{-15}m$. This force holds the protons and neutrons together in the nucleus of an atom.

4. **Weak Nuclear Force**

Weak nuclear force is important in certain types of nuclear process such as −decay. This force is not as weak as the gravitational force.

3.1.3 System of Forces

Like vectors, when several forces act simultaneously on a body, they constitute a system of forces. These system are named, depending on the position of line of action of the forces as following:

- **Concurrent forces** If the lines of action of all the forces acting on a body pass through a single point, the forces are termed as concurrent forces.
- **Collinear forces** If the line of action of all the forces acting on a body lie along a single line, the forces are called collinear forces. Example is, forces on a rope in a tug of war.
- **Coplanar forces** If all the forces acting on a body lie in a single plane, they are called coplanar forces. They may be:
  (a) coplanar and concurrent forces.
  (b) coplanar and non-concurrent forces.
- **Non-Coplanar forces** If all the forces acting on a body do not lie in a single plane they are called non-coplanar forces or forces in space. They may be:
  (a) non-coplanar and concurrent forces.
  (b) non-coplanar and non-concurrent forces.

3.1.4 Classification of Forces

Forces acting on a rigid body can be classified as:

(a) External forces

(b) Internal forces

External force

A force acting on a body from some external agency is called external force. External Forces are further classified into two types

1. **Applied or Active Force**

Applied forces, or contact forces, are also external forces.
We apply external forces when we push a swing, pull an elastic, or throw a ball.

2. **Reactive Force or Force of Constraint** When a particle or body is made to move along or rest on a curve or surface, the force exerted by such curve or surface is called a force of constraint or reactive force.

If a particle rests on or moves along an inclined plane, the reaction of the plane is a reactive force but the weight of the particle is an active force.

Internal force

Internal forces are those forces which the different parts of a system exert on each other and such forces obey Newtons Third Law of Motion.
If we regard the earth and the moon as a system, then mutual attraction will be internal force but the attraction of the sun exerted on each of them will be am external force. **Examples of Internal Forces** There are four types of internal forces:

1. Tension
2. Compression
3. Torsion
4. Shear

1. **Tension**

Tension is an internal force that pulls the particles of a stretched object apart. The example is an elastic band. When it is stretched by fingers (the external force), the internal force of tension causes all of the particles of the band to pull apart. Tension acts on many objects, such as a trampoline or guitar strings.
If an elastic band is stretched too far, it breaks. The particles in an elastic material can stretch only to a certain point. This point is called the **breaking point**.

2. **Compression**

An object that is pressed or squeezed experiences compression.
Compression is an internal force that presses the particles of an object together. Compression happens when you kick a soccer ball or lay your head on a pillow. The springs inside a mattress compress when you lie on a bed.
Compressed objects usually return to their original shape when the external force is removed.

3. **Torsion**

Torsion acts in an object when the object is twisted. It can be created by twisting one or both ends of the object.
Torsion happens when we turn a doorknob or when a skater twists in a jump

4. **Shear**

Shear forces happen when two forces in an object push or pull in opposite directions. Shear forces can bend, tear, and cut objects.
A strong wind can cause shear forces within a tree. The forces can bend or break the tree.

3.1.5 Some other Well known Forces

- **Action and Reaction**

When two bodies are in contact, each exerts a force on the other, known as action and reaction. They are equal in magnitude and opposite in direction.

- **Friction**

When a body moves or tends to move upon another body, an opposing force (opposite to the direction of motion) appears between their surfaces, known as the force of friction. It will be discussed in detail in chapter 5.

3.2 Composition and Resolution of Forces

As force is a vector quantity, so its composition and resolution will be dealt by vector laws, hence all properties of vectors should be in your mind.
Resultant Force If a number of forces act simultaneously on a particle, then it is possible to find out a single force which can replace them. This single force has the same effect as produced by all the given forces. This single force is called resultant force. It is generally denoted by $\vec{R}$.
Composition Of Forces The process by which the resultant force of a number of given forces is obtained, is known as composition of forces.
Component of a Force A force can be resolved into two or more parts, without changing its effect on the body. These parts are called components of a force. Each component has shared effect on the body in some direction.
Resolution Of Forces The process of dividing a given force into a number of components, is called resolution of a force. It is opposite process of composition of forces.
In this chapter, first we will discuss resultant force, for this recall addition of vectors.
Two or more forces can be added geometrically (graphically) and analytically to obtain their resultant. A number of methods are available to add forces. A few of them are listed below.

3.3 Resultant of Coplanar and non-Concurrent Forces

The resultant of two or more coplanar and non-concurrent forces can be calculated by head to tail method.

3.3.1 Head to Tail Method

Head to tail method or graphical method is one of the easiest method used to find the resultant force of two or more than two forces. Given two forces $\vec{P}$ and $\vec{Q}$. Their resultant $\vec{P}+\vec{Q}$ is obtained by joining the tail of $\vec{Q}$ with the head of $\vec{P}$ without making any alteration in the direction of vectors. Draw a vector from the tail of $\vec{P}$ to the head of $\vec{Q}$. This vector is $\vec{P}+\vec{Q}$. Also we can obtain $\vec{Q}+\vec{P}$, the same vector, let it be $\vec{R}$. Hence we can say

$$\vec{R} = \vec{P} + \vec{Q} \quad = \quad \vec{Q} + \vec{P}$$

This means vector addition is commutative. Geometrically this sum is illustrated in Fig.

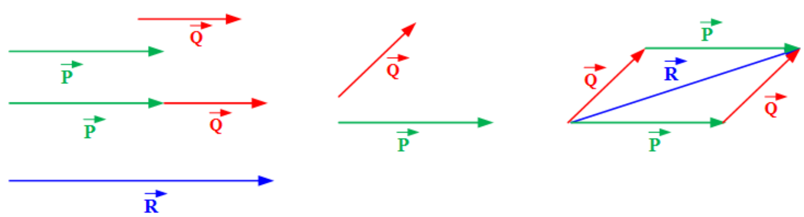

Figure 3.1: Resultant force by head to tail method

3.1, on left side two forces are acting in same directions and on right side two vectors are not acting in same directions. The length of $\vec{P}+\vec{Q}$ is the magnitude and its inclination with $\vec{P}$ is its direction relative to P.
Any number of forces can be added by this method.

Example 3.3.1. *Consider two forces with magnitudes* $3N$ *and* $4N$ *are acting on a body. Find their resultant vector using head to tail rule.*

Solution: We label the forces as

$$\begin{aligned} P &= 3\ N \\ Q &= 4\ N \end{aligned}$$

Extending the forces in backward direction, we find the angle between them is $60°$. Consider 2-dimensional rectangular coordinate system. Select a suitable scale to represent the forces.

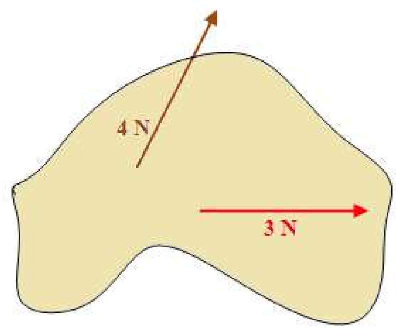

Figure 3.2: Two force acting on body

Let $1N = 1cm$. Draw a line of length of $3cm$ in the direction of P. Draw a line of length of $4cm$ starting from the tip of P in the direction of Q. Draw a line starting from the tail of P and ends at the tip of Q. This is the resultant of the given forces. By measuring, we find its length is $5.1cm$, hence the magnitude of the resultant is $5.1N$. By using protector

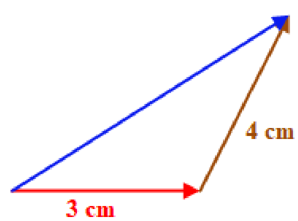

Figure 3.3: Resultant by head to tail rule.

we find the resultant makes an angle $\theta = 35°$ with force P.
When the two forces are not acting in same direction, the Head to tail method is known as triangle method and is defined as following.

Triangle Method

If two forces are represented in magnitude and direction by two sides of a triangle taken in order, then their resultant is the closing side of the triangle taken in the opposite order.

Proof Two forces $\vec{P}$ and $\vec{Q}$ are represented completely by two sides OA and AB of a triangle OAB. Then by vector addition (head to tail rule) the resultant vector $\vec{R}$ of two

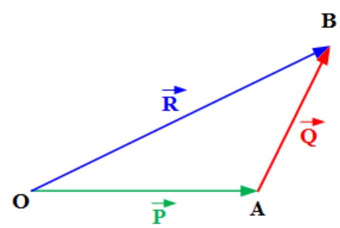

Figure 3.4: triangular Method

vectors $\vec{P}$ and $\vec{Q}$ is

$$\vec{R} = \vec{P} + \vec{Q}$$

Geometrically this sum is illustrated in Fig. 3.4. The length of $\vec{R}$ is the magnitude and its inclination with $\vec{P}$ is its direction.

Example 3.3.2. *Consider two concurrent and coplanar forces $\vec{P} = \langle 2, 3\rangle$ and $\vec{Q} = \langle 2, -3\rangle$ are acting on a body. Find their resultant force.*

Solution:The system is shown in Fig. 3.5. Here

$$\vec{P} = \langle 2, 3\rangle$$
$$\vec{Q} = \langle 2, -3\rangle$$

Their resultant is

$$\begin{aligned}\vec{R} &= \vec{P} + \vec{Q}\\ &= \langle 2, 3\rangle + \langle 2, -3\rangle\\ &= \langle 4, 0\rangle\end{aligned}$$

Geometrically the resultant is shown in Fig. 3.6.

Polygon Method

The triangle rule can be made more general to apply to any geometrical shape or polygon. This then becomes the polygon law. It can be stated as:
If a number of forces are represented both in magnitude and direction by the sides of a polygon taken in the same order, then their resultant is represented both in magnitude and direction by the closing side of the polygon taken in the opposite order.

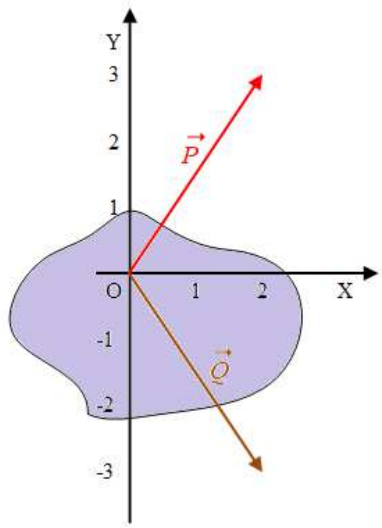

Figure 3.5: Forces $\vec{P}$ and $\vec{Q}$.

Proof We consider four forces $\vec{F_1}, \vec{F_2}, \vec{F_3}$ and $\vec{F_4}$, that are represented by four sides $OA.AB, BC$ and CD of a polygon $OABCD$. Then by vector addition (head to tail rule) the resultant force $\vec{R}$ of four forces is

$$\vec{R} \;=\; \vec{F_1} + \vec{F_2} + \vec{F_3} + \vec{F_4}$$

Geometrically this sum is illustrated in Fig. 3.7. The length of $\vec{R}$ is the magnitude and its inclination with $\vec{F_1}$ is its direction relative to $\vec{F_1}$.
Since vector addition is associative, the resultant force obtained by the polygon rule is independent of the order of composition of forces.

3.3.2 Subtraction of forces

The subtraction of a force from another is a force obtained by adding one force to the negative of the other. It is also called difference of forces.
Given two forces $\vec{P}$ and $\vec{Q}$. Their difference $\vec{P} - \vec{Q}$ is obtained by joining the tail of $-\vec{Q}$ with the head of $\vec{P}$ without making any alteration in the direction of forces. Draw a vector from the tail of $\vec{P}$ to the head of $-\vec{Q}$. This vector is $\vec{R} = \vec{P} - \vec{Q}$.

$$\vec{R} = \vec{P} - \vec{Q} \;=\; \vec{P} + \left(-\vec{Q}\right)$$

Geometrically this difference force is illustrated in Fig. 3.8, on left side the two forces are parallel and on right side the two forces are not parallel. The length of $\vec{R}$ is the magnitude

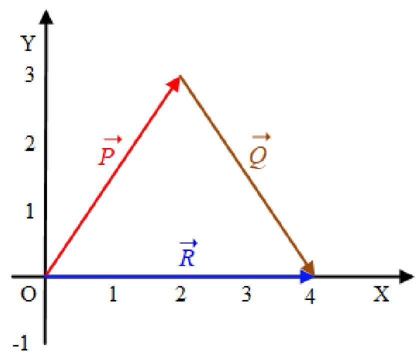

Figure 3.6: Resultant of $\vec{P}$ and $\vec{Q}$

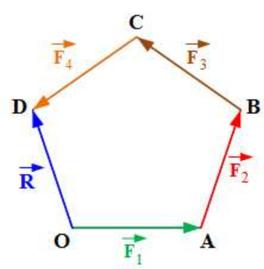

Figure 3.7: Polygon method

and its inclination with $\vec{P}$ is its direction relative to $\vec{P}$.

Example 3.3.3. *Consider two concurrent and coplanar forces $\vec{P} = \langle 2, 3\rangle$ and $\vec{Q} = \langle 2, -3\rangle$ are acting on a body. Find their difference force $P - Q$.*

Solution:The system is shown in Fig. 3.9. Here

$$\begin{aligned}\vec{P} &= \langle 2, 3\rangle \\ \vec{Q} &= \langle 2, -3\rangle\end{aligned}$$

and

$$-\vec{Q} = \langle -2, 3\rangle$$

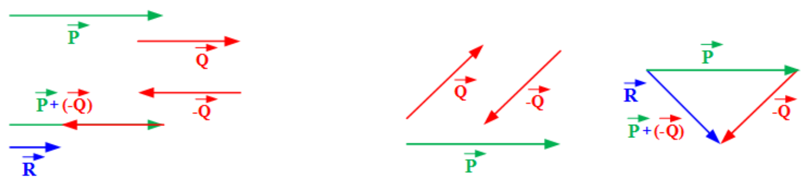

Figure 3.8: Subtraction of two forces.

Their difference $P - Q$ is

$$\begin{aligned} \vec{R} &= \vec{P} - \vec{Q} \\ &= \langle 2, 3 \rangle + \langle -2, 3 \rangle \\ &= \langle 0, 6 \rangle \end{aligned}$$

Geometrically the difference force is shown in Fig. 3.10.

3.4 Resultant of Concurrent and Coplanar Forces

Here the forces acting on a particle are concurrent and coplanar. We will find the resultant of two concurrent and coplanar forces acting on the body, first by parallelogram law and next by ratio theorem.

3.4.1 Parallelogram Law

If two forces acting simultaneously on a particle, are represented in magnitude and direction by two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram, passing through the point of intersection of the forces as shown in Fig. 3.11. and is given by (3.4.1)

$$\vec{R} = \vec{P} + \vec{Q} \qquad (3.4.1)$$

If α is the angle between these two forces then the magnitude of the resultant is

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha} \qquad (3.4.2)$$

and if the resultant makes an angle θ with the force $\vec{P}$, then

$$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha}$$

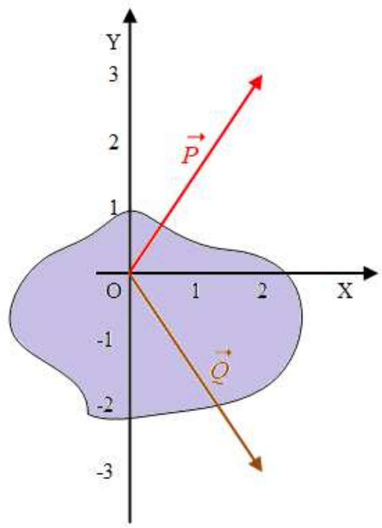

Figure 3.9: Forces $\vec{P}$ and $\vec{Q}$.

or

$$\theta = \arctan\left(\frac{Q\sin\alpha}{P+Q\cos\alpha}\right) \tag{3.4.3}$$

Note: We usually refer (3.4.2) for magnitude and (3.4.3) for direction of the resultant.
Proof The two forces $\vec{P}$ and $\vec{Q}$ are represented completely by two adjacent sides OA and OC of a parallelogram $OABC$. The vector $\vec{Q}$ can be represented by AB side, then by vector addition (head to tail rule) the resultant vector $\vec{R}$ of two vectors $\vec{P}$ and $\vec{Q}$ is

$$\vec{R} = \vec{P} + \vec{Q}$$

In Fig. 3.12, a triangle OAB is formed by three vectors $\vec{P}$, $\vec{Q}$, $\vec{R}$. In this triangle, $\angle A = \pi - \alpha$, $\angle O = \theta$, and let $\angle B = \phi$, then by law of sines we can write

$$\frac{R}{\sin(\pi-\alpha)} = \frac{P}{\sin\phi} = \frac{Q}{\sin\theta}$$

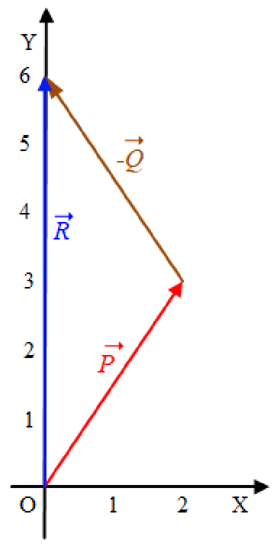

Figure 3.10: Difference $\vec{P} - \vec{Q}$ force.

Again consider triangle OAB, by law of cosines we can write

$$\begin{aligned}
\cos(\angle OAB) &= \frac{|\overline{OA}|^2 + |\overline{AB}|^2 - |\overline{OB}|^2}{2|\overline{OA}||\overline{AB}|} \\
\cos(\pi - \alpha) &= \frac{P^2 + Q^2 - R^2}{2PQ} \\
-\cos\alpha &= \frac{P^2 + Q^2 - R^2}{2PQ} \\
-2PQ\cos\alpha &= P^2 + Q^2 - R^2
\end{aligned}$$

Then R can be written as

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

Draw a perpendicular BD from B on OA, which meets OA at D produced as shown in Fig. 3.13. In right angle triangle ADB

$$\cos\alpha = \frac{|\overline{AD}|}{|\overline{AB}|}$$

then

$$\begin{aligned}
|\overline{AD}| &= |\overline{AB}|\cos\alpha \\
&= Q\cos\alpha
\end{aligned} \tag{3.4.4}$$

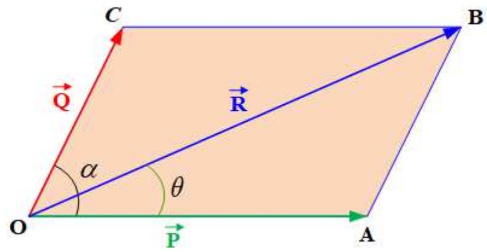

Figure 3.11: Parallelogram of vectors

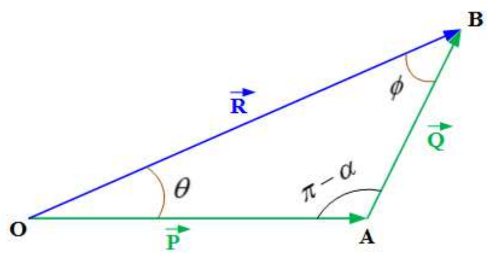

Figure 3.12: Resultant by head to tail rule.

Similarly

$$|\overline{BD}| \quad = \quad Q \sin\alpha \tag{3.4.5}$$

Let the resultant $\vec{R}$ makes an angle θ with the vector $\vec{P}$ as shown in Fig. 3.13. In right angle triangle ODB, $\angle BOD = \theta$, then

$$\tan\theta \quad = \quad \frac{|\overline{BD}|}{|\overline{OB}|} \tag{3.4.6}$$

Since

$$|\overline{OD}| \quad = \quad |\overline{OA}| + |\overline{AD}|$$

then (3.4.6) becomes

$$\tan\theta \quad = \quad \frac{|\overline{BD}|}{|\overline{OA}| + |\overline{AD}|} \tag{3.4.7}$$

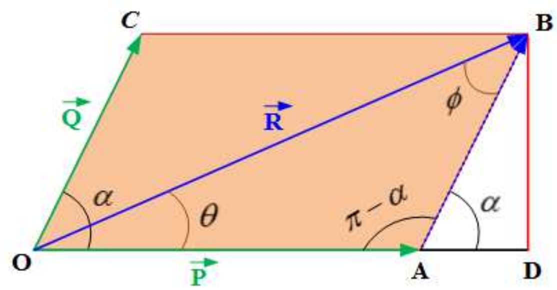

Figure 3.13: Parallelogram law

Using (3.4.4) and (3.4.5), (3.4.7) becomes

$$\tan\theta = \frac{Q\sin\alpha}{P+Q\cos\alpha}$$

or

$$\theta = \arctan\left(\frac{Q\sin\alpha}{P+Q\cos\alpha}\right) \qquad (3.4.8)$$

If the resultant $\vec{R}$ makes an angle θ with the force $\vec{Q}$, the resultant will remain the same, but in (3.4.8), P and Q will interchange.

Particular Cases Here some particular cases can be discussed for different values of the angle between the forces.

Case 1: If the two forces are acting in the same direction, then the angle between them is $\alpha = 0\ rad$, as shown in Fig. 3.14 and by (3.4.2), their resultant is

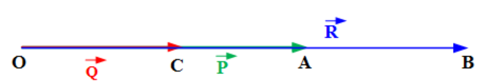

Figure 3.14: Vectors acting in the same direction

$$\begin{aligned} R &= \sqrt{P^2+Q^2+2PQ\cos(0)} \\ &= \sqrt{P^2+Q^2+2PQ} \\ &= \sqrt{(P+Q)^2} \\ &= P+Q \qquad (3.4.9) \end{aligned}$$

(3.4.9) gives the magnitude of the resultant. For direction, consider (3.4.3) with $\alpha = 0$

$$\begin{aligned} \tan\theta &= \frac{Q\sin(0)}{P + Q\cos(0)} \\ &= 0 \end{aligned} \tag{3.4.10}$$

(3.4.10) gives the direction of the resultant. In this case the magnitude of the resultant of the forces is the sum of the individual magnitudes of the forces and it acts in the direction of the forces. Such resultant is known as resultant of greatest magnitude.

Case 2: If the two forces are perpendicular, then the angle between them is $\alpha = \frac{\pi}{2}\ rad$, as shown in Fig. 3.15 and by (3.4.2), their resultant is

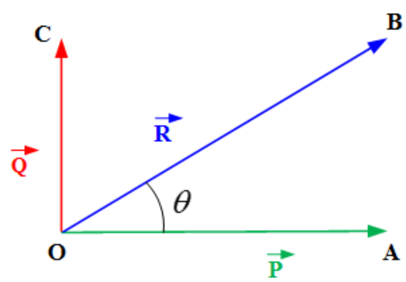

Figure 3.15: Vectors are orthogonal

$$\begin{aligned} R &= \sqrt{P^2 + Q^2 + 2PQ\cos(\frac{\pi}{2})} \\ &= \sqrt{P^2 + Q^2} \end{aligned} \tag{3.4.11}$$

(3.4.11) gives the magnitude of the resultant. For direction, consider (3.4.3) with $\alpha = \frac{\pi}{2}$

$$\begin{aligned} \tan\theta &= \frac{Q\sin(\frac{\pi}{2})}{P + Q\cos(\frac{\pi}{2})} \\ &= \frac{Q}{P} \end{aligned}$$

or

$$\theta = \arctan\left(\frac{Q}{P}\right) \tag{3.4.12}$$

(3.4.12) gives the direction of the resultant.

Case 3: If the two vectors are acting in the opposite direction, then the angle between them is $\alpha = \pi\ rad$, and by (3.4.2), their resultant is

$$\begin{aligned} R &= \sqrt{P^2 + Q^2 + 2PQ\cos(\pi)} \\ &= \sqrt{P^2 + Q^2 - 2PQ} \\ &= \sqrt{(P-Q)^2} \\ &= |P - Q| \end{aligned} \quad (3.4.13)$$

(3.4.13) gives the magnitude of the resultant, when the vectors are acting in opposite direction, as shown in Fig. 3.16. For direction, consider (3.4.3) with $\alpha = \pi$

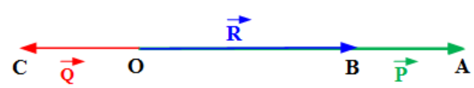

Figure 3.16: Vectors acting in opposite direction

$$\begin{aligned} \tan\theta &= \frac{Q\sin(\pi)}{P + Q\cos(\pi)} \\ &= 0 \end{aligned}$$

and

$$\theta = 0 \text{ or } \pi \quad (3.4.14)$$

(3.4.14) gives the direction of the resultant. In this case the magnitude of the resultant of the vectors is the difference of the individual magnitudes of the vectors and it acts in the direction of a forces with greater magnitude. This resultant is known as resultant of least magnitude.

Example 3.4.1. *Consider two concurrent and coplanar forces with magnitudes* $3N$ *and* $4N$ *are acting on a body. The angle between them is* 60 *degrees. Find their resultant vector using parallelogram law of vectors.*

Solution: We can label the forces as

$$\begin{aligned} P &= 3\ N \\ Q &= 4\ N \end{aligned}$$

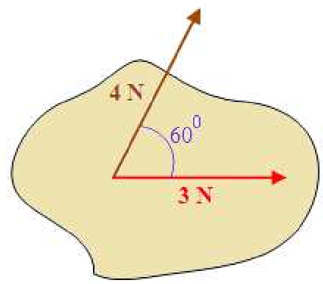

Figure 3.17: Two concurrent forces acting on a body.

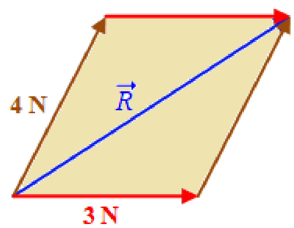

Figure 3.18: Resultant by parallelogram method.

and the angle between them is

$$\alpha = 60^\circ$$

Using (3.4.2), the resultant is

$$\begin{aligned} R &= \sqrt{3^2 + 4^2 + 2(3)(4)\cos 60} \\ &= \sqrt{9 + 16 + 12} \\ &= \sqrt{37} = 6.0828 \approx 6N \end{aligned}$$

Let P is the initial vector and Q is the terminal vector, then using (3.4.3), the angle of resultant with P is

$$\begin{aligned} \theta &= \arctan\left(\frac{4\sin(60)}{3 + 4\cos(60)}\right) \\ &= \arctan\left(\frac{3.4641}{5}\right) \\ &= \arctan(0.6928) = 34.72^\circ \approx 35^\circ \end{aligned}$$

The resultant makes an angle $\theta = 35°$ with force P.

Corollary 3.4.1. *If two concurrent and coplanar forces P and Q act at such an angle that their resultant $R = P$, show that if P is doubled and Q remained same then the new resultant is at right angle to Q.*

Proof Let two concurrent and coplanar forces P and Q act at O and α is the angle between them as shown in Fig. 3.19. Following the law of parallelogram of vector addition,

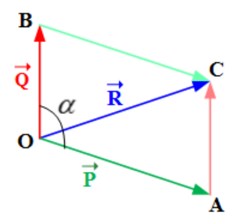

Figure 3.19: $R = P$ is resultant of P and Q

their resultant is given by

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha} \tag{3.4.15}$$

But from given condition $R = P$, (3.4.15) becomes

$$P = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha} \tag{3.4.16}$$

Squaring both sides, we get

$$P^2 = P^2 + Q^2 + 2PQ\cos\alpha$$

or

$$Q\left(Q + 2P\cos\alpha\right) = 0$$

Since $Q \neq 0$, so we have

$$Q + 2P\cos\alpha = 0 \tag{3.4.17}$$

Next the other given condition implies that we have to find the sum of $2P$ and Q. Let it be R_1 which makes an angle θ with Q that can be calculated by using (3.4.8).

$$\theta = \arctan\left(\frac{2P\sin\alpha}{Q + 2P\cos\alpha}\right) \tag{3.4.18}$$

Using (3.4.17) in (3.4.18), the angle is

$$\theta = \arctan(\infty) = \frac{\pi}{2} \tag{3.4.19}$$

(3.4.19) shows that new resultant is at right angle to Q as shown in Fig. 3.20.

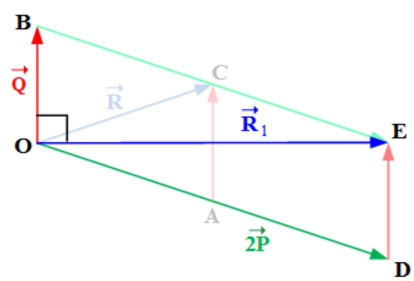

Figure 3.20: Resultant of $2P$ and Q is orthogonal with Q

Example 3.4.2. *Consider two concurrent and coplanar forces with magnitude* $5N$ *each are acting on a body. The angle between them is* 120 *degrees. The magnitude of their resultant is also* $5N$*. The system is shown in Fig. 3.21. Its angle with one of the force is* 60 *degrees. Next the magnitude of this force is doubled then show that the new resultant is at right angle to the force whose magnitude is fixed.*

Solution: After doubling the magnitude of a force, we can label them as

$$P = 10\ N$$
$$Q = 5\ N$$

and the angle between them is

$$\alpha = 120^\circ$$

Then we have to show that the new resultant is at right angle to force Q. Its angle with force Q can be calculated by using (3.4.3)

$$\begin{aligned}\theta &= \arctan\left(\frac{10\sin(120)}{5 + 10\cos(120)}\right)\\ &= \arctan\left(\frac{5\sqrt{3}}{5-5}\right)\\ &= \arctan(\infty) = 90^\circ\end{aligned}$$

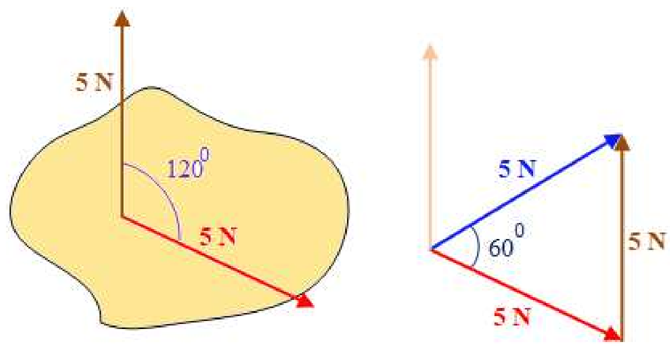

Figure 3.21: Forces and their resultant.

The resultant is at right angle to force Q and is shown in Fig. 3.22.

3.4.2 Ratio Theorem

As the ratio is chosen as λ, μ, so this theorem is also known as λ, μ theorem. In this theorem, the two forces are concurrent and coplanar. Their resultant force also acts at the point of action of these forces. A line lies on the lines of action of these forces that helps to determine the resultant.

Theorem 3.4.2. *If A and B are two terminal points of forces $\vec{a}$ and $\vec{b}$ in a system with O as point of application. Next C is a point which divides AB in ratio $\lambda : \mu$, then their resultant $\vec{c}$ with terminal point C relative to O is*

$$\vec{c} = \frac{\mu\vec{a} + \lambda\vec{b}}{\lambda + \mu} \tag{3.4.20}$$

equivalently

$$(\lambda + \mu)\,\vec{OC} = \mu\vec{OA} + \lambda\vec{OB} \tag{3.4.21}$$

Proof Two points A and B having position vectors $\vec{a}$ and $\vec{b}$ relative to O are shown in Fig. 3.27. If C divides AB in ratio $\lambda : \mu$, then from Fig. 3.27, we can write

$$AC = \frac{\lambda}{\lambda + \mu}AB$$

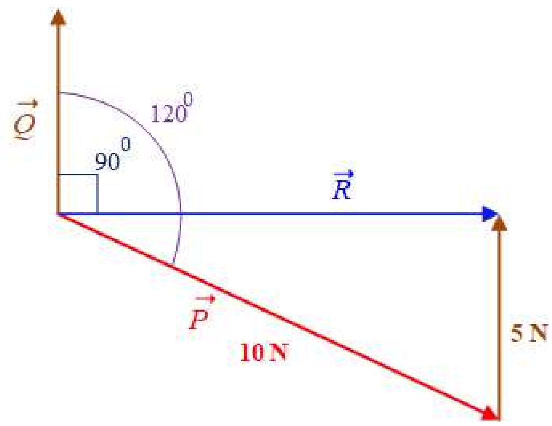

Figure 3.22: Resultant of $2P$ and Q is orthogonal with Q

Hence

$$\vec{AC} = \frac{\lambda}{\lambda+\mu}\vec{AB} \tag{3.4.22}$$

Following *head to tail rule* of vector addition, $\vec{OB}$ can be written as a sum of $\vec{OA}$ and $\vec{AB}$ (see Fig. 3.27). Then

$$\vec{AB} = \vec{OB} - \vec{OA} \tag{3.4.23}$$

Using (3.4.23), (3.4.22) can be written as

$$\vec{AC} = \frac{\lambda}{\lambda+\mu}\left(\vec{OB} - \vec{OA}\right) \tag{3.4.24}$$

Again following *head to tail rule* of vector addition, $\vec{OC}$ can be written as (see Fig. 3.27)

$$\vec{OC} = \vec{OA} + \vec{AC} \tag{3.4.25}$$

Since $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$ and $\vec{OC} = \vec{c}$ is given by (3.4.24). Then (3.4.25) becomes

$$\begin{aligned}\vec{c} &= \vec{a} + \frac{\lambda}{\lambda+\mu}\left(\vec{b} - \vec{a}\right) \\ &= \frac{(\lambda+\mu-\lambda)\,\vec{a} + \lambda\vec{b}}{\lambda+\mu} \\ &= \frac{\mu\vec{a} + \lambda\vec{b}}{\lambda+\mu}\end{aligned}$$

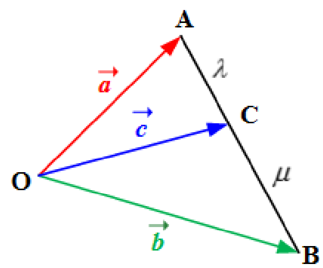

Figure 3.23: λ μ Theorem.

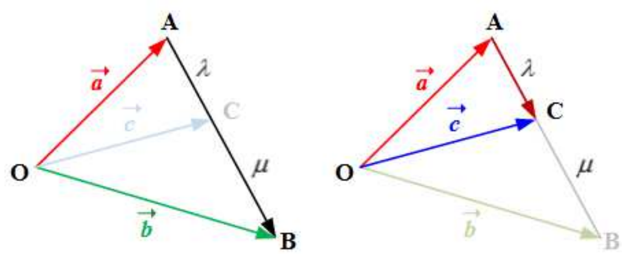

Figure 3.24: Vector addition by head to tail rule.

Hence the result.
Special case: When $\lambda = \mu$, then C is the mid point of AB and $\vec{c}$ is

$$\vec{c} \;=\; \frac{\vec{a} + \vec{b}}{2}$$

Example 3.4.3. *Let the force along OA is $\vec{a} = -\hat{i} + 3\hat{j} + \hat{k}$ and along OB is $\vec{b} = 2\hat{i} - 2\hat{k}$. If C divides the line AB in ratio $2:1$ find their resultant $\vec{c}$.*

Solution Here $\lambda = 2$ and $\mu = 1$ and $\lambda + \mu = 3$,
the vectors $\mu\vec{a} = -\hat{i} + 3\hat{j} + \hat{k}$ and $\lambda\vec{b} = 2\vec{b} = 4\hat{i} - 4\hat{k}$

then the resultant is

$$\begin{aligned}\vec{c} &= \frac{\left(-\hat{\imath} + 3\hat{\jmath} + \hat{k}\right) + \left(4\hat{\imath} - 4\hat{k}\right)}{3} \\ &= \frac{3\hat{\imath} + 3\hat{\jmath} - 3\hat{k}}{3} \\ &= \hat{\imath} + \hat{\jmath} - \hat{k}\end{aligned}$$

Example 3.4.4. *In triangle OAB, $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$ as shown in Fig. 3.25. If C*

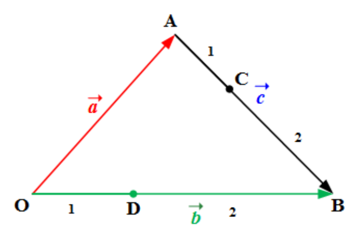

Figure 3.25: triangle of vectors

divides the line AB in ratio $1:2$ and D divides the line OB in ratio $1:2$. Find $\vec{DC}$ and hence show that $\vec{DC}$ is parallel to $\vec{OA}$.

Solution Following $\lambda,\ \mu$ theorem, the sum of $\vec{OA}$ and $\vec{OB}$ is $\vec{OC}$ as shown in Fig. 3.26 (a) with $\lambda = 1,\ \mu = 2$. Hence $\vec{OC}$ is

$$\vec{OC} = \frac{2\vec{a} + \vec{b}}{3}$$

In Fig. 3.26 (b) $\vec{DO}$ is

$$\vec{DO} = -\frac{1}{3}\vec{b}$$

following *head to tail rule* of vector addition, in Fig. 3.26 (b) $\vec{DC}$ can be written as

$$\begin{aligned}\vec{DC} &= \vec{DO} + \vec{OC} \\ &= -\frac{1}{3}\vec{b} + \frac{2}{3}\vec{a} + \frac{1}{3}\vec{b} \\ &= \frac{2}{3}\vec{a}\end{aligned}$$

Since $\vec{DC}$ is a scalar multiple of $\vec{a}$ and is acting in the direction of $\vec{a}$; that is $\vec{DC}$ is parallel to $\vec{OA}$.

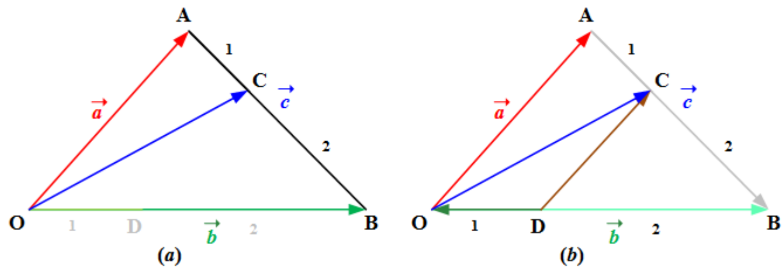

Figure 3.26: Vector addition by head to tail rule.

Corollary 3.4.3. *Two concurrent and coplanar forces $\vec{P}$ and $\vec{Q}$ act at point O and their resultant is $\vec{R}$. If any transversal cuts the lines of action of the forces in points A, B, C respectively, prove that*

$$\frac{R}{OC} = \frac{P}{OA} + \frac{Q}{OB} \tag{3.4.26}$$

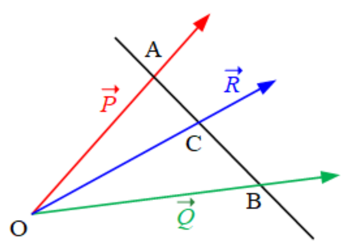

Figure 3.27: A line cuts lines of action of forces.

Proof First we will calculate directions of these forces, that will be the unit vectors. As $\vec{P}$ acts along $\vec{OA}$, unit vector in this direction is

$$e_1 = \frac{\vec{OA}}{OA}$$

Then $\vec{P}$ is

$$\begin{aligned} \vec{P} &= Pe_1 = P\frac{\vec{OA}}{OA} \\ &= \frac{P}{OA}\vec{OA} \end{aligned} \tag{3.4.27}$$

Similarly $\vec{Q}$ is

$$\begin{aligned}\vec{Q} &= Qe_2 = Q\frac{\vec{OB}}{OB} \\ &= \frac{Q}{OB}\vec{OB} \qquad (3.4.28)\end{aligned}$$

and $\vec{R}$ is

$$\begin{aligned}\vec{R} &= Re_3 = R\frac{\vec{OC}}{OC} \\ &= \frac{R}{OC}\vec{OC} \qquad (3.4.29)\end{aligned}$$

Using (3.4.21) from ratio theorem, we can write

$$\frac{P}{OA}\vec{OA} + \frac{Q}{OB}\vec{OB} = \left(\frac{P}{OA} + \frac{Q}{OB}\right)\vec{OC}$$

Using (3.4.27) and (3.4.28), we can write

$$\vec{P} + \vec{Q} = \left(\frac{P}{OA} + \frac{Q}{OB}\right)\vec{OC} \qquad (3.4.30)$$

As $\vec{R}$ is resultant of $\vec{P}$ and $\vec{Q}$, so

$$\vec{P} + \vec{Q} = \vec{R}$$

Using (3.4.29), we can write

$$\vec{P} + \vec{Q} = \frac{R}{OC}\vec{OC} \qquad (3.4.31)$$

From (3.4.32) and (3.4.31), we can write

$$\frac{R}{OC}\vec{OC} = \left(\frac{P}{OA} + \frac{Q}{OB}\right)\vec{OC}$$

As both vectors are equal, it means

$$\frac{R}{OC} = \frac{P}{OA} + \frac{Q}{OB}$$

Hence the result.

3.5 Resolution Of Forces

A force $\vec{F}$ can be resolved into an infinite number of pairs of possible components. For this, a parallelogram is constructed with $\vec{F}$ as diagonal and whose sides are along any two distinct lines which are coplanar with the force $\vec{F}$ and pass through its line of action. If the directions of the components are specified, the resolution of force into components is unique. The two directions in which we resolve a given force may or may not be mutually perpendicular. First we consider arbitrary directions.

3.5.1 Resolved Components of a Force in Two given Directions

Let $\vec{R}$ be a given force and $\vec{P}$ and $\vec{Q}$ be its resolved parts making angles θ and ϕ with it. Completing parallelogram $OACB$ as shown in Fig. 3.28. In this figure, consider the triangle OAB, in which the sides OA, AB and OB represents the forces P, Q and R in magnitude respectively. Then by law of sine's we have

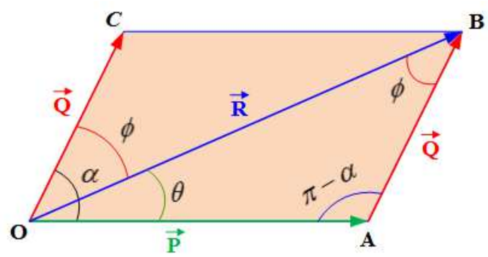

Figure 3.28: Resolved components of a vector in two given directions

$$\frac{R}{\sin(\pi-\alpha)} = \frac{P}{\sin\phi} = \frac{Q}{\sin\theta}$$

and we can write

$$P = \frac{R\sin\phi}{\sin(\pi-\alpha)} \tag{3.5.1}$$

and

$$Q = \frac{R\sin\theta}{\sin(\pi-\alpha)} \tag{3.5.2}$$

Then P and Q given by (3.5.1) and (3.5.3) respectively gives the magnitudes of resolved parts of a force.

Example 3.5.1. *Consider a force of magnitude* $50N$ *acts on a body. Let this force has two components, one on its right making an angle* $30°$ *with it and second on its left making an angle* $45°$ *with it. The system is shown in Fig. 3.29. Find their magnitudes.*

Solution: Let R be the force acting on the body. Let P be its component acting on its right making an angle $30°$ with it and Q be its component acting on its left making an angle $45°$ with it. The system is shown in Fig. 3.30. We can say

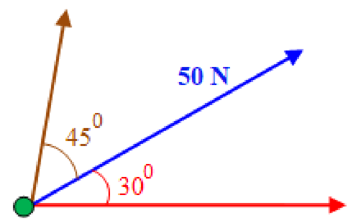

Figure 3.29: Resolved components of a force

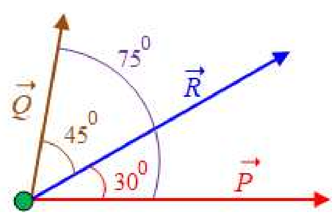

Figure 3.30: Resolved components of a force

$$\begin{aligned}\theta &= 30^\circ \\ \phi &= 45^\circ \\ \alpha &= 75^\circ\end{aligned}$$

then

$$\pi - \alpha = 180 - 75 = 105$$

Using (3.5.1), the magnitude of P is

$$\begin{aligned}P &= \frac{50\sin(45)}{\sin(105)} \\ &= \frac{50(0.707)}{0.966} \\ &= 36.6N\end{aligned}$$

and (3.5.3), the magnitude of Q is

$$\begin{aligned}Q &= \frac{50\sin(30)}{\sin(105)} \\ &= \frac{50(0.5)}{0.966} \\ &= 25.9N\end{aligned}$$

3.5.2 Rectangular Components of a Force

If the resolved parts of a force are at right angle with each other, then these are known as rectangular components of a force. In this case the angle between the forces $\alpha = \frac{\pi}{2}$, then $\phi = \frac{\pi}{2} - \theta$, then

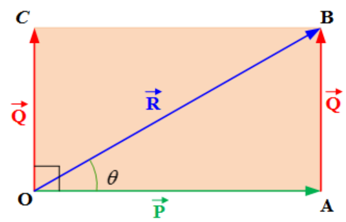

Figure 3.31: Resolved components of a vector in orthogonal directions

$$\sin(\pi - \alpha) = \sin\left(\frac{\pi}{2}\right) = 1$$

and

$$\sin\phi = \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

Then the resolved components of a force are

$$P = R\cos\theta \tag{3.5.3}$$

and

$$Q = R\sin\theta \tag{3.5.4}$$

(3.5.3) and (3.5.4) are known as rectangular components of a force and are shown in Fig. 3.31.

Example 3.5.2. *Consider a force of magnitude $4N$ acts on a body. Let this force has two components, each making an angle 45° with it. Find their magnitudes.*

Solution: Let R be the force acting on the body. Let its component P acts on its right side and its component Q acts on its left side. The system is shown in Fig. 3.33. We can say that the resultant makes an angle $\theta = 45^\circ$ with its component P. Then the magnitude of P is

$$\begin{aligned} P &= R\cos\theta \\ &= 4\cos(45^\circ) \\ &= 2\sqrt{2} \approx 2.828N \end{aligned}$$

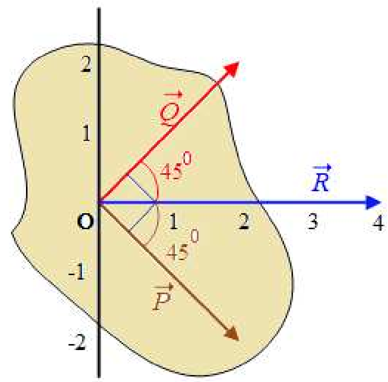

Figure 3.32: Resolved components of a force

and the magnitude of Q is

$$
\begin{aligned}
Q &= R\sin\theta \\
&= 4\sin(45^\circ) \\
&= 2\sqrt{2} \approx 2.828N
\end{aligned}
$$

Here the magnitude of each rectangular components is $2.828N$

3.5.3 Resolved Components of a Force in Rectangular Coordinate System

A force is, generally, resolved along two mutually perpendicular directions. If we consider rectangular coordinate system, the component along x *axis* is called horizontal component and the component along y *axis* is called vertical component.
We can consider a force in rectangular coordinate system.

Resolved Components of a Force in Two Dimensional Space

If we consider two dimensional rectangular coordinate system, the component along x *axis* is called horizontal component and the component along y *axis* is called vertical component.
A force in $2 - space$ is

$$
\begin{aligned}
\vec{F} &= \langle F_1, F_2 \rangle \\
&= F_1\hat{i} + F_2\hat{j} \qquad (3.5.5)
\end{aligned}
$$

If a force $\vec{F}$ is applied at origin making an angle θ with $x - axis$, then the force with the combination of its orthogonal components can be written as

$$\vec{F} = F\cos\theta\hat{i} + F\sin\theta\hat{j} \tag{3.5.6}$$

Example 3.5.3. *Consider a force of magnitude $4N$ acts on a body along positive $x - axis$. Let this force has two rectangular components, each making an angle $45°$ with it. Find these components.*

Solution: Let R be the force acting on the body. Let its component P lies in first quadrant and its component Q in fourth quadrant. The system is shown in Fig. 3.33. We can say that the resultant makes an angle $\theta = 45°$ with its component P. Then the

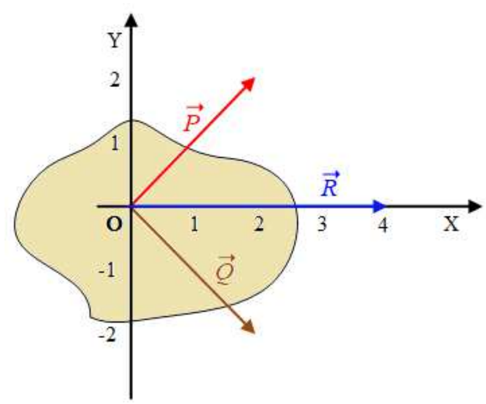

Figure 3.33: Resolved components of a force

magnitude of P is

$$\begin{aligned} P &= R\cos\theta \\ &= 4\cos(45°) \\ &= 2\sqrt{2} \approx 2.828N \end{aligned}$$

As P lies in first quadrant and it makes an angle $\phi = 45°$ with $x - axis$. By using (3.5.6), its $x - component$ P_x is

$$\begin{aligned} P_x &= P\cos\phi \\ &= 2\sqrt{2}\cos(45°) \\ &= 2 \end{aligned}$$

and its $y-component$ P_y is

$$\begin{aligned} P_y &= P\sin\phi \\ &= 2\sqrt{2}\sin(45^\circ) \\ &= 2 \end{aligned}$$

Hence the component of given force in first quadrant is

$$\vec{P} = \langle 2, 2 \rangle$$

Next the magnitude of Q is

$$\begin{aligned} Q &= R\sin\theta \\ &= 4\sin(45^\circ) \\ &= 2\sqrt{2} \approx 2.828N \end{aligned}$$

As Q lies in fourth quadrant and it makes an angle $\psi = -45^\circ$ with $x - axis$. By using (3.5.6), its $x - component$ Q_x is

$$\begin{aligned} Q_x &= Q\cos\psi \\ &= 2\sqrt{2}\cos(-45^\circ) \\ &= 2 \end{aligned}$$

and its $y - component$ Q_y is

$$\begin{aligned} Q_y &= Q\sin\psi \\ &= 2\sqrt{2}\sin(-45^\circ) \\ &= -2 \end{aligned}$$

Hence the component of given force in fourth quadrant is

$$\vec{Q} = \langle 2, -2 \rangle$$

Resolved Components of a Force in Three Dimensional Space

A force in three dimensional rectangular coordinate system is

$$\begin{aligned} \vec{F} &= \langle F_1, F_2, F_3 \rangle \\ &= F_1\hat{i} + F_2\hat{j} + F_3\hat{k} \end{aligned} \tag{3.5.7}$$

If force $\vec{F}$ makes angle α with $x - axis$, β with $y - axis$ and γ with $z - axis$ and taking dot product of (3.5.7) with $\hat{i}$, $\hat{j}$ and $\hat{k}$, we have

$$\begin{aligned} F_1 &= F\cos\alpha \\ F_2 &= F\cos\beta \\ F_3 &= F\cos\gamma \end{aligned}$$

then (3.5.7) can be written as

$$\begin{aligned}\vec{F} &= F\cos\alpha\hat{i} + F\cos\beta\hat{j} + F\cos\gamma\hat{k} \\ &= F\left(\cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}\right) \qquad (3.5.8) \\ &= F\hat{F}\hat{F} \qquad (3.5.9)\end{aligned}$$

Here

$$\hat{F} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k} \qquad (3.5.10)$$

is unit vector which determines the direction of the force.

Example 3.5.4. *Find the rectangular components of a force* $\vec{F} = \langle 1, 1, 1\rangle$.

The magnitude of the given force is

$$\begin{aligned}F &= \sqrt{1^2 + 1^2 + 1^2} \\ &= \sqrt{3}\end{aligned}$$

and a unit vector in this direction is

$$\hat{F} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

Comparing it with (3.5.10), we find

$$\cos\alpha = \frac{1}{\sqrt{3}}$$

and

$$\begin{aligned}\alpha &= \arccos\left(\frac{1}{\sqrt{3}}\right) = \beta = \gamma \\ &= 0.955 \quad \text{radian} = 54.7^\circ\end{aligned}$$

The force makes an angle of 54.7° with all its components.

3.6 Resultant of Coplanar Forces Using Rectangular Components

The rectangular components of resultant force of two or more coplanar forces can be obtained by summing rectangular components of individual forces. For two dimensional system, the horizontal components of resultant vector of two or more vectors is the sum of horizontal components of individual vectors and the vertical components of resultant vector of two or more vectors is the sum of vertical components of individual vectors. First consider a system of two vectors in $2 - space$.

3.6.1 Resultant of Two Coplanar Vectors

Consider two coplanar vectors $\vec{P}$ and $\vec{Q}$ as shown in Fig. 3.34. Their representations in

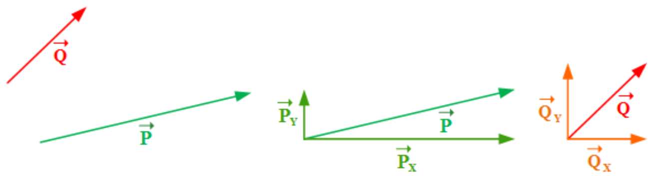

Figure 3.34: Two coplanar vectors.

rectangular components are

$$\vec{P} = P_X\hat{\imath} + P_Y\hat{\jmath}$$

and

$$\vec{Q} = Q_X\hat{\imath} + Q_Y\hat{\jmath}$$

Then by vector addition (head to tail rule) the resultant vector $\vec{R}$ of two vectors $\vec{P}$ and $\vec{Q}$ is

$$\vec{R} = \vec{P} + \vec{Q} \tag{3.6.1}$$

$\vec{R}$ in rectangular components is

$$\vec{R} = R_X\hat{\imath} + R_Y\hat{\jmath}$$

In rectangular components, (3.6.1) can be written as

$$\begin{aligned} R_X\hat{\imath} + R_Y\hat{\jmath} &= \left(P_X\hat{\imath} + P_Y\hat{\jmath}\right) + \left(Q_X\hat{\imath} + Q_Y\hat{\jmath}\right) \\ &= (P_X + Q_X)\,\hat{\imath} + (P_Y + Q_Y)\,\hat{\jmath} \end{aligned}$$

Comparing components, we have

$$R_X = P_X + Q_X$$

The horizontal component of resultant is sum of horizontal components of individual vectors. And

$$R_Y = P_Y + Q_Y$$

the vertical component of resultant is sum of vertical components of individual vectors. The magnitude of resultant is

$$R = \sqrt{R_X^2 + R_Y^2} \tag{3.6.2}$$

And the direction is

$$\theta = \arctan\left(\frac{R_Y}{R_X}\right) \tag{3.6.3}$$

Geometrically this sum is illustrated in Fig. 3.35. The length of $\vec{R}$ is the magnitude and

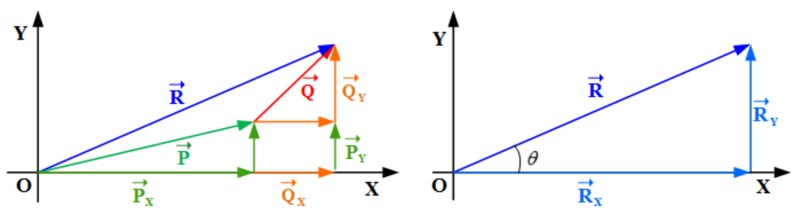

Figure 3.35: Addition of two coplanar vectors.

its inclination with horizontal is its direction.

3.6.2 Resultant of n Coplanar Vectors

For n coplanar vectors $\vec{F_1}, \vec{F_2}, ...\vec{F_n}$, the rectangular components of resultant vector are

$$R_X = F_{1X} + F_{2X} + ... + F_{nX} = \sum_{i=1}^{n} F_{iX}$$

And

$$R_Y = F_{1Y} + F_{2Y} + ... + F_{nY} = \sum_{i=1}^{n} F_{iY}$$

The magnitude and direction of resultant vector can be calculated by using (3.6.2) and (3.6.3) respectively.

3.6.3 Resultant of n Non-coplanar Vectors

Next consider a system of n vectors $\vec{F_1}, \vec{F_2}, ...\vec{F_n}$in $3 - space$.
The rectangular components of resultant vector are

$$
\begin{aligned}
R_X &= F_{1X} + F_{2X} + ... + F_{nX} = \sum_{i=1}^{n} F_{iX} \\
R_Y &= F_{1Y} + F_{2Y} + ... + F_{nY} = \sum_{i=1}^{n} F_{iY}
\end{aligned}
$$

And

$$
R_Z = F_{1Z} + F_{2Z} + ... + F_{nZ} = \sum_{i=1}^{n} F_{iZ}
$$

The magnitude of resultant is

$$
R = \sqrt{R_X^2 + R_Y^2 + R_Z^2} \tag{3.6.4}
$$

If the resultant R makes angles α with $x\ axis$, β with $y\ axis$ and γ with $z\ axis$, then direction cosines are

$$
\begin{aligned}
\cos\alpha &= \frac{R_X}{R} \\
\cos\beta &= \frac{R_Y}{R}
\end{aligned}
$$

and

$$
\cos\gamma = \frac{R_Z}{R}
$$

which will help to determine the direction of the resultant.

3.7 Resultant of Concurrent and Coplanar Vectors

Following subsection 2.5.4, the rectangular components of a vector can be obtained. The rectangular components of resultant vector of two or more concurrent and coplanar vectors can be obtained by summing rectangular components of individual vectors. For two dimensional system, the horizontal components of resultant vector of two or more vectors is the sum of horizontal components of individual vectors and the vertical components of resultant vector of two or more vectors is the sum of vertical components of individual vectors. First consider a system of two vectors in $2 - space$.

3.7.1 Resultant of Two Concurrent and Coplanar Vectors

Consider two concurrent and coplanar vectors $\vec{P}$ and $\vec{Q}$. Let their point of application is origin O of cartesian coordinate system. Vector $\vec{P}$ makes an angle α_1 and vector $\vec{Q}$ makes an angle α_2 with x *axis* as shown in Fig. 3.36. Their representations in rectangular

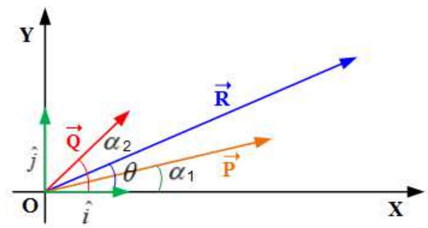

Figure 3.36: Resultant of two concurrent and coplanar vectors

components are

$$\begin{aligned}\vec{P} &= P_X\hat{i} + P_Y\hat{j} \\ &= P\cos\alpha_1\hat{i} + P\sin\alpha_1\hat{j}\end{aligned}$$

and

$$\begin{aligned}\vec{Q} &= Q_X\hat{i} + Q_Y\hat{j} \\ &= Q\cos\alpha_2\hat{i} + Q\sin\alpha_2\hat{j}\end{aligned}$$

If $\vec{R}$ is their resultant making an angle θ with x *axis* as shown in Fig. 3.36 and is given by (3.6.1). Its representation in rectangular components is

$$\begin{aligned}\vec{R} &= R_X\hat{i} + R_Y\hat{j} \\ &= R\cos\theta\hat{i} + R\sin\theta\hat{j}\end{aligned}$$

In rectangular components, (3.6.1) can be written as

$$\begin{aligned}R_X\hat{i} + R_Y\hat{j} &= \left(P_X\hat{i} + P_Y\hat{j}\right) + \left(Q_X\hat{i} + Q_Y\hat{j}\right) \\ &= (P_X + Q_X)\,\hat{i} + (P_Y + Q_Y)\,\hat{j}\end{aligned}$$

Comparing components, we have

$$\begin{aligned}R_X &= P_X + Q_X \\ &= P\cos\alpha_1 + Q\cos\alpha_2\end{aligned}$$

The horizontal component of resultant is sum of horizontal components of individual vectors. And

$$\begin{aligned} R_Y &= P_Y + Q_Y \\ &= P\sin\alpha_1 + Q\sin\alpha_2 \end{aligned}$$

the vertical component of resultant is sum of vertical components of individual vectors. The length of $\vec{R}$ is the magnitude and its inclination with x *axis* is its direction can be calculated by using (3.6.2) and (3.6.3) respectively.

Example 3.7.1. *Consider two forces magnitudes $5N$ and $8N$ are acting on a body. Let the first force makes an angle $15°$ with horizontal and second force makes an angle $45°$ with horizontal. Find their resultant force.*

Solution: The given data is

$$\begin{aligned} F_1 &= 5N \\ F_2 &= 8N \\ \theta_1 &= 15° \\ \theta_2 &= 45° \end{aligned}$$

The resultant can be obtained by using addition of vectors by rectangular components. The x components of first force is

$$\begin{aligned} F_{1X} &= F_1\cos 15° \\ &= 5(0.9659) = 4.8296N \end{aligned}$$

The x components of second force is

$$\begin{aligned} F_{2X} &= F_2\cos 45° \\ &= 8(0.7071) = 5.6568N \end{aligned}$$

The y components of first force is

$$\begin{aligned} F_{1Y} &= F_1\sin 15° \\ &= 5(0.2588) = 1.294N \end{aligned}$$

The y components of second force is

$$\begin{aligned} F_{2Y} &= F_2\sin 45° \\ &= 8(0.7071) = 5.6568N \end{aligned}$$

Let F be the resultant of two forces. Its x components is

$$\begin{aligned} F_X &= F_{1X} + F_{2X} \\ &= 4.8296 + 5.6568 = 10.4864N \end{aligned}$$

and y components is

$$\begin{aligned} F_Y &= F_{1Y} + F_{2Y} \\ &= 1.294 + 5.6568 = 6.9508N \end{aligned}$$

Using (3.6.2) magnitude of resultant force is

$$\begin{aligned} F &= \sqrt{(F_X)^2 + (F_Y)^2} \\ &= \sqrt{(10.4864)^2 + (6.9508)^2} \\ &= \sqrt{158.2781} = 12.5808 \\ &\simeq 12.6N \end{aligned}$$

Let the resultant force makes an angle θ with x *axis*. Using (3.6.3) direction of resultant force is

$$\begin{aligned} \theta &= \arctan\left(\frac{F_Y}{F_X}\right) \\ &= \arctan\left(\frac{6.9508}{10.4864}\right) = \arctan(0.6628) \\ &= 33.5379 \simeq 36^\circ \end{aligned}$$

3.7.2 Resultant of n Concurrent and Coplanar Vectors

For n vectors $\vec{F}_1, \vec{F}_2, ...\vec{F}_n$ making angles $\alpha_1, \alpha_2, ...\alpha_n$ with x *axis* as shown in Fig. 3.37. If $\vec{R}$ is their resultant making an angle θ with x *axis* as shown in Fig. 3.37. Rectangular

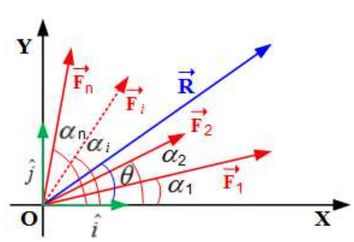

Figure 3.37: Resultant of n concurrent and coplanar vectors

components of resultant vector are

$$\begin{aligned} R_X &= F_{1X} + F_{2X} + ... + F_{nX} = \sum_{i=1}^{n} F_{iX} \\ R\cos\theta &= F_1\cos\alpha_1 + F_2\cos\alpha_2 + ... + F_{nX}\cos\alpha_n = \sum_{i=1}^{n} F_i\cos\alpha_i \end{aligned}$$

And

$$\begin{aligned} R_Y &= F_{1Y} + F_{2Y} + ... + F_{nY} = \sum_{i=1}^{n} F_{iY} \\ R\sin\theta &= F_1\sin\alpha_1 + F_2\sin\alpha_2 + ... + F_{nX}\sin\alpha_n = \sum_{i=1}^{n} F_i\sin\alpha_i \end{aligned}$$

The magnitude and direction of resultant vector can be calculated by using (3.6.2) and (3.6.3) respectively.

Corollary 3.7.1. *Three forces $\vec{F_1}$, $\vec{F_2}$ and $\vec{F_3}$ act at a point parallel to the sides of a triangleABC taken in the same order. Show that the magnitude of the resultant is*

$$\sqrt{F_1^2 + F_2^2 + F_3^2 - 2F_2F_3\cos A - 2F_1F_3\cos B - 2F_1F_2\cos C} \tag{3.7.1}$$

where A is angle between forces F_1 and F_3, B is angle between forces F_1 and F_2 and C is angle between forces F_2 and F_3. The system is shown in the Fig. 3.38.

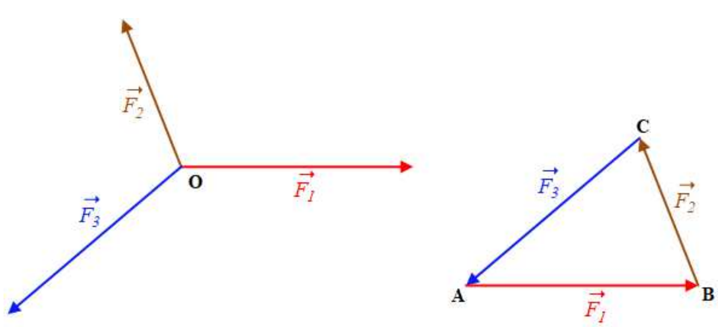

Figure 3.38: Forces acting at a point along sides of a triangle

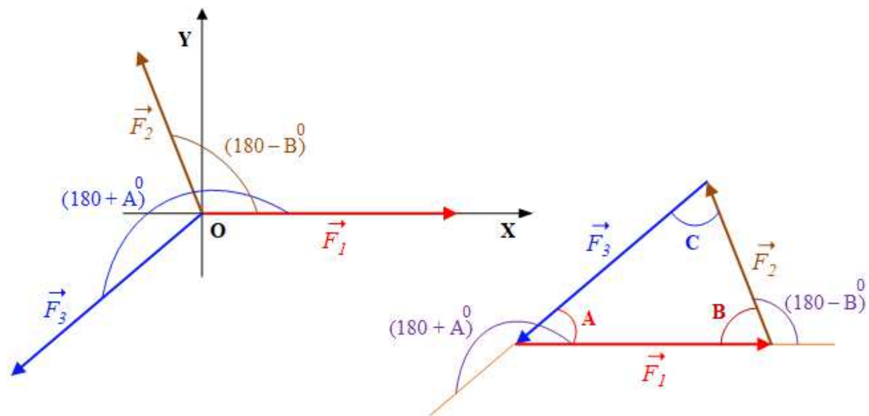

Figure 3.39: Forces acting at a point along sides of a triangle

Proof: Take $2-$*dimensional* rectangular coordinate system with O (origin) as the point of application of the forces. For simplicity consider side AB parallel to x *axis* as shown in the Fig. 3.39. The resultant can be calculated by finding its rectangular components. First we find rectangular components of these forces. The force F_1 makes 0 degree with x *axis*, then its rectangular components are

$$F_{1x} = F_1$$
$$F_{1y} = 0$$

The force F_2 makes $180 - B$ degree with x *axis* hence lying in II quadrant, then its rectangular components are

$$F_{2x} = F_2 \cos(180 - B) = -F_2 \cos B$$
$$F_{2y} = F_2 \sin(180 - B) = F_2 \sin B$$

The force F_3 makes $\pi + A$ radian with x *axis* hence lying in III quadrant, then its rectangular components are

$$F_{3x} = F_3 \cos(\pi + A) = -F_3 \cos A$$
$$F_{3y} = F_3 \sin(\pi + A) = -F_3 \sin A$$

The *xcomponent* of resultant is

$$\begin{aligned} R_x &= F_{1x} + F_{2x} + F_{3x} \\ &= F_1 - F_2 \cos B - F_3 \cos A \end{aligned}$$

The *ycomponent* of resultant is

$$\begin{aligned} R_y &= F_{1y} + F_{2y} + F_{3y} \\ &= F_2 \sin B - F_3 \sin A \end{aligned}$$

Using (3.6.2) magnitude of resultant force is

$$R = \sqrt{(R_X)^2 + (R_Y)^2} \tag{3.7.2}$$

The square of x *component* is

$$\begin{aligned} (R_X)^2 &= (F_1 - F_2 \cos B - F_3 \cos A)^2 \\ &= F_1^2 + F_2^2 \cos^2 B + F_3^2 \cos^2 A - 2F_1F_2 \cos B - 2F_1F_3 \cos A + 2F_2F_3 \cos B \cos A \end{aligned}$$

and the square of y *component* is

$$\begin{aligned} (R_Y)^2 &= (F_2 \sin B - F_3 \sin A)^2 \\ &= F_2^2 \sin^2 B + F_3^2 \sin^2 A - 2F_2F_3 \sin B \sin A \end{aligned}$$

The sum of these squared component is

$$\begin{aligned} (R_X)^2 + (R_Y)^2 &= F_1^2 + F_2^2 \cos^2 B + F_3^2 \cos^2 A - 2F_1F_2 \cos B \\ &- 2F_1F_3 \cos A + 2F_2F_3 \cos B \cos A \\ &+ F_2^2 \sin^2 B + F_3^2 \sin^2 A - 2F_2F_3 \sin B \sin A \\ &= F_1^2 + F_2^2 \left(\cos^2 B + \sin^2 B\right) + F_3^2 \left(\cos^2 A + \sin^2 A\right) \\ &- 2F_1F_2 \cos B - 2F_1F_3 \cos A + 2F_2F_3 \left(\cos A \cos B - \sin A \sin B\right) \end{aligned}$$

Using trigonometric identities, we can write

$$\begin{aligned} (R_X)^2 + (R_Y)^2 &= F_1^2 + F_2^2 + F_3^2 - 2F_1F_2 \cos B - 2F_1F_3 \cos A \\ &+ 2F_2F_3 \cos(A + B) \end{aligned} \tag{3.7.3}$$

In triangle ABC, angle $A + B = (180 - C)$, and $\cos(180 - C) = -\cos C$, then (3.7.3) becomes

$$\begin{aligned} (R_X)^2 + (R_Y)^2 &= F_1^2 + F_2^2 + F_3^2 - 2F_1F_2 \cos B - 2F_1F_3 \cos A \\ &- 2F_2F_3 \cos C \end{aligned} \tag{3.7.4}$$

Using (3.7.4), the magnitude of the resultant is

$$R = \sqrt{F_1^2 + F_2^2 + F_3^2 - 2F_2F_3 \cos A - 2F_1F_3 \cos B - 2F_1F_2 \cos C}$$

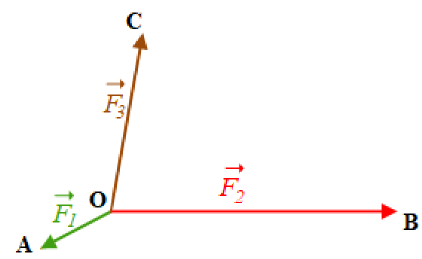

Figure 3.40: Three concurrent and non-coplanar vectors

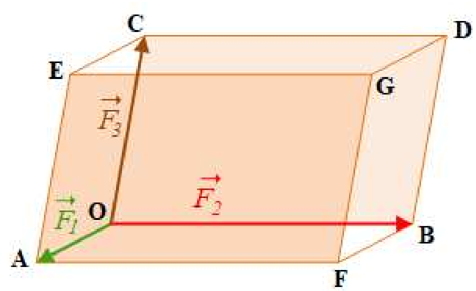

Figure 3.41: Three concurrent and non-coplanar vectors

3.8 Resultant of Three Concurrent and Non-coplanar Forces

Consider three concurrent and non-coplanar forces $\vec{F_1}$, $\vec{F_2}$ and $\vec{F_3}$ are acting at point O, are represented by the vectors $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$ and $\vec{OC} = \vec{c}$ respectively as shown in Fig 3.40. Complete the parallelepiped $OABCDEFG$ whose edges OA, OB and OC represents the vectors $\vec{F_1}$, $\vec{F_2}$ and $\vec{F_3}$ respectively as shown in Fig. 3.41. Consider parallelogram $OAFB$, whose two adjacent sides OA and OB represents the vectors $\vec{F_1}$ and $\vec{F_2}$. Then by law of parallelogram of vector addition, OF represents the sum of $\vec{F_1}$ and $\vec{F_2}$. Let it be $\vec{u}$.

$$\vec{OA} + \vec{OB} = \vec{OF}$$

or

$$\vec{F_1} + \vec{F_2} = \vec{u}$$

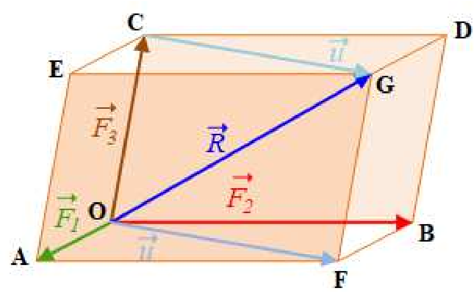

Figure 3.42: Addition of three concurrent and non-coplanar vectors

Next consider parallelogram $OFGC$, whose two adjacent sides OF and OC represents the vectors $\vec{u}$ and $\vec{F_3}$. Then by law of parallelogram of vector addition, OG represents the sum of $\vec{u}$ and $\vec{c}$. Let it be $\vec{R}$.

$$\vec{OF} + \vec{OC} \quad = \quad \vec{OG}$$

or

$$\vec{u} + \vec{F_3} \quad = \quad \vec{R}$$

so that

$$\vec{OA} + \vec{OB} + \vec{OC} \quad = \quad \vec{OG}$$

or

$$\vec{F_1} + \vec{F_2} + \vec{F_3} \quad = \quad \vec{R}$$

Hence the resultant of three concurrent and non-coplanar forces, acting at O, is represented by the diagonal, drawn through O, of a parallelepiped with the given forces for its edges.

3.9 Moment of a Force or Torque

The tendency of a force to produce rotation of a body about some reference axis or point is called the moment of a force. It is also known as torque or turning effect of a force. For example turning pencil in a sharpener, turning stopcock of a water tap, turning doorknob and so on. The moment of a force is positive if the rotation is in anti clock wise sense and is negative if the rotation is in clock wise sense. In SI system its unit of measure is Newton-meter $(N - m)$.

3.9.1 Moment of a Force About a Point

Consider a rigid body. Let a force $\vec{F}$ is acting on it which tends to rotate it about a fixed point O as shown in Fig. 3.45. The line along which the force acts is called the **line of action of the force** and the line OC is known as the axis of rotation. Take a point A on

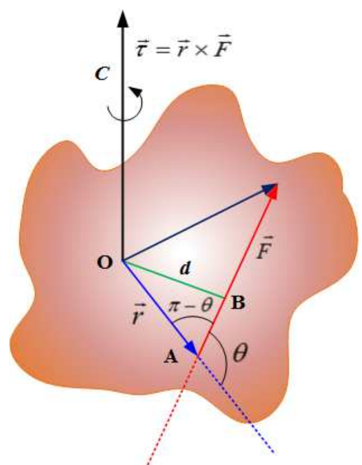

Figure 3.43: Moment of a force

the line of action of the force having position vector $\vec{r}$ with respect to O. Moment of a force $\vec{F}$ about O is a vector quantity and is defined as

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{3.9.1}$$

Hence $\vec{\tau}$ is a vector perpendicular to the plane containing $\vec{r}$ and $\vec{F}$. If θ is the angle between $\vec{r}$ and $\vec{F}$, then

$$\vec{r} \times \vec{F} = rF \sin\theta \hat{n} \tag{3.9.2}$$

Where $\hat{n}$ is a unit vector perpendicular to the plane containing $\vec{r}$ and $\vec{F}$. The magnitude of the moment of force $\vec{F}$ about O is

$$\begin{aligned} \tau &= |\vec{r} \times \vec{F}| \\ &= rF\sin\theta \end{aligned} \tag{3.9.3}$$

The perpendicular distance from O to the line of action of the force is called the **moment arm of the force**. To calculate it, consider the right angle triangle ABO

$$\begin{aligned} d &= r\sin(\pi - \theta) \\ &= r\sin\theta \end{aligned} \tag{3.9.4}$$

Using (4.1.13) in (4.1.12), we have

$$\tau = Fd$$

(4.1.14) gives the magnitude of the moment of a force $\vec{F}$ about O. From (4.1.14), it is clear greater the force, greater is the moment of the force and longer the moment arm, greater is the moment of the force. It is more understandable by the following examples.
Examples To open or close a door, it is more easy to apply a force at the outer edge of a door rather than near the hinge.
It is more easy to open or tighten a nut or a bolt with a spanner having long arm rather than a spanner having short arm with same applied force.

3.9.2 Moment of a Force About a Point Lying on the Line of Action of the Force

If the fixed point O lies on the line of action of the force, see Fig. 3.45. Take an other point

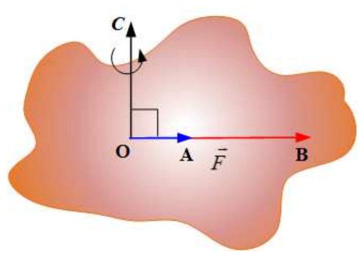

Figure 3.44: Moment of a force

A on the line of action of the force having position vector $\vec{r}$ with respect to O. In this case

$\vec{r}$ and $\vec{F}$ have the same direction (or have opposite direction), so $\theta = 0$ or $(\theta = \pi)$, then moment of a force $\vec{F}$ about O is

$$\begin{aligned} \vec{\tau} &= \vec{r} \times \vec{F} \\ &= rF \sin\theta \hat{n} \\ &= 0\hat{n} \end{aligned} \tag{3.9.5}$$

And the magnitude of the moment of force $\vec{F}$ about O is also zero.

Corollary 3.9.1. *The moment of a force is independent of the choice of the point of application of the force.*

Proof Consider a rigid body. Let a force $\vec{F}$ is acting it on at point A whose position vector is $\vec{r}$ with respect to O.. Due to this force the body tends to rotate about a fixed point O, then moment of a force $\vec{F}$ about O is

$$\vec{\tau} = \vec{r} \times \vec{F}$$

If B is another point on the line of action of the force having position vector $\vec{r_1}$ with respect

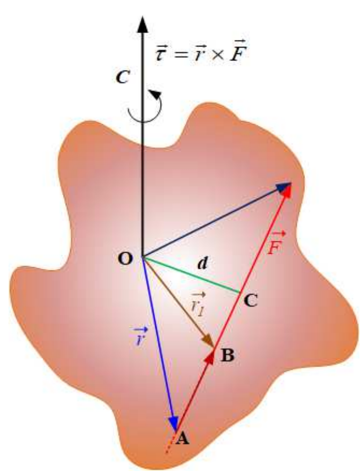

Figure 3.45: Moment of a force

to O. By head to tail rule $\vec{r_1}$ can be written as

$$\vec{r_1} = \vec{r} + \vec{AB}$$

then moment of a force $\vec{F}$ about O is

$$\begin{aligned}\vec{\tau} &= \vec{r_1} \times \vec{F} \\ &= \left(\vec{r} + \vec{AB}\right) \times \vec{F} \\ &= \vec{r} \times \vec{F} + \vec{AB} \times \vec{F}\end{aligned}$$

As A lies on the line of action of the force hence the moment $\vec{AB} \times \vec{F}$ is zero and we are left with

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Hence the moment of a force is independent of the choice of the point of application of the force.

Theorem 3.9.2. *The moment about a point of the resultant of a system of concurrent forces is equal to the sum moments of all forces of the system about the same point. This theorem is known as Varignon's theorem, due to the French mathematician P. Varignon (1654-1722)*

Proof Let $\vec{F}_1,\ \vec{F}_2,\ \vec{F}_3, \ldots \vec{F}_n$ be forces acting at A. Let $\vec{r}$ be the position vector of A relative to origin O of a reference system. Let $\vec{R}$ be the resultant of these forces as shown in Fig. 3.46, then $\vec{R}$ is

$$\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_n$$

Next moment of $\vec{R}$ about A is

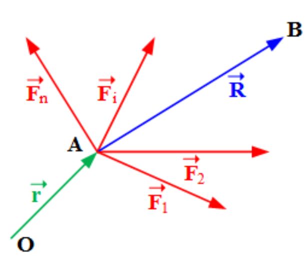

Figure 3.46: Sum of moments of n forces

$$\begin{aligned}\vec{\tau} = \vec{r} \times \vec{R} &= \vec{r} \times \left(\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_n\right) \\ &= \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2 + \vec{r} \times \vec{F}_3 + \ldots + \vec{r} \times \vec{F}_n\end{aligned}$$

Note: If these forces have distinct position vector, then the sum of moments of all the forces is

$$\begin{aligned}\vec{\tau} &= \vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + \ldots + \vec{\tau}_n \\ &= \vec{r_1} \times \vec{F_1} + \vec{r_2} \times \vec{F_2} + \vec{r_3} \times \vec{F_3} + \ldots + \vec{r_n} \times \vec{F_n}\end{aligned}$$

These concepts are illustrated in the following examples.

Example 3.9.1. *The forces $\vec{F_1} = -\hat{\imath} + \hat{\jmath}$, $\vec{F_2} = \hat{\jmath}$, $\vec{F_3} = -2\hat{\imath} - 4\hat{\jmath}$ and $\vec{F_4} = 2\hat{\imath}$ are acting at $O(0,0)$, $A(2,0)$, $B(0,-2)$ and $C(3,2)$ respectively as shown in the Fig. 3.47. Find*

a) moments of all forces about O.

b) sum of moments of all forces about O.

c) moments of all forces about A, when no force is acting at A.

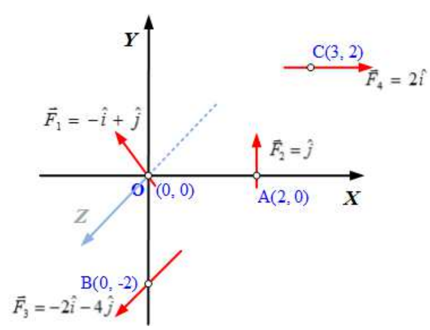

Figure 3.47: Moment of a force

Solution The given data is

$$\begin{aligned}\vec{F_1} &= -\hat{\imath} + \hat{\jmath} \\ \vec{F_2} &= \hat{\jmath} \\ \vec{F_3} &= -2\hat{\imath} - 4\hat{\jmath} \\ \vec{F_4} &= 2\hat{\imath}\end{aligned}$$

a) moments of all forces about O.

Considering O as the reference point, the position vectors of all these forces are

$$\begin{aligned}\vec{r_1} &= 0\hat{i} + 0\hat{j}\\ \vec{r_2} &= 2\hat{i}\\ \vec{r_3} &= -2\hat{j}\\ \vec{4_4} &= 3\hat{i} + 2\hat{j}\end{aligned}$$

respectively and are shown in Fig. 3.48. Using 4.1.10, the moment of force $\vec{F_1} = -\hat{i} + \hat{j}$

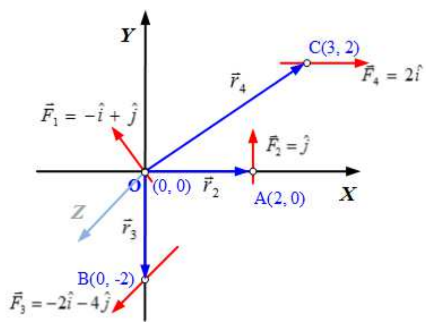

Figure 3.48: Moment of a force

about O is

$$\begin{aligned}\vec{\tau_1} &= \vec{r_1} \times \vec{F_1}\\ &= (0\hat{i} + 0\hat{j}) \times (-\hat{i} + \hat{j})\\ &= 0\hat{k}\ N - m\end{aligned}$$

The magnitude of the moment of force $\vec{F_1}$ about O is $0\ N - m$. The moment of force $\vec{F_2} = \hat{j}$ about O is

$$\begin{aligned}\vec{\tau_2} &= \vec{r_2} \times \vec{F_2}\\ &= (2\hat{i} + 0\hat{j}) \times (0\hat{i} + \hat{j}) = 2(\hat{i} \times \hat{j})\\ &= 2\hat{k}\ N - m\end{aligned}$$

The moment of force $\vec{F_3} = -2\hat{i} - 4\hat{j}$ about O is

$$\begin{aligned}\vec{\tau_3} &= \vec{r_3} \times \vec{F_3}\\ &= (0\hat{i} - 2\hat{j}) \times (-2\hat{i} - 4\hat{j}) = 4(\hat{j} \times \hat{i})\\ &= -4\hat{k}\ N - m\end{aligned}$$

The moment of force $\vec{F_4} = 2\hat{i}$ about O is

$$\begin{aligned}\vec{\tau_4} &= \vec{r_4} \times \vec{F_4} \\ &= (3\hat{i} + 2\hat{j}) \times (2\hat{i}) = 4(\hat{j} \times \hat{i}) \\ &= -4\hat{k}\ N - m\end{aligned}$$

b) sum of moments of all forces about O.

Let $\vec{\tau}$ be the moment of all forces about O, then it is the sum of all moments

$$\begin{aligned}\vec{\tau} &= \vec{\tau_1} + \vec{\tau_2} + \vec{\tau_3} + \vec{\tau_4} \\ &= (0 + 2 - 4 - 4)\hat{k} \\ &= -6\hat{k}\ N - m\end{aligned}$$

c) moments of all forces about A, when no force is acting at A.

Considering A as the reference point, the position vectors of all these forces are

$$\begin{aligned}\vec{r_1} &= -2\hat{i} \\ \vec{r_2} &= \hat{i} + 2\hat{j} \\ \vec{r_3} &= -2\hat{j} - 2\hat{j}\end{aligned}$$

respectively and are shown in Fig. 3.49.
Using 4.1.10, the moment of force $\vec{F_1} = -\hat{i} + \hat{j}$ about A is

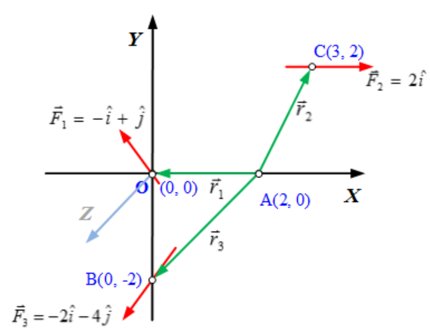

Figure 3.49: Moment of a force

$$\begin{aligned}\vec{\tau_1} &= \vec{r_1} \times \vec{F_1} \\ &= 2\hat{i} \times (-\hat{i} + \hat{j}) \\ &= 2\hat{k}\ N - m\end{aligned}$$

The moment of force $\vec{F_4} = 2\hat{\imath}$ about A is

$$\begin{aligned}\vec{\tau_2} &= \vec{r_2} \times \vec{F_2} \\ &= (\hat{\imath} + 2\hat{\jmath}) \times (2\hat{\imath}) \\ &= -4\hat{k}\ N - m\end{aligned}$$

The moment of force $\vec{F_3} = -2\hat{\imath} - 4\hat{\jmath}$ about A is

$$\begin{aligned}\vec{\tau_3} &= \vec{r_3} \times \vec{F_3} \\ &= (-2\hat{\imath} - 2\hat{\jmath}) \times (-2\hat{\imath} - 4\hat{\jmath}) \\ &= 4\hat{k}\ N - m\end{aligned}$$

Example 3.9.2. *A mechanic tightens the nut of a bicycle by exerting a force of* $100N$ *at the outer edge of a* $0.1\ m$ *long spanner. Find the moment of a force that has tightened it.*

Solution The given data is

$$\begin{aligned}F &= 100\ N \\ d &= 0.1\ m\end{aligned}$$

Let position of the nut be the fixed point O, using (4.1.14), the moment of a force about O is

$$\begin{aligned}\tau &= 100(0.1) \\ &= 10\ N - m\end{aligned}$$

3.10 Couples

The concept of couple arise, when the lines of action of forces acting on a body do not intersect. Example is, a system of two parallel forces. The magnitude and direction of the resultant of two such forces can be found by vector addition, but the line of action of the resultant can not be determined by parallelogram law of vector addition. Couple can be defined as:
A system of two parallel forces equal in magnitude and opposite in direction is said to form a couple. If the magnitude of a force is F*, then the couple is written as* $(F, -F)$ *and is shown in Fig. 3.50.* The plane through the line of action of the forces $\vec{F}$ and $-\vec{F}$ is called plane of the couple and the perpendicular distance between them is called arm of the couple. (See Fig. 3.51)

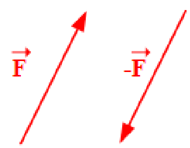

Figure 3.50: Couple of forces

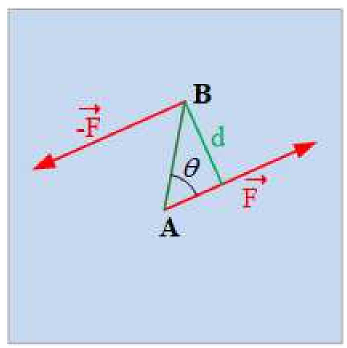

Figure 3.51: Plane and arm of a Couple of forces

3.10.1 Moment of a Couple

Consider a rigid body in 3 *space* system with O as origin and a couple $(\vec{F}, -\vec{F})$ is acting on it. Let $\vec{F}$ acts at A whose position vector is $\vec{r}_1$ relative to O and $-\vec{F}$ acts at B whose position vector is $\vec{r}_2$ relative to O as shown in Fig. 3.52. Then the sum of moments of the two forces about O is

$$\begin{aligned}\vec{\tau} &= \vec{\tau}_1 + \vec{\tau}_2 \\ &= \vec{r_1} \times \vec{F} + \vec{r_2} \times -\vec{F} \\ &= (\vec{r_1} - \vec{r_2}) \times \vec{F}\end{aligned}$$

If

$$\vec{AB} = \vec{r} = \vec{r_1} - \vec{r_2}$$

then the sum of moments of the two forces $\vec{F}$ and $-\vec{F}$ about O is

$$\vec{\tau} = \vec{r} \times \vec{F}$$

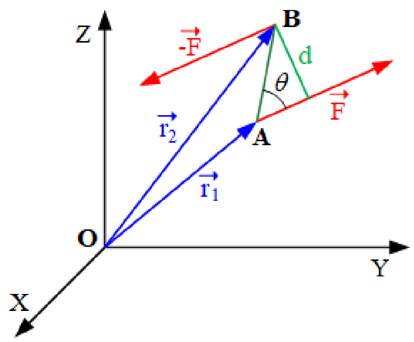

Figure 3.52: Moment of a couple

If d is the perpendicular distance between the forces and θ is an angle at the point of application of a force (see Fig. 3.52), then the magnitude of this moment is

$$\begin{aligned} \tau &= rF\sin\theta \\ &= Fd \end{aligned}$$

The moment of a couple is determined by the product of the magnitude of one of the force of the couple and the length of its arm taken with appropriate sign.

Theorem 3.10.1. *The effect of a couple upon a rigid body is unaltered if is replaced by any other couple of the same moment lying in the same plane.*

Proof Consider a rigid body in 3 *space* system with O as origin and a couple $(\vec{F}, -\vec{F})$ of arm length d_1 is acting on it. Let $\vec{F}$ acts at A and $-\vec{F}$ acts at B as shown in Fig. **??**. From A and B draw two arbitrary parallel lines AC and BD with d_2 is the distance between them. Resolve the force Let $\vec{F}$ at A into two components, $\vec{P}$ along AC and $\vec{Q}$ along AB. Similarly resolve the force $-\vec{F}$ at B into two components, $-\vec{P}$ along DB and $-\vec{Q}$ along BA. These components of these forces are as shown in Fig. 3.53.
The forces $\vec{Q}$ and $-\vec{Q}$ acting along the line AB, are in equilibrium and can be removed. The forces $\vec{P}$ and $-\vec{P}$ acting at A and B form a couple. Thus the couple of forces $\vec{P}$ and $-\vec{P}$ and arm length d_1 is replaced by another couple of forces $\vec{P}$ and $-\vec{P}$ and arm length d_2.
Next we show that moments of the couple $\vec{F}$ and $-\vec{F}$ and $\vec{P}$ and $-\vec{P}$ are equal. We consider force $\vec{P}$ acting at A with components, $\vec{P}$ along AC and $\vec{Q}$ along AB. Then by Varignon's theorem the moment of $\vec{F}$ about B is equal to the sum of moments of $\vec{P}$ and $\vec{Q}$ about B.

$$\begin{aligned} \vec{\tau} &= \vec{\tau_1} + \vec{\tau_2} \\ \vec{AB} \times \vec{F} &= \vec{AB} \times \vec{P} + \vec{AB} \times \vec{Q} \end{aligned}$$

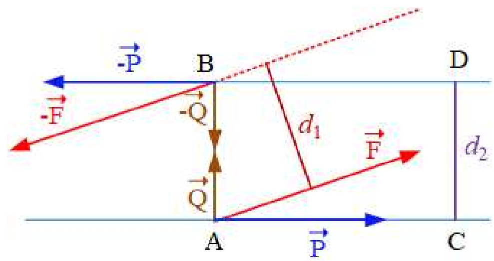

Figure 3.53: Equivalent couples

As B lies on the line of action of the force $\vec{Q}$ hence the moment $\vec{AB} \times \vec{Q}$ is zero and we are left with

$$\vec{AB} \times \vec{F} = \vec{AB} \times \vec{P}$$

In magnitude we can write

$$Fd_1 = Pd_2$$

Location of a Couple Let the points of application of forces $\vec{P}$ and $-\vec{P}$ be transferred from A and B to any points C and D in their lines of action. Since the location of the points A and B and the directions of AC and BD are arbitrary, the location of couple of forces $\vec{P}$ and $-\vec{P}$ in the plane is also arbitrary (see Fig. 3.58).

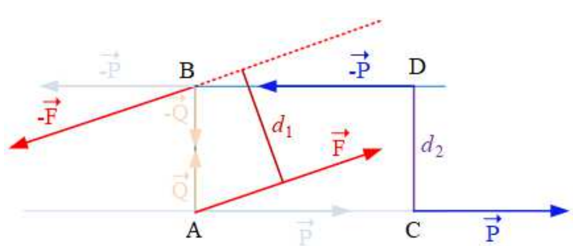

Figure 3.54: Location of these couples is arbitrary

3.11 Composition of Couples

n Coplanar couples of moments $\vec{\tau_1}, \vec{\tau_2}, ...\vec{\tau_n}$ are equivalent to a single couple lying in the same plane, whose momentum $\vec{\tau}$ is given by

$$\vec{\tau} = \vec{\tau_1} + \vec{\tau_2} + ... + \vec{\tau_n}$$

By the theorem of equivalent couples we can replace the couples of moments $\vec{\tau_1}, \vec{\tau_2}, ...\vec{\tau_n}$ by the couples $(\vec{F_1}, -\vec{F_1})$, $(\vec{F_2}, -\vec{F_2})$, $\dots(\vec{F_n}, -\vec{F_n})$ with a common arm d provided the magnitudes of the forces $\vec{F_1}, \vec{F_2}, \vec{F_3}, \dots\vec{F_n}$ are given by

$$\begin{aligned} F_1 d &= \tau_1 \\ F_2 d &= \tau_2 \\ &. \\ &. \\ &. \\ F_n d &= \tau_n \end{aligned}$$

The forces $\vec{F_1}, \vec{F_2}, \dots\vec{F_n}$ act in one straight line and $-\vec{F_1}, -\vec{F_2}, \dots - \vec{F_n}$ in a parallel line. Let $\vec{R}$ be the sum of these forces, then

$$\vec{R} = \vec{F_1} + \vec{F_2} + \vec{F_3} + \dots + \vec{F_n}$$

and their magnitude is

$$R = F_1 + F_2 + F_3 + \dots + F_n$$

Thus we get a couple of forces $(\vec{R}, -\vec{R})$ whose arm is of length d and its magnitude is

$$\begin{aligned} \tau &= Rd \\ &= (F_1 + F_2 + F_3 + \dots + F_n)\, d \\ &= F_1 d + F_2 d + F_3 d + \dots + F_n d \\ &= \tau_1 + \tau_2 + ... + \tau_n \end{aligned}$$

3.12 A Force and a Couple

Theorem 3.12.1. *A force $\vec{F}$ acting on a rigid body can be moved to any point O of the rigid body provided a couple is added, whose moment is equal to the moment of $\vec{F}$ about O.*

Proof Let the given force $\vec{F}$ acts at a point A of the rigid body. At the point O of the body we introduce two forces $\vec{F}$ and $-\vec{F}$ without altering the effect of the original force $\vec{F}$

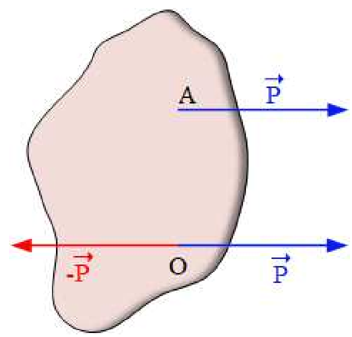

Figure 3.55: A couple and a force

on the body. The force $\vec{F}$ at A and $-\vec{F}$ at O form a couple $(\vec{F}, -\vec{F})$. Hence the given force $\vec{F}$ at A is equivalent to a force $\vec{F}$ at O together with a couple $(\vec{F}, -\vec{F})$ whose moment is equal to the moment about O of the force $\vec{F}$ at A.
Converse of theorem 3.12.1 is as under

Theorem 3.12.2. *A single force and a couple acting in the same plane upon a rigid body are equivalent to a single force acting in a direction parallel to its original direction.*

Proof Let the given system consists of a force $\vec{P}$ and a couple $(\vec{Q}, -\vec{Q})$. First suppose that the force $\vec{P}$ is not parallel to force $\vec{Q}$. The force $\vec{P}$ acts at O, $\vec{Q}$ at A and $-\vec{Q}$ at B. Shift $\vec{P}$ at A, to calculate the resultant of $\vec{P}$ and $\vec{Q}$, let it be $\vec{R}$

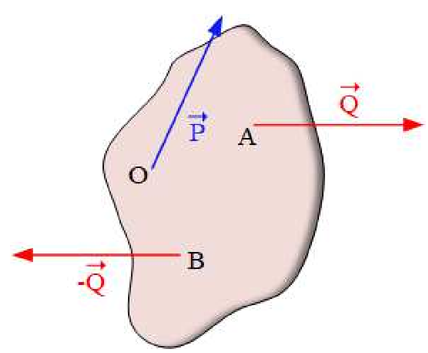

Figure 3.56: A force and a couple.

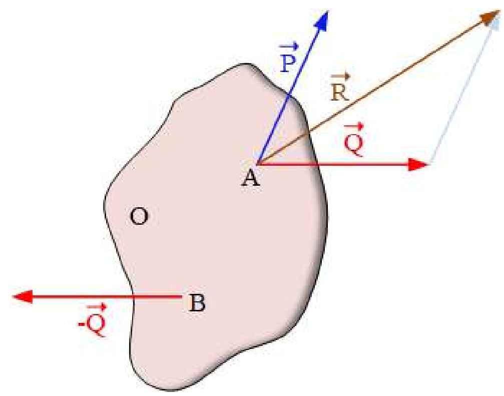

Figure 3.57: Resultant of P and Q at A.

$$\vec{R} = \vec{P} + \vec{Q}$$

It acts at A along the line AC. We can move $\vec{R}$ with its components at D as shown in the Fig. Now the forces $\vec{Q}$ and $-\vec{Q}$ acting along BD balance each other. Therefore the given system is equivalent to a single force $\vec{P}$ acting at D.

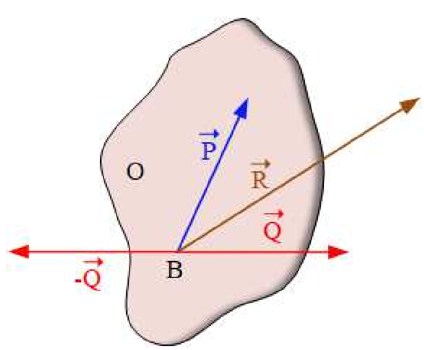

Figure 3.58: Resultant of P and Q at B.

3.13 Reduction of a System of Coplanar Forces to one Force and one Couple

Consider the forces $\vec{F_1}, \vec{F_2}, \vec{F_3}, \ . \ . \ .\vec{F_n}$ acting in one plane upon a rigid body at the points the forces $A_1, A_2, A_3, \ . \ . \ .A_n$. Let O be any point coplanar with the forces. According to

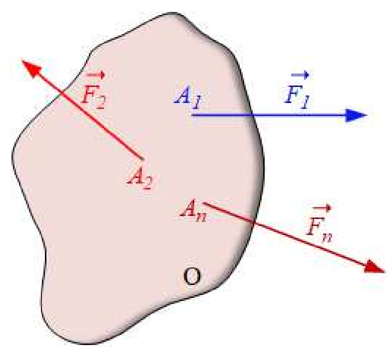

Figure 3.59: n forces are acting on a rigid body.

theorem 3.12.1 any force $\vec{F_i}$ acting at A_i is equivalent to a force $\vec{F_i}$ acting at O together with a couple whose moment τ_i is equal to the moment of $\vec{F_i}$ about O. Transforming all the forces to act at O we obtain a system of forces $\vec{F_1}, \vec{F_2}, \vec{F_3}, \ . \ . \ .\vec{F_n}$ acting at O and a system of couples of moments $\vec{\tau_1}, \vec{\tau_2}, ...\vec{\tau_n}$.

The concurrent forces applied at O can by replaced by their resultant $\vec{R}$ acting at the same

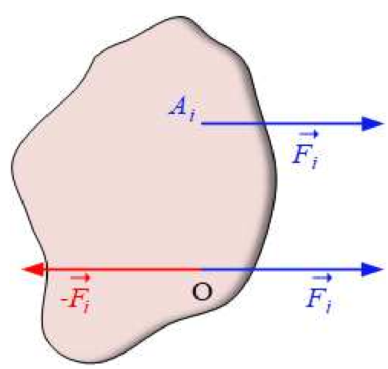

Figure 3.60: A couple and a force.

point. Similarly , by the theorem of the composition of couples, all the coplanar couples can be replaced by a single couple of moment τ.
The force $\vec{R}$ and the couple of moment $\vec{\tau}$ are given by

$$\vec{R} = \sum_{i=1}^{n} \vec{F_i}$$

$$\vec{\tau} = \sum_{i=1}^{n} \vec{\tau_i}$$

Exercises

1. Two concurrent and coplanar forces of magnitudes 10 N and 15 N are acting on a body. Find their resultant if the angle between them is
   a) 30°.
   b) 45°.
   c) 60°.
   d) 75°.
   e) 90°.
   f) 180°.
2. Two concurrent and coplanar forces $\vec{F_1}$ and $\vec{F_2}$ are acting on a body. If $\vec{F_1}$ makes an angle of 30° and $\vec{F_2}$ makes an angle of 40° with the resultant of magnitude 75 N. Find F_1 and F_2.
3. Consider two forces of equal magnitude are acting on a body. If the forces are acting at such an angle that their resultant also has same magnitude.
   a) Find the angle between the forces.
   b) Find the angle between each force and the resultant force.
4. The forces $\vec{F_1} = 2\hat{\imath}$, $\vec{F_2} = \hat{\imath} + \hat{\jmath}$, $\vec{F_3} = 2\hat{\imath} - 4\hat{\jmath}$ and $\vec{F_4} = 2\hat{\imath} + \hat{\jmath}$ are acting on a body. Determine their resultant.
5. Two concurrent and coplanar forces are acting on a body. If the magnitude of one force is double of the other, then their resultant is 160 N. If the direction of larger force is reversed and the other remained unaltered, then the magnitude of their resultant is 120 N. Determine the magnitude of each force and the angle between them.
6. In example 3.9.1 find moments of all forces about B and C.
7. The forces $\vec{F_1} = -\hat{\imath} + \hat{\jmath}$, $\vec{F_2} = \hat{\jmath}$, $\vec{F_3} = -2\hat{\imath} - 4\hat{\jmath}$ and $\vec{F_4} = 2\hat{\imath}$ are acting at $D(-1, 1)$, $C(1, 2)$, $A(2, 0)$ and $B(0, -2)$ respectively as shown in the Fig. 3.61. Find
   a) moments of all forces about O.
   b) sum of moments of all forces about O.
   c) moments of all forces about A, B, C and D.
8. A force $\vec{F} = 4N\ \hat{\imath} + 3N\ \hat{\jmath}$ is applied on an object of mass 4 kg at point $P(5m,\ 4m)$. If $z - axis$ is a fixed axis in the object find the magnitude and the direction of the torque.
9. A force $\vec{F} = 4N\ \hat{\imath} + 3N\ \hat{\jmath} - 2N\ \hat{k}$ is applied on an object of mass 2 kg at point $P(5m,\ 4m,\ 2m)$. Find the torque of the force $\vec{F}$ about A whose position vector relative to origin O is $\vec{r}_A = 1m\ \hat{\imath} + 3m\ \hat{\jmath} + 2m\ \hat{k}$.

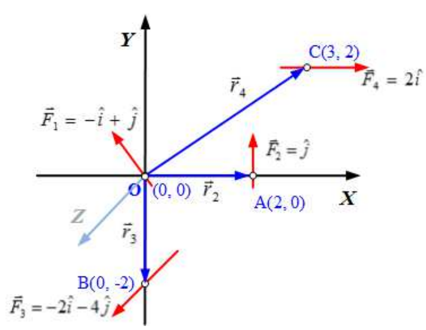

Figure 3.61: Moment of a force

Chapter 4

Equilibrium

According to Newtons second law of motion, a body will move with a velocity if it is acted upon by a force. When a number of concurrent forces acts on a body, it will move in the direction of the resultant force. However, if another force, which is equal in magnitude of the resultant but opposite in direction, is applied to a body, the body comes to rest. Then the sum of all the forces acting on the body is zero and the body is said to be in equilibrium. Equilibrium is also known as balancing as the forces are balanced .
Equilibrant of a system of forces is a single force, which acts along with the other forces to keep the body in equilibrium.

4.1 Equilibrium

The external forces acting on the body may be

(a) Concurrent and coplanar forces.

(b) Non-concurrent and coplanar forces.

(c) Concurrent forces in space.

4.1.1 Conditions of Equilibrium

If a system of concurrent and coplanar forces is acting on a body. The body will be in equilibrium if the following conditions are satisfied.

1. The vector sum of all the external forces that act on the particle/body must be zero.

2. The vector sum of all external torques that act on the body, measured about any possible point, must also be zero.

First condition of equilibrium:
The vector sum of all the external forces that act on the particle/body must be zero.
According to first condition, a particle is in equilibrium if the resultant of all the forces acting on it is zero. If $\vec{R}$ is the resultant of n forces $\vec{F_1}, \vec{F_2}, ...\vec{F_n}$ acting on a body, then accordingly

$$\vec{R} = \sum_{k=1}^{n} \vec{F_k} = \vec{0} \tag{4.1.1}$$

If $R_X,\ R_Y$ and R_Z are rectangular components of $\vec{R}$, then (4.1.1) can be written as

$$\langle R_X,\ R_Y, R_Z\rangle = \langle 0, 0, 0\rangle \tag{4.1.2}$$

or

$$\begin{aligned} R_X &= 0 \\ R_Y &= 0 \\ R_Z &= 0 \end{aligned}$$

are the necessary and sufficient conditions of equilibrium of equilibrium of a particle.
If the acting forces are coplanar, then (4.1.2) reduces to

$$\langle R_X,\ R_Y\rangle = \langle 0, 0\rangle \tag{4.1.3}$$

$$\begin{aligned} R_X &= 0 \\ R_Y &= 0 \end{aligned}$$

Next are some theorems regarding the equilibrium of a particle.

Theorem 4.1.1. ***The Triangle of Forces:*** *If a particle is in equilibrium under the action of three concurrent and coplanar forces, these forces may be represented in magnitude and direction (but not in position) by the sides of a triangle, taken in order.*

Proof: Let $\vec{P}$, $\vec{Q}$ and $\vec{R}$ are three concurrent and coplanar forces acting in a body at O and keep it in equilibrium. By parallelogram law, the resultant of two forces P and Q is given by the diagonal of parallelogram $OADB$ as shown in Fig 4.1 (b). Since the body is in equilibrium, this resultant will be equal in magnitude and opposite in direction to $\vec{R}$. Hence If a particle is in equilibrium under the action of three concurrent and coplanar forces, these forces may be represented in magnitude and direction (but not in position) by the sides of a triangle, taken in order as shown in Fig 4.1 (d).

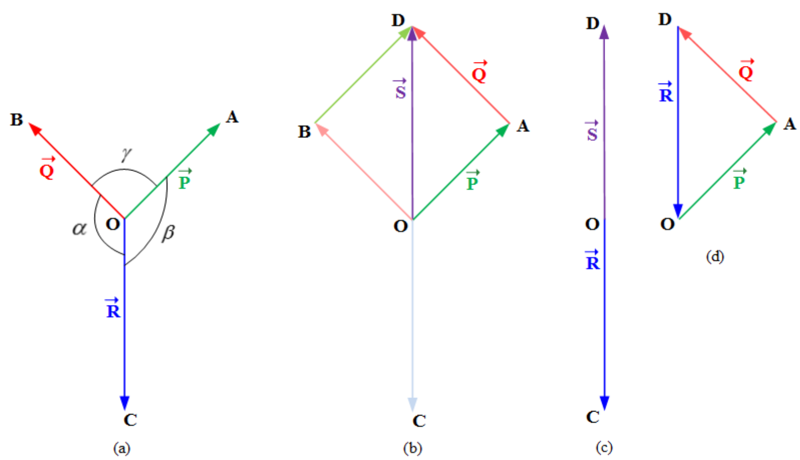

Figure 4.1: Triangle of forces

Theorem 4.1.2. ***The Polygon of Forces:*** *If a particle is in equilibrium under the action of several forces, these forces may be represented in magnitude and direction (but not in position) by the sides of a closed polygon, taken in order.*

Theorem 4.1.3. ***Lami's Theorem:*** *If three coplanar forces acting at a point in a body and keep it in equilibrium, then magnitude each force is directly proportional to the sine of the angle between the other two forces.*

Proof: Let $\vec{P}$, $\vec{Q}$ and $\vec{R}$ are three coplanar forces acting in a body at O and keep it in equilibrium. Let α be the angles between forces $\vec{Q}$ and $\vec{R}$, β be the angles between forces $\vec{R}$ and $\vec{P}$ and γ be the angles between forces $\vec{P}$ and $\vec{Q}$. Then by Lami's theorem:

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma} \tag{4.1.4}$$

Next by Law of triangle of forces, these forces may be represented in magnitude and direction (but not in position) by three sides of a triangle taken in order. We obtain a triangle of forces OAD as shown in Fig. 4.2. In this triangle
$OA = P$, $AD = Q$ and $DO = R$.
Its internal angles are $\angle ODA = \pi - \alpha$, $\angle AOD = \pi - \beta$ and $\angle OAD = \pi - \gamma$.
Also the forces $\vec{P}$ and $\vec{Q}$ are coplanar, completing the parallelogram, its diagonal will give the resultant of $\vec{P}$ and $\vec{Q}$.

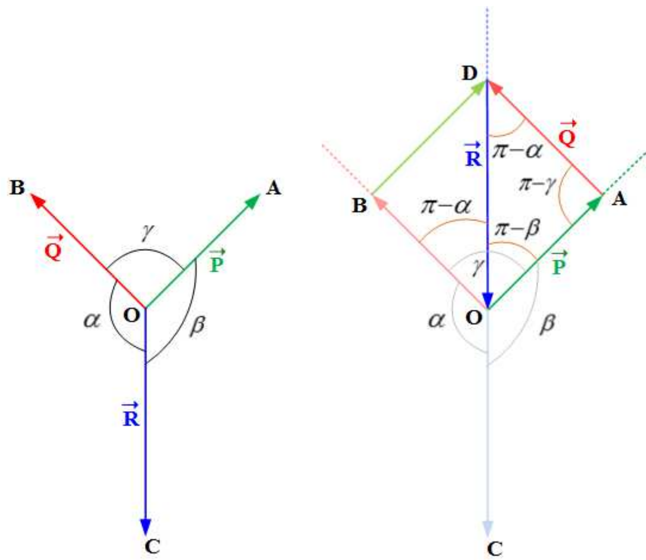

Figure 4.2: Lami's theorem

Next by law of sines to triangle OAD, we can write

$$\begin{aligned}\frac{QA}{\sin(\pi-\alpha)} &= \frac{AD}{\sin(\pi-\beta)} = \frac{DO}{\sin(\pi-\gamma)}\\ \frac{P}{\sin\alpha} &= \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}\end{aligned}$$

Example 4.1.1. *Three forces $\vec{F_1}$, $\vec{F_2}$ and $\vec{F_3}$ keeps a body in equilibrium. $\vec{F_1}$ acts towards north, $\vec{F_2}$ towards east-south and $\vec{F_3}$ towards south-west. If the magnitude of $\vec{F_1}$ is 8 N, find the magnitude of other two forces by using*

(a) Equilibrium conditions.

(b) Law of triangle of force.

(c) Lami's theorem.

Solution

(a) Magnitude of other two forces by using equilibrium conditions.

Three forces acting on a particle are shown in Fig. 4.3 (a). The horizontal components of

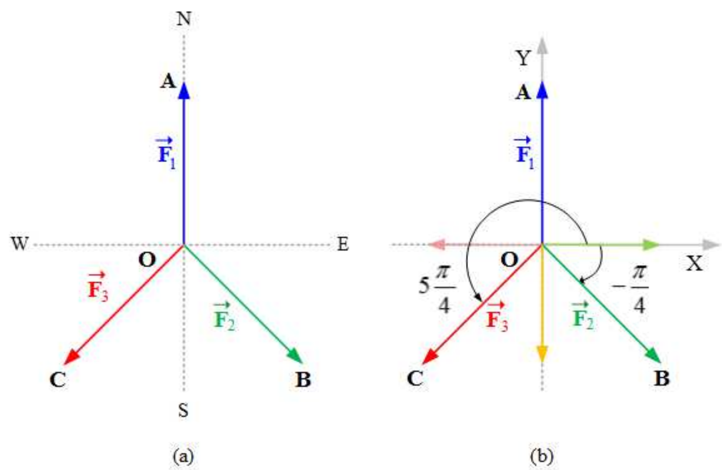

Figure 4.3: Triangle of forces

these forces are

$$\begin{aligned} F_{1X} &= 0 \\ F_{2X} &= F_2 \cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} F_2 \\ F_{3X} &= F_3 \cos\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} F_3 \end{aligned}$$

The vertical components of these forces are

$$\begin{aligned} F_{1Y} &= F_1 = 8\ N \\ F_{2Y} &= F_{2Y} \sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} F_2 \\ F_{3Y} &= F_{3Y} \sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} F_3 \end{aligned}$$

Since particle is in equilibrium under the action of these forces, then by (4.1.3), we have

$$\frac{1}{\sqrt{2}} F_2 - \frac{1}{\sqrt{2}} F_3 = 0 \tag{4.1.5}$$

$$8 - \frac{1}{\sqrt{2}} F_2 - \frac{1}{\sqrt{2}} F_3 = 0 \tag{4.1.6}$$

(4.1.5) implies that

$$F_2 = F_3 \tag{4.1.7}$$

Using (4.1.7) in (4.1.6), we have

$$F_2 = F_3 = 4\sqrt{2}\ N \tag{4.1.8}$$

(4.1.8) gives the other two forces which are equal in magnitude.

(b) Magnitude of other two forces by using law of triangle of force.

Since the particle is in equilibrium under the action of three concurrent and coplanar forces, so these forces can be represented in magnitude and direction (but not in position) by three sides of a triangle, taken in order. This triangle of forces is shown in Fig. 4.4. In triangle

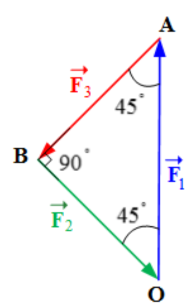

Figure 4.4: Triangle of forces

OAB

$$\begin{aligned} \vec{OA} &= \vec{F}_1 \\ \vec{BO} &= \vec{F}_2 \\ \vec{AB} &= \vec{F}_3 \end{aligned}$$

and

$$\begin{aligned} \angle OAB &= 45^\circ \\ \angle AOB &= 45^\circ \\ \angle ABO &= 90^\circ \end{aligned}$$

Since two angles are equal, each is 45°. Their corresponding sides will also be equal $i.e$

$$\begin{aligned} BO &= AB \\ F_2 &= F_3 \end{aligned}$$

From right triangle OAB, we can write

$$F_1 = \sqrt{F_2^2 + F_3^2}$$
$$8 = \sqrt{2}F_2$$

or

$$F_2 = F_3 = 4\sqrt{2}\ N$$

(c) Magnitude of other two forces by using Lami's theorem.

The force F_1 is along vertical and $F_2,\ F_3$ are making angles 45° with the horizontal, so the angles between these forces are (see Fig. 4.5):

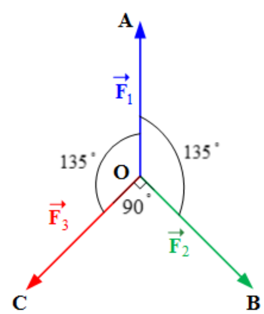

Figure 4.5: Lami's theorem

$$\text{Angle between } F_1 \text{ and } F_2 = 135^\circ$$
$$\text{Angle between } F_1 \text{ and } F_3 = 135^\circ$$
$$\text{Angle between } F_2 \text{ and } F_3 = 90^\circ$$

Since particle is in equilibrium under the action of these forces, then by (4.1.4), we have

$$\frac{F_1}{\sin 90^\circ} = \frac{F_2}{\sin 135^\circ} = \frac{F_3}{\sin 135^\circ}$$
$$\frac{8}{1} = \frac{F_2}{\frac{1}{\sqrt{2}}} = \frac{F_3}{\frac{1}{\sqrt{2}}} \tag{4.1.9}$$

From (4.1.9), we can write

$$F_2 = F_3 = 4\sqrt{2}\ N$$

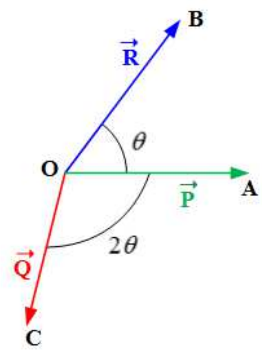

Figure 4.6: Equilibrium under three forces

Example 4.1.2. *A particle is in equilibrium under the action of three concurrent and coplanar forces $\vec{P}$, $\vec{Q}$ and $\vec{R}$. If $\vec{P}$ acts along horizontal towards right, the angle between $\vec{P}$ and $\vec{Q}$ is double of the angle between $\vec{P}$ and $\vec{R}$ as shown in Fig. 4.6, then prove that*

$$R^2 = Q(Q-P)$$

by using

(a) Equilibrium conditions.

(b) Law of triangle of force.

(c) Lami's theorem.

Solution

(a) Equilibrium conditions.

Take cartesian coordinate system, and the origin as the point of application of the forces. Let the force $\vec{P}$ acts along positive $x\ axis$. Let the angle between $\vec{P}$ and $\vec{R}$ is θ, then the angle between $\vec{P}$ and $\vec{Q}$ will be -2θ as shown in Fig. 4.11. We can say that $\vec{P}$ makes zero angle with $x\ axis$, $\vec{R}$ makes θ angle with $x\ axis$ and $\vec{Q}$ makes 2θ angle with $x\ axis$ as shown in Fig. 4.11. The horizontal components of these forces are

$$\begin{aligned} P_X &= P \\ R_X &= R\cos\theta \\ Q_X &= Q\cos(2\pi - 2\theta) = Q\cos 2\theta \end{aligned}$$

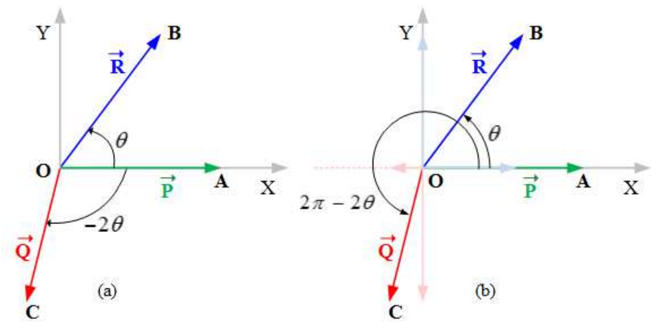

Figure 4.7: Equilibrium under three forces

The vertical components of these forces are

$$\begin{aligned} P_Y &= 0 \\ R_Y &= R \sin\theta \\ Q_Y &= Q \sin(2\pi - 2\theta) = -Q \sin 2\theta \end{aligned}$$

Since particle is in equilibrium under the action of these forces, then by (4.1.3), we have

$$P + Q\cos 2\theta + R\cos\theta = 0 \tag{4.1.10}$$

$$R\sin\theta - Q\sin 2\theta = 0 \tag{4.1.11}$$

Since $\sin 2\theta = 2\sin\theta\cos\theta$, then (4.1.11) can be written as

$$2Q\sin\theta\cos\theta = R\sin\theta \tag{4.1.12}$$

Also $\theta \neq 0$ $i.e$ $\sin\theta \neq 0$, then (4.1.12) can be written as

$$\cos\theta = \frac{R}{2Q} \tag{4.1.13}$$

Since $\cos 2\theta = 2\cos^2\theta - 1$, then (4.1.10) can be written as

$$P + Q\left(2\cos^2\theta - 1\right) + R\cos\theta = 0 \tag{4.1.14}$$

Using (4.1.13) in (4.1.14), we have

$$P + Q\left(2\left(\frac{R}{2Q}\right)^2 - 1\right) + R\frac{R}{2Q} = 0$$

After simplification we get

$$R^2 = Q(Q - P)$$

(b) Law of triangle of force.

Since the particle is in equilibrium under the action of three concurrent and coplanar forces, so these forces can be represented in magnitude and direction (but not in position) by three sides of a triangle, taken in order. This triangle of forces is shown in Fig. 4.8. In triangle

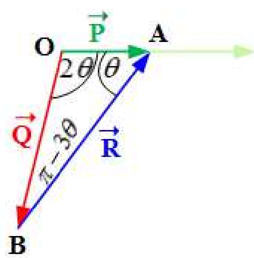

Figure 4.8: Triangle of forces

OAB, the forces are not in order. One force seems to be the resultant of other two forces.

(c) Lami's theorem.

Take cartesian coordinate system, and the origin as the point of application of the forces. Let the force $\vec{P}$ acts along positive $x\ axis$. Let the angle between $\vec{P}$ and $\vec{R}$ is θ, then the angle between $\vec{P}$ and $\vec{Q}$ is θ. The angle between $\vec{Q}$ and $\vec{R}$ is $2\pi - 3\theta$. Since particle is in

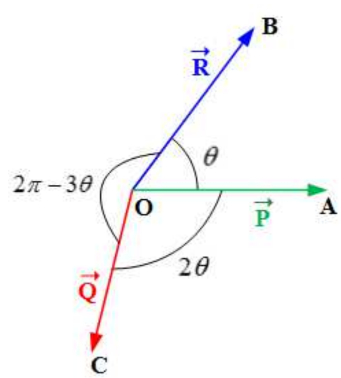

Figure 4.9: Lami's theorem

equilibrium under the action of these forces, then by (4.1.4), we have

$$\begin{aligned}\frac{P}{\sin(2\pi-3\theta)} &= \frac{Q}{\sin\theta}=\frac{R}{\sin 2\theta}\\ \frac{P}{\sin(-3\theta)} &= \frac{Q}{\sin\theta}=\frac{R}{\sin 2\theta}\\ -\frac{P}{\sin 3\theta} &= \frac{Q}{\sin\theta}=\frac{R}{\sin 2\theta}\end{aligned} \tag{4.1.15}$$

From (4.1.15), we can write

$$-\frac{P}{\sin 3\theta} = \frac{Q}{\sin\theta}$$

which implies that

$$P\sin\theta+Q\sin 3\theta = 0 \tag{4.1.16}$$

Also from (4.1.15), we can write

$$\frac{Q}{\sin\theta}=\frac{R}{\sin 2\theta}$$

$$R\sin\theta-Q\sin 2\theta = 0 \tag{4.1.17}$$

which is (4.1.12), then we have (4.1.13), see (a).
Next use trigonometric relation $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ in (4.1.16)

$$P\sin\theta+Q\left(3\sin\theta-4\sin^3\theta\right) = 0 \tag{4.1.18}$$

Since $\sin\theta\neq 0$, then (4.1.18) can be written as

$$\begin{aligned}P+Q\left(3-4\sin^2\theta\right) &= 0\\ P+Q\left(3-4+4\cos^2\theta\right) &= 0\\ P+Q\left(-1+4\cos^2\theta\right) &= 0\end{aligned}$$

Next using (4.1.13) in (4.1.19) will give the desired result.

Example 4.1.3. *A particle is in equilibrium under the action of three concurrent and coplanar forces $\vec{P}$, $\vec{Q}$ and $\vec{R}$ as shown in Fig. 4.10, then show that*

$$R^2 = Q(Q+P)$$

by using

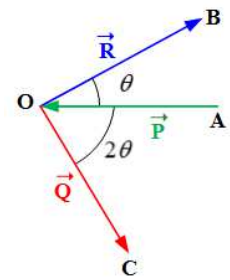

Figure 4.10: Equilibrium under three forces

(a) Equilibrium conditions.

(b) Law of triangle of force.

(c) Lami's theorem.

Solution

(a) Equilibrium conditions.

Take cartesian coordinate system, and the origin as the point of application of the forces. Let the force $\vec{P}$ acts along positive $x\ axis$. Let the angle between $\vec{P}$ and $\vec{R}$ is θ, then the angle between $\vec{P}$ and $\vec{Q}$ will be -2θ as shown in Fig. 4.11. We can say that $\vec{P}$ makes zero angle with $x\ axis$, $\vec{R}$ makes θ angle with $x\ axis$ and $\vec{Q}$ makes 2θ angle with $x\ axis$ as shown in Fig. 4.11. The horizontal components of these forces are

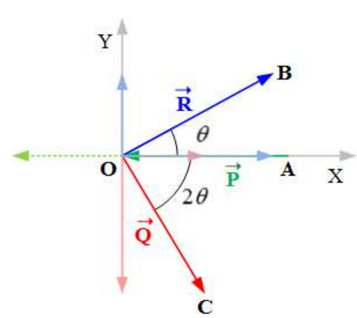

Figure 4.11: Equilibrium under three forces

$$\begin{aligned} P_X &= -P \\ R_X &= R\cos\theta \\ Q_X &= Q\cos(2\pi - 2\theta) = Q\cos 2\theta \end{aligned}$$

The vertical components of these forces are

$$\begin{aligned} P_Y &= 0 \\ R_Y &= R\sin\theta \\ Q_Y &= Q\sin(2\pi - 2\theta) = -Q\sin 2\theta \end{aligned}$$

Since particle is in equilibrium under the action of these forces, then by (4.1.3), we have

$$-P + Q\cos 2\theta + R\cos\theta = 0 \quad (4.1.19)$$

$$R\sin\theta - Q\sin 2\theta = 0 \quad (4.1.20)$$

Since $\sin 2\theta = 2\sin\theta\cos\theta$, then (4.1.20) can be written as

$$2Q\sin\theta\cos\theta = R\sin\theta \quad (4.1.21)$$

Also $\theta \neq 0$ *i.e* $\sin\theta \neq 0$, then (4.1.21) can be written as

$$\cos\theta = \frac{R}{2Q} \quad (4.1.22)$$

Since $\cos 2\theta = 2\cos^2\theta - 1$, then (4.1.19) can be written as

$$-P + Q\left(2\cos^2\theta - 1\right) + R\cos\theta = 0 \quad (4.1.23)$$

Using (4.1.22) in (4.1.23), we have

$$-P + Q\left(2\left(\frac{R}{2Q}\right)^2 - 1\right) + R\frac{R}{2Q} = 0$$

After simplification we get

$$R^2 = Q(Q+P)$$

(b) Law of triangle of force.

Since the particle is in equilibrium under the action of three concurrent and coplanar forces, so these forces can be represented in magnitude and direction (but not in position) by three sides of a triangle, taken in order. This triangle of forces is shown in Fig. 4.12.

Apply *law of sine's* on triangle OAC

$$\frac{P}{\sin(\pi - 3\theta)} = \frac{Q}{\sin\theta} = \frac{R}{\sin 2\theta}$$

$$\frac{P}{\sin 3\theta} = \frac{Q}{\sin\theta} = \frac{R}{\sin 2\theta} \quad (4.1.24)$$

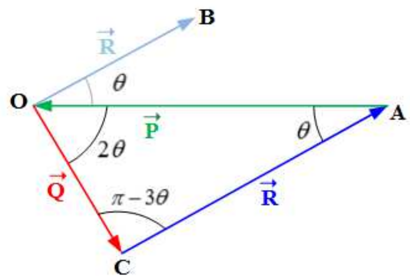

Figure 4.12: Triangles of forces

From (4.1.24), we can write

$$\frac{P}{\sin 3\theta} = \frac{Q}{\sin\theta}$$

which implies that

$$P\sin\theta - Q\sin 3\theta = 0 \tag{4.1.25}$$

Also from (4.1.24), we can write

$$\frac{Q}{\sin\theta} = \frac{R}{\sin 2\theta}$$

$$R\sin\theta - Q\sin 2\theta = 0$$

which gives (4.1.21), then we have (4.1.22), see (a).
Next use trigonometric relation $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ in (4.1.25).

$$P\sin\theta - Q\left(3\sin\theta - 4\sin^3\theta\right) = 0 \tag{4.1.26}$$

Since $\sin\theta \neq 0$, then (4.1.26) can be written as

$$\begin{aligned} P - Q\left(3 - 4\sin^2\theta\right) &= 0 \\ P - Q\left(3 - 4 + 4\cos^2\theta\right) &= 0 \\ P - Q\left(-1 + 4\cos^2\theta\right) &= 0 \end{aligned} \tag{4.1.27}$$

Next using (4.1.22) in (4.1.27) will give the desired result.

(c) Lami's theorem.

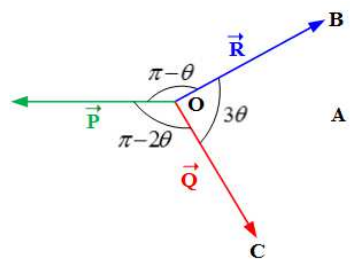

Figure 4.13: Lami's theorem

Take cartesian coordinate system, and the origin as the point of application of the forces. Let the force $\vec{P}$ acts along negative x *axis*. Let the angle between $\vec{P}$ and $\vec{R}$ is $\pi - \theta$, then the angle between $\vec{P}$ and $\vec{Q}$ is $\pi - \theta$. The angle between $\vec{Q}$ and $\vec{R}$ is 3θ as shown in Fig. 4.13. Since particle is in equilibrium under the action of these forces, then by (4.1.4), we have

$$\frac{P}{\sin 3\theta} = \frac{Q}{\sin(\pi - \theta)} = \frac{R}{\sin(\pi - 2\theta)}$$
$$\frac{P}{\sin 3\theta} = \frac{Q}{\sin\theta} = \frac{R}{\sin 2\theta}$$
$$\frac{P}{\sin 3\theta} = \frac{Q}{\sin\theta} = \frac{R}{\sin 2\theta}$$

which is (4.1.24), next we can use (b) to obtain the desire result.

Example 4.1.4. *In example 4.1.3, if $\theta = 30°$ and $P = 10\ N$, then find the other two forces.*

Solution Put $\theta = 30°$ and $P = 10\ N$ in (4.1.24)

$$\frac{10}{\sin 3(30°)} = \frac{Q}{\sin 30°} = \frac{R}{\sin 2(30°)}$$
$$\frac{10}{\sin 90°} = \frac{Q}{\sin 30°} = \frac{R}{\sin 60°}$$
$$10 = 2Q = 1.1547R$$

and finally we have

$$Q = 5\ N$$
$$R = 8.66\ N$$

Note: Triangle OAC is right angle triangle. Then by Pythagoras theorem we have

$$(10)^2 = (5)^2 + (8.66)^2$$

4.2 Moment of a Force and Equilibrium

A body is balanced if the sum of clockwise moments acting on the body is equal to the sum of anticlockwise moments acting on it.

Example 4.2.1. *A beam of length 2 meter, is supported at its middle point O as shown in Fig. 4.14. A body of weight 50 N is suspended at point A, 80 cm away from O. To the*

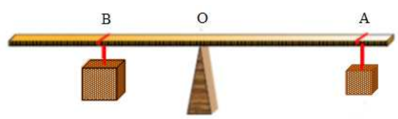

Figure 4.14: Balancing weights

other side, at B, another block is suspended 50 cm away from O to balance the rod. Find the weight of the second block.

Solution The given problem can be solved by considering clockwise and anticlockwise moments acting on the body with O as the fixed point. The given data is

$$\begin{aligned} W_1 &= 50\ N \\ OA = d_1 &= 80\ cm = 0.8\ m \\ OB = d_2 &= 50\ cm = 0.5\ m \end{aligned}$$

see Fig. 4.15, and we have to find W_2 Let τ_1 be the anticlockwise moment of W_1 about O

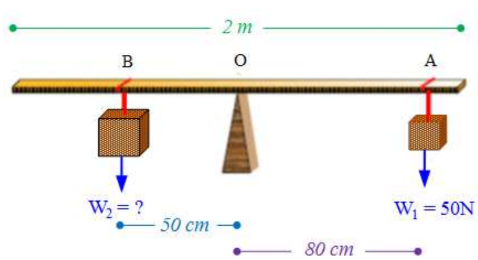

Figure 4.15: Balancing weights

and τ_2 be the clockwise moment of W_2 about O. Then

$$\begin{aligned} \tau_1 &= 50(0.8) = 40\ N - m \\ \tau_2 &= W_2(0.5)\ N - m \end{aligned}$$

By principle of moments, we can write

$$\begin{aligned} \tau_1 &= \tau_2 \\ 40 &= 0.5W_2 \end{aligned}$$

or

$$W_2 = 80\ N$$

Thus, weight of the other block is 80 N

Exercises

1. Three forces $\vec{F_1}$, $\vec{F_2}$ and $\vec{F_3}$ keeps a body in equilibrium. $\vec{F_1}$ acts towards south, $\vec{F_2}$ towards east-north and $\vec{F_3}$ towards north-west. If the magnitude of $\vec{F_1}$ is 10 N, find the magnitude of other two forces by using
   (a) Equilibrium conditions.
   (b) Law of triangle of force.
   (c) Lami's theorem.
2. The three forces in the diagram are in equilibrium. Find $\vec{F}$ and θ? by using

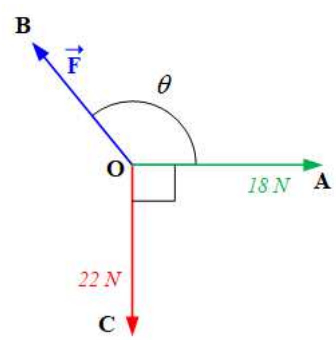

Figure 4.16: Three forces

   (a) Equilibrium conditions.
   (b) Law of triangle of force.
   (c) Lami's theorem.
3. A beam of length 3 meter, is supported at its middle point O as shown in Fig. 4.17. A body of weight 40 N is suspended at point A, 1.5 m away from O. To the other

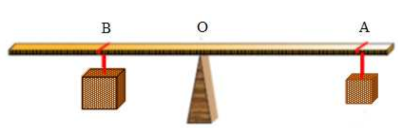

Figure 4.17: Balancing weights

side, at B, another block is suspended 1 m away from O to balance the rod. Find the weight of the second block.

Chapter 5

Friction

5.1 Friction

When a body moves or tends to move upon another body, an opposing force (opposite to the direction of motion) appears between their surfaces, known as the force of friction. Its line of action is tangential to the contacting surfaces. The magnitude of this force depends on the roughness of surfaces.
Friction is a part of our daily life, may be useful or not. It appears in almost every movement. We can walk on the ground because of friction. Friction is useful in power transmission by belts. It is useful in appliances like brakes, bolts, screw jacks, etc. It is undesirable in bearing and moving machine parts where it results in loss of energy and, thereby, reduces efficiency of the machine.
Smooth Contact
If the force of friction between two bodies in contact with each other is zero, the contact is said to be smooth.
Rough Contact
If the force of friction between two bodies in contact with each other is not zero, the contact is said to be rough.
In nature no perfectly smooth body exist. Each body is capable of exerting some force of friction, although it may be quite small as in the case of glass, steel etc.

5.1.1 Types of Friction

There are two types of friction:

1. **Dry friction**: When the surfaces in contact are un-lubricated surfaces or dry surfaces, the friction is dry friction.

2. **Fluid friction**: When the surfaces in contact are lubricated surfaces, the friction is fluid friction. In this case there is no direct contact between the surfaces.

The dry friction can be further subdivided based on how the two surfaces are at rest or moving relative to each other.

a) **Static friction**

The friction that exists between two surfaces that are not moving relative to each other.

b) **Kinetic friction**

The friction that exists between two surfaces that are moving relative to each other.
In any situation, the static friction is greater than the kinetic friction.

5.1.2 Static and Kinetic Friction

Consider a block of mass m rests on a plane surface as shown in Fig. 5.1. The weight $\vec{W}$ acts downward and is balanced by an upward normal reaction $\vec{R}$ offered by the surface. Next a pull force $\vec{F}$ is applied to the block such that it tends to move. The motion is opposed by a frictional force $\vec{F_r}$, equal and opposite to $\vec{F}$. The resultant of normal reaction and the applied force is called resultant reaction denoted be $\vec{S}$, making an angle λ with the normal reaction. If we increase the pull or attractive force, the fractional force will also increase

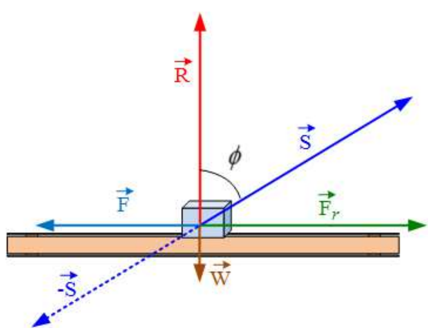

Figure 5.1: Friction

and hence the resultant reaction and inclination will increase. Thus, with increase of the pull force, the frictional force, the resultant reaction and its inclination will increase, till the body tends to move. The frictional resistance offered upto that instant is known as static friction.
When the body is just on the point of sliding, but actual sliding has not started, the ultimate value of static friction is called limiting friction or maximum static friction. The condition, when all the forces are just in equilibrium and the body has a tendency to move, is called limiting equilibrium position. When a body moves relative to another body, the resisting force between them is called kinetic or sliding friction.
Thus four possible cases arise when two rough bodies are in contact with each other:

1. No force of friction is acting between them.

2. The force of friction is acting but neither body is on the point of sliding along the other. The friction in this case is non-limiting.

3. One of the bodies is on the point of sliding along the other. The friction in this case is limiting.

4. One body slide along the other. The force of friction in such a case which opposes motion is the kinetic friction.

Table 5.1: Coefficients of Static and Kinetic friction between different surfaces.

$Surfaces$	μ_s	μ_k
steel on steel	0.74	0.57
aluminum on steel	0.61	0.47
copper on steel	0.53	0.36
rubber on concrete	1.0	0.8
wood on wood	$0.25 - 0.5$ depends on the type of wood	0.2
glass on glass	0.94	0.4
waxed wood on wet snow	0.14	0.1
waxed wood on dry snow	–	0.04
metal on metal (lubricated)	0.15	0.06
ice on ice	0.14	0.03
teflon on teflon	0.04	0.04
synovial joints in humans	0.01	0.003

5.1.3 Laws of Dry Friction

The force of friction is self-adjusting force, obeying some definite laws. These laws are stated as below.

1. The direction of friction is opposite to the direction in which the body moves or tends to move.

2. The magnitude of friction upto a certain extent, is equal to the force, tending to produce motion.

3. The frictional force depends upon the nature of the surfaces in contact.

4. The amount of friction is independent of the areas and shape of the surfaces in contact provided the normal pressure remains unaltered.

5. For moderate speeds, frictional force is independent of the relative velocities of the bodies in contact.

6. Only a certain amount of friction can be called into play and in each particular case it can not exceed a certain limit. The maximum amount of friction which can be called into play is called **limiting friction**.

7. The magnitude of the limiting friction (for given surfaces) bears a constant ratio μ to the normal pressure between the surfaces

The magnitude of limiting friction is denoted by F_r and the normal reaction by R. Then the coefficient of friction is

$$\mu = \frac{F_r}{R} \tag{5.1.1}$$

or

$$F_r = \mu R$$

Limiting Equilibrium Equilibrium under the influence of limiting friction is called limiting equilibrium.

5.1.4 Angle of Friction

If F_r is the magnitude of limiting fraction and R be the normal reaction as shown in the Fig. 5.2. In case of limiting friction the angle between the resultant reaction and the normal reaction is called the angle of friction and is denoted by ϕ. From Fig. 5.2, we can write

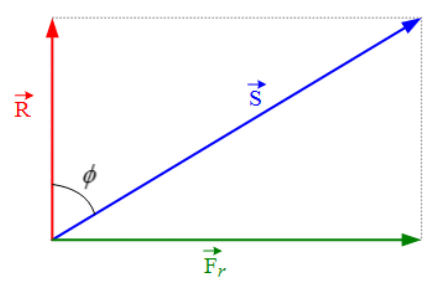

Figure 5.2: Angle of friction

$$R = S\cos\phi$$
$$F_r = S\sin\phi$$

Using above results, (5.1.1) can be written as

$$\begin{aligned}\mu &= \frac{F_r}{R} = \frac{S\sin\phi}{S\cos\phi} \\ &= \tan\phi \end{aligned} \tag{5.1.2}$$

(5.1.2) gives the relation between the angle of friction and the coefficient of friction. Its inverse is

$$\phi = \tan^{-1}\mu \tag{5.1.3}$$

gives the angle of friction.
The magnitude of the angle of friction is usually denoted by λ so that $\lambda = \tan^{-1}\mu$ or $\mu = \tan\lambda$. The resultant reaction can make any angle with the normal with magnitude λ, but cannot make a greater angle, because F_r cannot exceed μR.

5.1.5 Cone of Friction

It is defined as the right circular cone with vertex at the point of contact of the two surfaces, its axis is in the direction of normal reaction N and its semi vertical angle is equal to angle of friction. (see Fig. 5.3)
For any real contact of the bodies $\vec{S}$ lies within the cone and in case of limiting friction $\vec{S}$ lies on a generator of this cone.

5.1.6 Role of Friction (Benefits)

Friction plays an important role in our daily life. Some of them are as following:

- **Walking:** When a person walks, he pushes the ground backward with his feet to cause a forward reaction. If there were no friction our feet will slip and we would not be able to walk.

- **Rolling Motion:** All rolling motion is caused by static friction.
  The force of friction encompasses the entire operation of a car and makes the tires possible to turn on the road. Tires are designed with a degree of tread that helps maintain a high degree of friction to allow the tire to grip tightly to the road and keep control. If there was no tread, there would be no friction and the car would not be able to stop at the appropriate time.

- **Using computers mouse pad:** Friction occurs between the mouse and the desktop or mouse pad. Friction is required to move the mouse and have it respond appropriately. If you were to use a different kind of material for a mouse pad, such as a piece of sand paper, the mouse is harder to move. The piece of sand paper has more friction than the mouse pad.

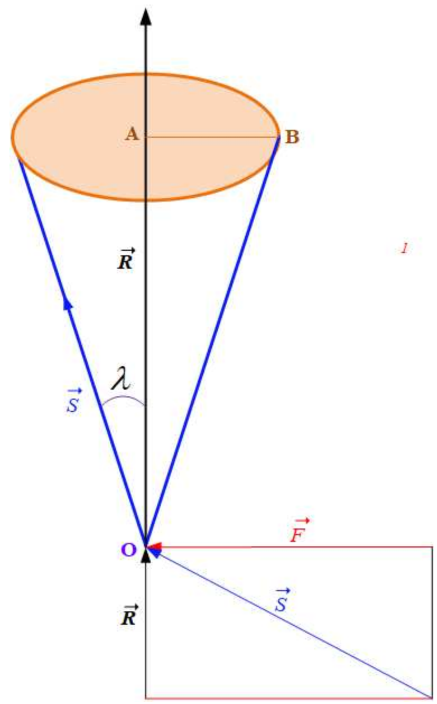

Figure 5.3: Cone of friction

If we lived in a world with no friction, things may be a bit more chaotic. Automobiles, airplanes and other vehicles would have a tough time trying to slow down or stop because they would be trying to brake on a frictionless surface. This would be like trying to stop on an ice skating rink. There would be nothing to grip to the surface.

5.2 Condition of Equilibrium of a Particle on a Rough Inclined Plane

Consider a rough inclined plane making an angle α with the horizontal. Let a block of mass m is resting on it. Various forces acting on the mass are shown in the Fig. 5.4. If the angle of inclination is slowly increased, a stage will come when the block will tend to slide down. This angle of the plane with horizontal plane is known as ***angle of repose***. For satisfying the conditions of limiting equilibrium and resolving the forces along and perpendicular to the plane. The forces along the plane are

$$\mu R - W \sin\alpha = 0 \quad (5.2.1)$$

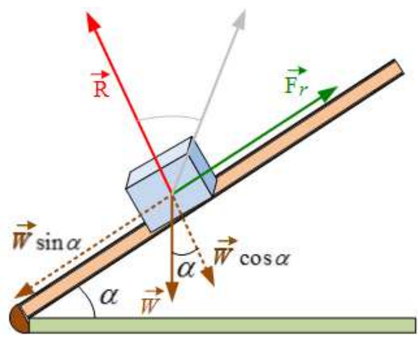

Figure 5.4: Angle of friction

And the forces perpendicular to the plane are

$$R - W\cos\alpha = 0$$

or

$$R = W\cos\alpha \tag{5.2.2}$$

Using (5.2.2), (5.2.1) can be written as

$$\mu W\cos\alpha = W\sin\alpha$$

or

$$\begin{aligned}\mu &= \frac{\sin\alpha}{\cos\alpha} \\ &= \tan\alpha \end{aligned} \tag{5.2.3}$$

Using (5.1.2), (5.2.3) can be written as

$$\begin{aligned}\tan\phi &= \tan\alpha \\ \phi &= \alpha \end{aligned} \tag{5.2.4}$$

(5.2.4) gives the condition for limiting equilibrium of a body on a rough inclined plane.

5.3 Least force required to drag a body on a rough plane

In this section we will find the least force required to drag a body on a rough horizontal plane and inclined plane. First consider horizontal plane.

5.3.1 Least Force Required to Drag a Body on a Rough Horizontal Plane

Consider a block of mass m is placed on a horizontal rough surface as shown in Fig. 5.5. A tractive force F is applied at an angle θ with the horizontal such that the block just tends to move. This force will have its components $F_x = F\cos\theta$ along and $F_y = F\sin\theta$ perpendicular to the plane. Hence, at the limiting equilibrium, the forces acting on the

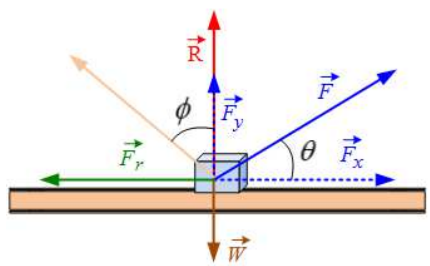

Figure 5.5: Friction

mass are:

1. The weight W acts downward and is balanced by a sum of vertical component of tractive force and an upward normal reaction R offered by the surface.

$$R + F\sin\theta = W \tag{5.3.1}$$

2. The horizontal component of tractive force is balanced by the force of friction F_r.

$$\begin{aligned} F\cos\theta &= F_r \\ &= \mu R \end{aligned} \tag{5.3.2}$$

Using (5.3.1), (5.3.2) can be written as

$$F\cos\theta = \mu\,(W - F\sin\theta) \tag{5.3.3}$$

Using (5.1.2), (5.3.3) can be written as

$$\begin{aligned} F\cos\theta &= \tan\phi\,(W - F\sin\theta) \\ &= \frac{\sin\phi}{\cos\phi}\,(W - F\sin\theta) \end{aligned}$$

or

$$F\,(\cos\theta\cos\phi + \sin\theta\sin\phi) = W\sin\phi$$

Hence the tractive force F can be written as

$$F = \frac{W \sin \phi}{\cos(\theta - \phi)} \tag{5.3.4}$$

The tractive force F will be least, if the denominator $\cos(\theta - \phi)$ must be maximum and it will be so if

$$\cos(\theta - \phi) = 1$$

then

$$(\theta - \phi) = 0$$

or

$$\theta = \phi \tag{5.3.5}$$

Hence, the force F will be the least if angle of its inclination with the horizontal θ is equal to the angle of friction ϕ. In this case the magnitude of the least tractive force is

$$F = W \sin \phi \tag{5.3.6}$$

5.3.2 Least Force Required to Drag a Body on a Rough Inclined Plane

Here we consider two cases

a) The body drags up the inclined plane.

b) The body drags down the inclined plane.

a) **The body drags up the inclined plane**.

Consider a block of mass m is placed on an inclined rough surface as shown in Fig. 5.6. A tractive force F is applied at an angle θ with the plane such that the block just tends to move up. This force will have its components $F \cos \theta$ along and $F \sin \theta$ perpendicular to the plane. Also the weight W has components $W \sin \alpha$ along and $W \cos \alpha$ perpendicular to the plane. Hence, at the limiting equilibrium, the forces acting on the mass and perpendicular to the plane are:

1. The weight component $W \cos \alpha$ is balanced by a sum of component of force $F \sin \theta$ and a normal reaction R offered by the surface.

$$R + F \sin \theta = W \cos \alpha$$

or

$$R = W \cos \alpha - F \sin \theta \tag{5.3.7}$$

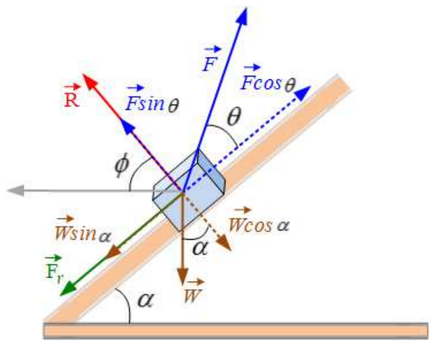

Figure 5.6: Friction

2. The component of force $F\cos\theta$ is balanced by a sum of the force of friction F_r and weight component $W\sin\alpha$.

$$\begin{aligned} F\cos\theta &= F_r + W\sin\alpha \\ &= \mu R + W\sin\alpha \end{aligned} \tag{5.3.8}$$

Using (5.3.7), (5.3.8) can be written as

$$F\cos\theta = \mu\left(W\cos\alpha - F\sin\theta\right) + W\sin\alpha \tag{5.3.9}$$

Using (5.1.2), (5.3.9) can be written as

$$\begin{aligned} F\cos\theta &= \tan\phi\left(W\cos\alpha - F\sin\theta\right) + W\sin\alpha \\ &= \frac{\sin\phi}{\cos\phi}\left(W\cos\alpha - F\sin\theta\right) + W\sin\alpha \end{aligned}$$

or

$$F\left(\cos\theta\cos\phi + \sin\theta\sin\phi\right) = W\left(\cos\alpha\sin\phi + \sin\alpha\cos\phi\right)$$

Hence the tractive force F can be written as

$$F = \frac{W\sin\left(\theta+\phi\right)}{\cos\left(\theta-\phi\right)} \tag{5.3.10}$$

The tractive force F will be least, if the denominator $\cos\left(\theta-\phi\right)$ must be maximum and it will be so if

$$\cos\left(\theta-\phi\right) = 1$$

then

$$\left(\theta-\phi\right) = 0$$

or

$$\theta = \phi$$

Hence, the force F will be the least if angle of its inclination with the horizontal θ is equal to the angle of friction ϕ. In this case the magnitude of the least tractive force is

$$F = W \sin(\theta + \phi) \tag{5.3.11}$$

b) **The body drags down the inclined plane**.

Consider a block of mass m is placed on an inclined rough surface as shown in Fig. 5.7. A tractive force F is applied at an angle θ with the plane such that the block just tends to move down. This force will have its components $F\cos\theta$ along and $F\sin\theta$ perpendicular to the plane. Also the weight W has components $W\sin\alpha$ along and $W\cos\alpha$ perpendicular to the plane. Hence, at the limiting equilibrium, the forces acting on the mass and perpendicular

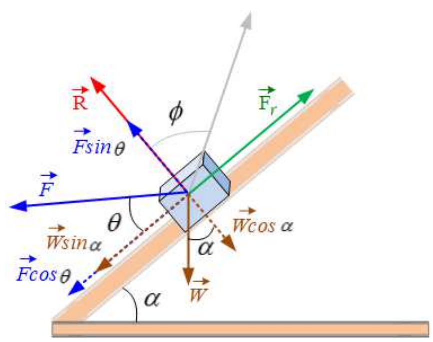

Figure 5.7: Friction

to the plane are:

1. The weight component $W\cos\alpha$ is balanced by a sum of component of force $F\sin\theta$ and a normal reaction R offered by the surface.

$$R + F\sin\theta = W\cos\alpha \tag{5.3.12}$$

2. The component of force $F\cos\theta$ is balanced by a sum of the force of friction F_r and weight component $W\sin\alpha$.

$$\begin{aligned} F\cos\theta + W\sin\alpha &= F_r \\ &= \mu R + W\sin\alpha \end{aligned} \tag{5.3.13}$$

Using (5.3.12), (5.3.13) can be written as

$$F\cos\theta = \mu\left(W\cos\alpha - F\sin\theta\right) - W\sin\alpha \tag{5.3.14}$$

Using (5.1.2), (5.3.14) can be written as

$$\begin{aligned} F\cos\theta &= \tan\phi\left(W\cos\alpha - F\sin\theta\right) - W\sin\alpha \\ &= \frac{\sin\phi}{\cos\phi}\left(W\cos\alpha - F\sin\theta\right) - W\sin\alpha \end{aligned}$$

or

$$F\left(\cos\theta\cos\phi + \sin\theta\sin\phi\right) = W\left(\cos\alpha\sin\phi + \sin\alpha\cos\phi\right)$$

Hence the tractive force F can be written as

$$\begin{aligned} F &= \frac{W\sin\left(\theta-\phi\right)}{\cos\left(\theta-\phi\right)} \\ &= W\tan\left(\theta-\phi\right) \end{aligned} \tag{5.3.15}$$

This means that the force F can be applied for $\phi < \theta$, and for $\theta < \phi$ body will move without applying force F.
Hence for $\phi < \theta$, the tractive force F will be least, if in (5.3.15), the denominator $\cos\left(\theta-\phi\right)$ must be maximum and it will be so if

$$\cos\left(\theta-\phi\right) = 1$$

then

$$\left(\theta-\phi\right) = 0$$

or

$$\theta = \phi$$

Hence, the force F will be the least if angle of its inclination with the horizontal θ is equal to the angle of friction ϕ. In this case the magnitude of the least tractive force is

$$F = W\sin\left(\theta-\phi\right) \tag{5.3.16}$$

Example 5.3.1. *Consider a block of weight $40N$ rests on a rough horizontal plane and can just be moved by a force of $10N$ acting horizontally. Find the coefficient and the angle of friction.*

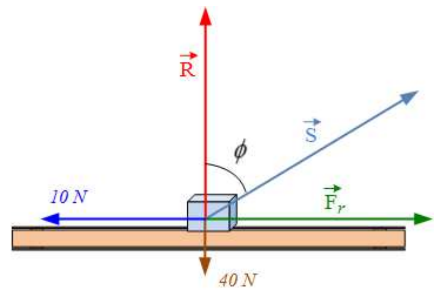

Figure 5.8: Friction

Solution Various forces acting on the block are shown in Fig. 5.8.
At limiting equilibrium, the forces acting perpendicular to the plane are the weight W balanced by the normal reaction R offered by the surface.

$$\begin{aligned} R &= W \\ &= 40N \end{aligned} \tag{5.3.17}$$

The forces acting along the plane are the pull force F balanced by the force of friction F_r.

$$\begin{aligned} F &= F_r \\ 10N &= \mu R \end{aligned} \tag{5.3.18}$$

Using (5.3.17) in (5.3.18), the coefficient of friction is

$$\mu = \frac{10}{40} = 0.25 \tag{5.3.19}$$

Using (5.1.3), the angle of friction is

$$\begin{aligned} \phi &= \tan^{-1}\mu = \tan^{-1}(0.25) \\ &= 14.03^\circ \end{aligned} \tag{5.3.20}$$

(5.3.20) gives the angle of friction.

Example 5.3.2. *Consider a block of mass* $3kg$ *rests on a floor (rough horizontal plane) and can just be moved by a force of* $12N$ *acting at an upward angle* θ *with the horizontal. The coefficient of static friction between the block and floor is* 0.4. *Find the least force required to drag a body on the floor. Also find its inclination with the horizontal.*

Solution The given data is

$$\begin{aligned} m &= 3kg \\ F &= 12N \\ \mu &= 0.4 \end{aligned}$$

The weight of the body is $W = 29.4N$
Various forces acting on the block are shown in Fig. 10.3.
Using (5.1.3), the angle of friction is

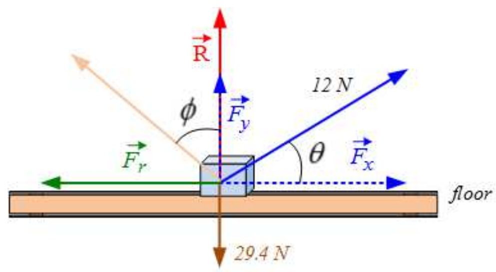

Figure 5.9: Friction

$$\begin{aligned} \phi &= \tan^{-1}\mu = \tan^{-1}(0.4) \\ &= 21.8^\circ = 22^\circ \end{aligned} \qquad (5.3.21)$$

A the limiting equilibrium, considering (5.3.5), the angle of least force is

$$\theta = \phi = 22^\circ$$

Finally, using (5.3.6), the magnitude of the least tractive force is

$$\begin{aligned} F &= W \sin\phi \\ &= 29.4\sin(22) \\ &= 1.611N \end{aligned}$$

Exercises

1. A $10kg$ piece of wood is placed on top of another piece of wood. There is $30N$ of maximum static friction measured between them. Determine the coefficient of static friction and the angle of friction between the two pieces of wood.

2. A rod, $4ft$. long, rests on a rough floor against the smooth edge of a table of height $3ft$. If the rod is on the point of slipping when inclined at an angle of $60°$ to the horizontal. Find the coefficient and the angle of friction.

3. A uniform ladder of length $30feet$, rests against a vertical wall, making an angle of $45°$ with the horizontal. The coefficient of friction between the ladder and the wall is 0.4 and the coefficient of friction between the ladder and the ground is 0.5. If a man, whose weight is equal to that of the ladder, ascends the ladder, where will he be when the ladder slips?

Chapter 6

Linear Momentum, Impulse and Collision

There are phenomena in which interaction between bodies is so fast that it is difficult to measure the forces that are produced between them or the time that the interaction lasts. For example, how long does the collision between two billiard balls last for? What force does one ball apply on the other? These questions are, no doubt, difficult to answer. In these cases, the notion of linear momentum and impulse, in addition to the conditions under which linear momentum is conserved, will allow us to make predictions of the speed and direction of the movement after the interaction. First consider some information about types of system.
Isolated system: No matter or energy is allowed to enter or leave in this system.
Closed system: A system in which no matter is allowed to enter or leave but energy can enter or leave is called closed system.
Open system: In this system energy and matter can enter or leave.

6.1 Linear Momentum

The momentum measure provides a sense of how difficult or easy it will be to change the motion of a particle. Assume a locomotive has a large mass m and a very small velocity. Despite the slow motion, it makes intuitive sense that it would be very difficult to stop the motion of this large object. The linear momentum $\vec{p}$ of the locomotive is large due to the large mass. Similarly, consider a bullet with a small mass moving with a very high velocity. Again, it makes intuitive sense that it would be difficult to deflect the motion of the bullet once it has been fired. In this case the linear momentum of the bullet is large not because of its mass, but because of its very large inertial velocity. We can say momentum as the quality of motion.
For a particle of mass m moving with velocity $\vec{v}$, then the linear momentum or simply

momentum is:

$$\vec{p} = m\vec{v}$$

It is a vector quantity. Its unit in SI is $kg.m/s$ or $N.s$.

6.1.1 Linear Momentum and Newtons Second Law of Motion

Newton expressed his second law of motion in terms of momentum as:
The time rate of change of the momentum of a particle is equal to the net force acting on the particle and is in the direction of the force. Mathematically

$$\vec{F}(t) = \frac{d}{dt}\vec{p}(t)$$

If the mass is not time-dependent, we have

$$\vec{F}(t) = m\frac{d}{dt}\vec{v}(t) = m\frac{d^2}{dt^2}\vec{r}(t)$$

The velocity is the time derivative of position vector

$$\vec{v} = \dot{\vec{r}} = \frac{d}{dt}\vec{r}$$

and acceleration is the two times derivative of position vector

$$\vec{a} = \ddot{r} = \frac{d^2}{dt^2}\vec{r}$$

Then Newton's second law of motion is

$$\begin{aligned}\vec{F} &= \dot{\vec{p}} = m\dot{\vec{v}} = m\ddot{\vec{r}}\\ &= m\vec{a}\end{aligned}$$

Thus Newton's second law of motion can also be stated as the force acting on a particle is directly proportional to the acceleration produced, considering m as a constant.
If a particle of constant mass m has a velocity changed from $\vec{v}_1$ to $\vec{v}_2$ in a time t by a force $\vec{F}$ acting then the acceleration $\vec{a}$ is

$$\vec{a} = \frac{\vec{v}_2 - \vec{v}_1}{t}$$

Then the force is

$$\begin{aligned}\vec{F} &= m\frac{\vec{v}_2 - \vec{v}_1}{t}\\ &= \frac{m\vec{v}_2 - m\vec{v}_1}{t}\end{aligned} \tag{6.1.1}$$

Example 6.1.1. *A car of mass* 1000 *kg has changed its speed from* 15 *m/s to* 12 *m/s in* 2 *s. How large was the retarding force?*

Solution:
The given data is

$$\begin{aligned} m &= 1000\ kg \\ v_1 &= 15\ m/s \\ v_2 &= 12\ m/s \\ t &= 2\ s \end{aligned}$$

A force is retarding force if its magnitude has negative sign. Using (6.1.1), the force acting on the car is

$$\begin{aligned} \vec{F} &= m\frac{\vec{v}_2 - \vec{v}_1}{t} \\ &= 1000\frac{12-15}{2} \\ &= -1500\ N \end{aligned}$$

The negative sign indicates that the acting force is retarding force.

Corollary 6.1.1. *If no force is acting on the particle, then the linear momentum is constant.*

Proof: According to Newton's second law of motion, the force acting on a particle is

$$\vec{F} = \dot{\vec{p}}$$

If $\vec{F} = 0$, then

$$\dot{\vec{p}} = \vec{O}$$

or

$$\vec{p} = \vec{C}\ (constant)$$

Hence the linear momentum is constant.

6.1.2 Law of Conservation of Linear Momentum

Consider an isolated system of two particles with masses m_1 and m_2. At an instant of time the particles are moving with velocities $\vec{v}_1$ and $\vec{v}_2$. Since the system is isolated, then by Newton's third law of motion, the action and reaction are the only forces coming into action, having same magnitude but opposite direction. If $\vec{F}_{12}$ be the force exerted by m_1 on m_2, then If $\vec{F}_{21}$ be the force exerted by m_2 on m_1, and we have

$$\vec{F}_{12} = -\vec{F}_{12}$$

or

$$\vec{F}_{12} + \vec{F}_{12} = \vec{0}$$

Over some time interval, the two masses will accelerate. Then, following Newton's second law of motion $\vec{F} = m\vec{a}$, we can write

$$m_1\vec{a}_1 + m_2\vec{a}_2 = \vec{0}$$

Since acceleration is the time derivative of velocity, so above relation can be written as

$$m_1\frac{d}{dt}\vec{v}_1 + m_2\frac{d}{dt}\vec{v}_2 = \vec{0}$$

If the masses m_1 and m_2 are constants, they can be brought into the derivatives, that is

$$\frac{d}{dt}(m_1\vec{v}_1) + \frac{d}{dt}(m_2\vec{v}_2) = \vec{0}$$
$$\frac{d}{dt}(m_1\vec{v}_1 + m_2\vec{v}_2) = \vec{0}$$

or

$$(m_1\vec{v}_1 + m_2\vec{v}_2) = \vec{C}\ (constant) \tag{6.1.2}$$

Hence the linear momentum is conserved.

6.2 Impulse of a Force

The impulse I of a constant force F acting for a time t is defined as the product of the force and time. $(i.e)$

$$\vec{I} = \vec{F} \cdot t \tag{6.2.1}$$

The impulsive of a force acting on a particle in any interval of time (t_1, t_2) is defined to be the momentum changed produced.

$$\vec{I} = \vec{p}_f - \vec{p}_i = \Delta\vec{p}$$

From Newton's second law of motion, we have

$$\vec{F} = \frac{d\vec{p}}{dt}$$

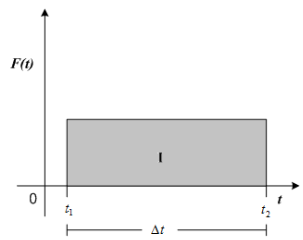

Figure 6.1: Impulse of a force

or

$$d\vec{p} = \vec{F}dt \tag{6.2.2}$$

Integrating Eq. (6.2.2) from a time t_1 to t_2, we have

$$\begin{aligned}\int_{t_1}^{t_2} d\vec{p} &= \int_{t_1}^{t_2} \vec{F}dt \\ \vec{p}_2 - \vec{p}_1 &= \int_{t_1}^{t_2} \vec{F}dt \\ \Delta\vec{p} &= \int_{t_1}^{t_2} \vec{F}dt \\ \vec{I} &= \int_{t_1}^{t_2} \vec{F}dt \qquad (6.2.3)\end{aligned}$$

Thus the impulse of the force $\vec{F}$ is the time integral of the force.
Another way to show this is as follow:
If a particle of constant mass m has a velocity changed from $\vec{v}_1$ to $\vec{v}_2$ in a time t by a force $\vec{F}$ acting then the impulse $\vec{I}$ is

$$\begin{aligned}\vec{I} &= \Delta\vec{p} = \vec{p}_2 - \vec{p}_1 \\ &= m\vec{v}_2 - m\vec{v}_1\end{aligned}$$

If m is constant and $\vec{v}_2 - \vec{v}_1$ can be written as the combination of differentiation and integration.

$$\begin{aligned}&= m\int_{t_1}^{t_2} \frac{d\vec{v}}{dt}dt = \int_{t_1}^{t_2} m\vec{a}dt \\ &= \int_{t_1}^{t_2} \vec{F}dt\end{aligned}$$

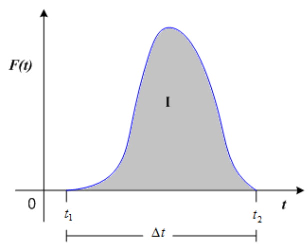

Figure 6.2: Impulse of a force

Thus the impulse of the force $\vec{F}$ is the time integral of the force. Also we can say that it is the area between the curve and t axis.

6.2.1 Linear Momentum and Kinetic Energy

The kinetic energy of a particle of mass m moving with speed v is

$$\begin{aligned} K &= \frac{1}{2}mv^2 \\ &= \frac{m^2v^2}{2m} \\ &= \frac{p^2}{2m} \end{aligned} \tag{6.2.4}$$

(6.2.4) is the relation between kinetic energy, momentum and mass.

Example 6.2.1. *The magnitude of an impulse I changes the speed of a particle of mass m from v_1 to v_2 in a time t by a force F. Show that the kinetic energy gained is $\frac{1}{2}I(v_2+v_1)$*

Solution: Since

$$I = mv_2 - mv_1$$

The kinetic energy gained is

$$\begin{aligned} K &= \frac{1}{2}m(v_2^2 - v_1^2) \\ &= \frac{1}{2}m(v_2 - v_1)(v_2 + v_1) \\ &= \frac{1}{2}I(v_2 + v_1) \end{aligned}$$

6.3 Collision and Impact

An important area of application of the conservation laws is the study of the collisions of various physical bodies. In many cases, it is hard to assess how exactly the colliding bodies interact with each other. However, in a closed system, the conservation laws often allow one to obtain the information about many important properties of the collision without going into the complicated details of the collision dynamics. The collision is of two types.

6.3.1 Elastic Collision

A collision between particles in which the total kinetic energy of the particles remains unchanged is called elastic collision.

6.3.2 Inelastic Collision

A collision where the total kinetic energy of the particles is not conserved is called inelastic collision. One or more of the particles may also be deformed after the collision.

6.4 Impulsive Forces

An impulsive force is a very great force acting for a very short time on a body, so that the change in the position of the body during the time the force acts on it may be neglected. Consider the relation (6.2.3)

$$\vec{I} = \Delta\vec{p} = \int_{t_1}^{t_2} \vec{F} dt$$

it follows that the net external force is responsible for change in momentum. When a collision (or crash) occur, the external force on the body has large magnitude, and suddenly changes the momentum of the body.
In many situation we do not know how the force varies with time but we do know the average magnitude $\vec{F}_{avg}$ of the force and the duration $\Delta t (= t_2 - t_1)$ of the collision. Thus we can write the magnitude of the impulse as

$$\vec{I} = \vec{F}_{avg} \Delta t \tag{6.4.1}$$

When the force $\vec{F}$ grows very large ($F \to \infty$) during very small interval of time ($t_2 - t_1 = \Delta t \to 0$), then

$$\vec{I} = \lim_{F\to\infty} \int_{t_1}^{t_2} \vec{F} dt = l \ (finite)$$

Then such forces are called impulsive forces. Their measurement as force in impracticable (not measurable) but one can measure the momentum change they produce.

Examples: The blow of a hammer, a bat hits a cricket ball, the collision of two billiard balls etc.

Example 6.4.1. *When a male bighorn sheep runs head-first into another male, the rate at which its speed drops to zero is dramatic. Figure (6.4) gives a typical graph of the*

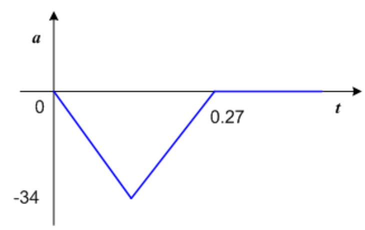

Figure 6.3: The acceleration versus time of a bighorn sheep during a collision with another male

acceleration a *versus time* t *for such a collision, with the acceleration taken as negative to correspond to an initially positive velocity. The peak acceleration has magnitude* $34m/s^2$ *and the duration of the collision is* $0.27s$. *Assuming that the sheep's mass is* $90kg$. *What are the magnitudes of the impulse and average force due to the collision?*

Solution: Since the impulse (magnitude) of the force F (magnitude) is the area between the curve and t axis. We can reform the given Figure (6.4) as:

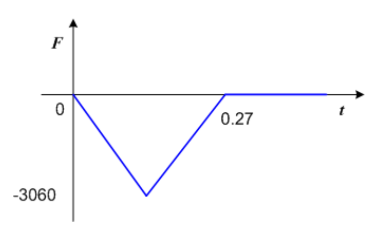

Figure 6.4: The force versus time of a bighorn sheep during a collision with another male

$$\begin{aligned} I &= area = \frac{1}{2}F \cdot t = \frac{1}{2}ma \cdot t \\ &= \frac{1}{2}(90)(34)(0.27) \cdot t \\ &= 413N.s \end{aligned}$$

For the magnitude of average force, from Eq. (6.4.2), we can write

$$\begin{aligned} F_{avg} &= \frac{I}{\Delta t} = \frac{413}{0.27} \\ &= 1500N \end{aligned}$$

6.4.1 Elastic Collision in One Dimension

Consider two smooth, non-rotating balls of masses m_1 and m_2. Let both are moving in the same directions with initial velocities (before impact) u_1 and u_2 respectively. After some time they collide and after collision they move in the same direction. Let m_1 moves with velocity v_1 and m_2 with velocity v_2. By law of conservation of linear momentum, we have

$$\begin{aligned} \text{momentum before collision} &= \text{momentum after collision} \\ m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2 \\ m_1\left(u_1 - v_1\right) &= m_2\left(v_2 - u_2\right) \end{aligned} \tag{6.4.2}$$

As the collision is elastic, the kinetic energy of the system is conserved. Hence by law of conservation of kinetic energy, we have

$$\begin{aligned} \text{kinetic energy before collision} &= \text{kinetic energy after collision} \\ \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \\ m_1\left(u_1^2 - v_1^2\right) &= m_2\left(v_2^2 - u_2^2\right) \end{aligned} \tag{6.4.3}$$

Dividing (6.4.3) by (6.4.2), we have

$$u_1 + v_1 = v_2 + u_2 \tag{6.4.4}$$

or

$$u_1 - u_2 = v_2 - v_1 = -\left(v_1 - v_2\right) \tag{6.4.5}$$

We note that before collision $(u_1 - u_2)$ is the velocity of the first ball relative to the second ball and $(v_1 - v_2)$ is the velocity of the first ball relative to the second ball after collision. (6.4.5) shows that these relative velocities have same magnitude but the order is reversed after the collision.

If all the data before collision (m_1, m_2, u_1 and u_2) is known, the velocities after collision (v_1 and v_2)can be calculated using (6.4.2) and (6.4.4).

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2} u_1 + \frac{2m_2}{m_1 + m_2} u_2 \tag{6.4.6}$$
$$v_2 = \frac{2m_1}{m_1 + m_2} u_1 + \frac{m_2 - m_1}{m_1 + m_2} u_2 \tag{6.4.7}$$

The velocities after collision also depend on the masses of both balls. Some cases arise depending on the nature of masses as follows.

case 1 If $m_1 = m_2$ then (6.4.6) and (6.4.7) gives the velocities after collision as

$$v_1 = u_2$$
$$v_2 = u_1$$

case 2 If $m_1 = m_2$ and $u_2 = 0$ then (6.4.6) and (6.4.7) gives the velocities after collision as

$$v_1 = 0$$
$$v_2 = u_1$$

case 3 If $m_1 \neq m_2$ and $u_2 = 0$ then (6.4.6) and (6.4.7) gives the velocities after collision as

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2} u_1$$
$$v_2 = \frac{2m_1}{m_1 + m_2} u_1$$

case 4 If m_1 is very very small as compared to m_2 $i.e$ $m_1 \ll m_2$ and $u_2 = 0$ then (6.4.6) and (6.4.7) gives the velocities after collision as

$$v_1 = u_1$$
$$v_2 = 0$$

case 5 If m_1 is very very large as compared to m_2 $i.e$ $m_1 \gg m_2$ and $u_2 = 0$ then (6.4.6) and (6.4.7) gives the velocities after collision as

$$v_1 = u_1$$
$$v_2 = 2\ u_1$$

Example 6.4.2. *Consider a ball of mass* $20\ g$ *is moving to the right with a velocity of* $9\ m/sec$ *and it collides with ball of mass* $1\ kg$ *which is at rest. Assume the collision was perfectly elastic. Find the velocity of each ball after collision.*

Solution The given data is

$$\begin{aligned} m_1 &= 20\ g = 0.02\ kg \\ m_2 &= 1\ kg \\ u_1 &= 9\ m/sec \\ u_2 &= 0 \end{aligned}$$

The problem is similar to case 3, then the velocity of first ball after collision is

$$\begin{aligned} v_1 &= \frac{m_1 - m_2}{m_1 + m_2} u_1 \\ &= \frac{0.02 - 1}{1 + 0.02} 9 \\ &= -8.65\ m/sec \end{aligned}$$

The first ball after collision moves to the left with a velocity of $8.65\ m/sec$. Next the velocity of second ball after collision is

$$\begin{aligned} v_2 &= \frac{2m_1}{m_1 + m_2} u_1 \\ &= \frac{2(0.02)}{1 + 0.02} 9 \\ &= 0.35\ m/sec \end{aligned}$$

The second ball after collision moves to the right with a velocity of $0.35\ m/sec$.

6.5 Impact of Elastic Bodies

A collision between two bodies is said to be impact, if the bodies are in contact for a short interval of time and exert very large force on each other during this short period. On impact, the bodies deform first and then recover due to elastic properties and start moving with different velocities. The velocity with which they they separate depends not only on their velocity of approach but also on the shape, size, elastic property and line of impact. Here the velocity of the bodies during the short period of impact in not considered. Only the velocities of the colliding bodies before impact and after impact are considered. First some some technical terms.

6.5.1 Definitions

.

- **Line of Impact**: Common normal to the colliding surfaces is known as line of impact.
- **Direct Impact**: If the motion of the two colliding bodies is directed along the line of impact
- **Oblique Impact**: If the motion of one or both of the colliding bodies is not directed along the line of impact.
- **Central Impact**: If the mass centres of colliding bodies are on the line of impact.
- **Eccentric Impact**: Even if mass centre of one of the colliding bodies is not on the line of impact.

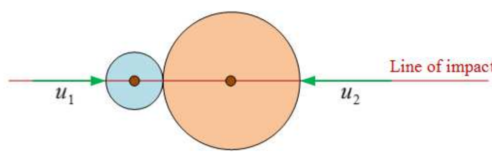

Figure 6.5: Direct Central Impact

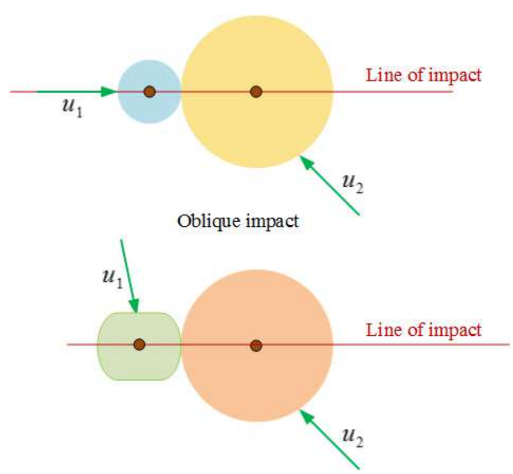

Figure 6.6: Oblique Central Impact

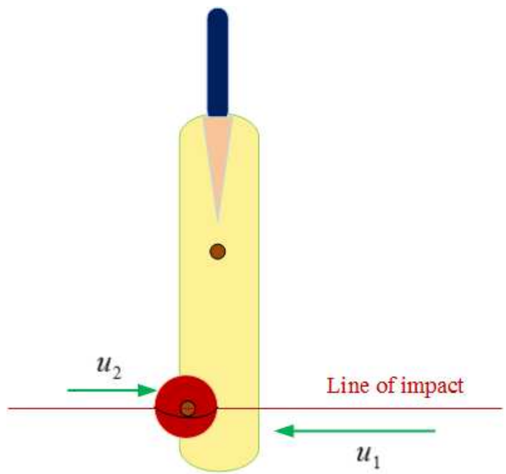

Figure 6.7: Direct Eccentric Impact

During the collision, the colliding bodies initially undergo a deformation for a small time interval and then recover the deformation in a further small time interval. So, the period of collision (or time of impact) consists of two time intervals.

1. Period of Deformation

2. Period of Restitution

- **Period of Deformation** is the time elapse between the instant of the initial contact and the instant of maximum deformation of the bodies.
- **Period of Restitution** is the time elapse between the instant of the maximum deformation condition and the instant of separation of the bodies. Thus,impulse during deformation $= F_D dt$
where F_D is the force that acts during the period of deformation. The magnitude of F_D varies from zero at the instant initial contact to the maximum value at the instant of maximum deformation.
Similarly, impulse during deformation $= F_R dt$ where F_R is the force that acts during the period of deformation. The magnitude of F_R varies from A maximum at the instant of maximum deformation condition to zero at the instant of just separation of the bodies.

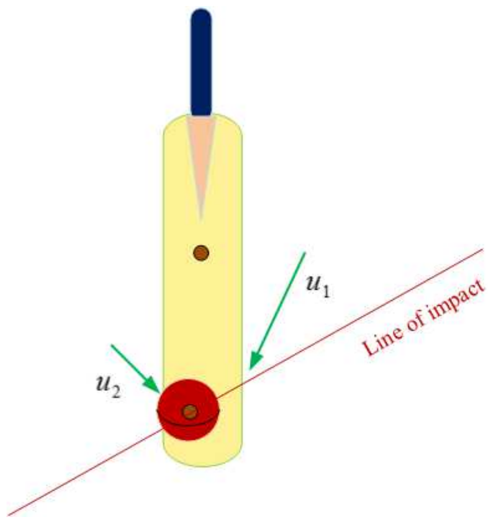

Figure 6.8: Oblique Eccentric Impact

6.6 Coefficient of Restitution.

Let
m_1 - mass of the first body
m_2 - mass of the second body
u_1 - velocity of the first body before impact
u_2 - velocity of the second body before impact
v_1 - velocity of the first body after impact
v_2 - velocity of the second body after impact
At the instant of maximum deformation, the colliding bodies will have same velocity. Let the velocity of the bodies at the instant of maximum deformation be $u_{D\ max}$.
Applying Impulse-Momentum principle for the first body

$$F_D dt = m_1 (u_{D\ max} - u_1) \tag{6.6.1}$$
$$F_R dt = m_1 (v_1 - u_{D\ max}) \tag{6.6.2}$$

Dividing (6.6.2) BY (6.6.2)

$$\frac{F_R dt}{F_D dt} = \frac{v_1 - u_{D\ max}}{u_{D\ max} - u_1} \tag{6.6.3}$$

Similarly the analysis for the second body gives,

$$\frac{F_R dt}{F_D dt} = \frac{u_{D\ max} - v_2}{u_2 - u_{D\ max}} \tag{6.6.4}$$

From (6.6.3) and (6.6.4)

$$\begin{aligned}
\frac{F_R dt}{F_D dt} &= \frac{v_1 - u_{D\ max}}{u_{D\ max} - u_1} = \frac{u_{D\ max} - v_2}{u_2 - u_{D\ max}} \\
&= \frac{v_1 - u_{D\ max} + u_{D\ max} - v_2}{u_{D\ max} - u_1 + u_2 - u_{D\ max}} \\
&= \frac{v_1 - v_2}{u_2 - u_1} = \frac{v_2 - v_1}{u_1 - u_2} \qquad (6.6.5) \\
&= \frac{\text{Relative velocity of separation}}{\text{Relative velocity of approach}}
\end{aligned}$$

Sir Isaac Newton conducted the experiments and observed that when collision of two bodies takes place relative velocity of separation bears a constant ratio to the relative velocity of approach, the relative velocities being measured along the line of impact. This constant ratio is called as the coefficient of restitution and is denoted by the letter e or C_R. Hence from (6.6.5), we have

$$e = C_R = \frac{F_R dt}{F_D dt} = \frac{v_1 - v_2}{u_2 - u_1} \qquad (6.6.6)$$

For perfectly elastic bodies, the magnitude of relative velocity after impact will be same as that before impact and hence the coefficient of restitution will be 1. Perfectly inelastic bodies cling together and hence the velocity of separation will be zero, and so the coefficient of restitution will also be 0. Hence the coefficient of restitution always lies between 0 and 1.
The value of coefficient of restitution depends not only on the material property but it also depends on the shape and size of the body. Hence the coefficient of restitution is the property of two colliding bodies but not merely of material of the colliding bodies.

Example 6.6.1. *Consider a $20N$ body is moving to the right with a velocity of $8m/sec$ and a $10N$ body moving to the left with a velocity of $12m/sec$. Direct central impact occurs between them (See Fig. 6.11). Find the velocity of each body after impact if the coefficient of restitution is 0.6.*

Solution

$$\begin{aligned}
\text{velocity of 20N body before impact} &= u_1 = 8\ m/sec \\
\text{velocity of 10N body before impact} &= u_2 = -12\ m/sec \\
\text{velocity of 20N body after impact} &= v_1\ m/sec \\
\text{velocity of 10N body after impact} &= v_2\ m/sec
\end{aligned}$$

Applying the principles of conservation of momentum to the colliding bodies, we get

$$\begin{aligned}
m_1 u_1 + m_2 u_2 &= m_1 v_1 + m_2 v_2 \\
\frac{20}{g}(8) + \frac{10}{g}(-12) &= \frac{20}{g} v_1 + \frac{10}{g} v_2
\end{aligned}$$

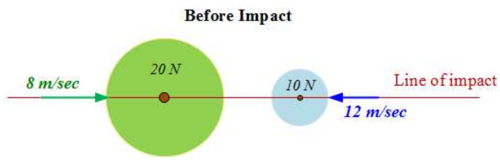

Figure 6.9: Direct Central Impact

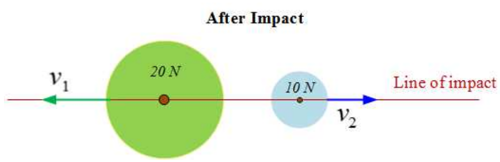

Figure 6.10: Direct Central Impact

or we have

$$2v_1 + v_2 = 4 \tag{6.6.7}$$

From (6.6.6), we have

$$\begin{aligned} e\,(u_1 - u_2) &= v_2 - v_1 \\ 0.6\,(8 - (-12)) &= v_2 - v_1 \\ v_2 - v_1 &= 12 \end{aligned} \tag{6.6.8}$$

From (6.6.11) and (6.6.8), we have

$$v_1 = -\frac{8}{3} = -2.67 \tag{6.6.9}$$

From (6.6.8), we can write

$$\begin{aligned} v_2 &= 12 + v_1 \\ v_2 &= 12 - 2.67 \\ &= 9.33\ m/sec \end{aligned} \tag{6.6.10}$$

After collision the $20N$ body is moving to the left with a velocity of 2.67 m/sec and a $10N$ body moving to the right with a velocity of 9.33 m/sec.

Example 6.6.2. *A body of* $20N$ *is moving to the right at a speed of* $7\ m/sec$ *and strikes a* $10N$ *body that is moving to the left at a speed of* $10\ m/sec$*. The final velocity of* $10N$ *body is* $4\ m/sec$ *to the right. Calculate the coefficient of restitution and final the velocity of the* $20N$ *body.*

Solution

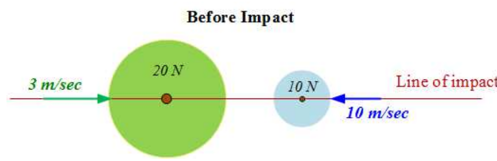

Figure 6.11: Direct Central Impact

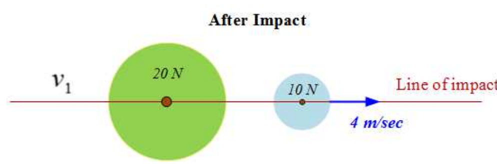

Figure 6.12: Direct Central Impact

$$\begin{aligned}
\text{velocity of 20N body before impact} &= u_1 = 7\ m/sec \\
\text{velocity of 10N body before impact} &= u_2 = -10\ m/sec \\
\text{velocity of 20N body after impact} &= v_1\ m/sec \\
\text{velocity of 10N body after impact} &= 4\ m/sec
\end{aligned}$$

Applying the principles of conservation of momentum to the colliding bodies, we get

$$\begin{aligned}
m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2 \\
\frac{20}{g}(7) + \frac{10}{g}(-10) &= \frac{20}{g}v_1 + \frac{10}{g}(4) \\
14 - 10 &= 2v_1 + 4
\end{aligned}$$

or we have

$$v_1 = 0 \qquad (6.6.11)$$

After collision, the $20N$ body comes to rest.
The coefficient of restitution is given by (6.6.6)

$$\begin{aligned} e &= \frac{v_1 - v_2}{u_2 - u_1} \\ &= \frac{0 - 4}{-10 - 7} \\ &= \frac{4}{17} = 0.235 \end{aligned} \tag{6.6.12}$$

Exercises

1. Consider a ball of mass 200 g is moving to the right with a velocity of 9 m/sec and it collides with ball of mass 80 g which is moving to the left with a velocity of 15 m/sec. Assume the collision was perfectly elastic. Find the velocity of each ball after collision.

2. A glass marble of weight $0.2N$ falls from a height of $10m$ and rebounds to a height of $8m$. Its time of contact with floor is $0.1s$. Find the impulse and the average force between the marble and the floor.

3. A $1N$ ball is bowled to a batsman. The velocity of the ball was $20m/s$ horizontally just before batsman hit it. After hitting it went away with a velocity of $48m/s$ at an inclination of 30^0 to the horizontal as shown in the Fig.(). Find the average force exerted on the ball by the bat if the impact lasts for 0.02 seconds.

4. An $80N$ body moving to the right at a speed of 3 m/sec strikes a $10N$ body that is moving to the left at a speed of 10 m/sec. The final velocity of $10N$ body is 4 m/sec to the right. Calculate the coefficient of restitution and final the velocity of the $80N$ body.

Chapter 7

Angular Momentum

7.1 Angular Momentum

Consider a particle of mass m is moving under a force $\vec{F}$. At time t the particle is at A, having position vector $\vec{r}$. The velocity $\vec{v}$ or linear momentum $\vec{p}$ of the particle is tangent to $\vec{r}$ at A. The angular momentum $\vec{L}$ of the particle is

$$\vec{L} = \vec{r} \times \vec{p} = rp \sin\theta \hat{n} \quad (7.1.1)$$

$$= \vec{r} \times m\vec{v} = mvr \sin\theta \hat{n} \quad (7.1.2)$$

Therefore, the angular momentum $\vec{L}$ is always perpendicular to the plane defined by the

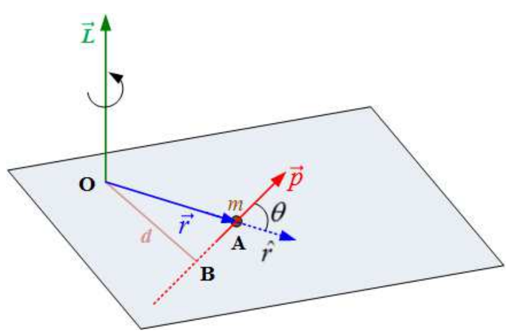

Figure 7.1: Angular Momentum.

particle's position vector $\vec{r}$ and velocity $\vec{v}$ as shown in Fig. 7.1. Note the magnitude and the direction of $\vec{L}$ depend on the choice of origin. The SI unit of angular momentum is

$kg - m^2/s$. If θ is the angle between $\vec{r}$ and $\vec{v}$, the magnitude of angular momentum is

$$\begin{aligned} L &= |\vec{r} \times \vec{p}| \\ &= rp \sin\theta = mvr \sin\theta \\ &= pd = mvd \end{aligned}$$

Where $d = r\sin\theta$ is the perpendicular distance of $\vec{v}$ from O.
Note:

- If $\vec{r}$ and $\vec{v}$ are parallel, the angular momentum is zero. In this case $\theta = 0$ or $\pi\ rad$. In other words, when the linear velocity of the particle is along a line that passes through the origin, the particle has zero angular momentum with respect to the origin.
- If $\vec{r}$ and $\vec{v}$ are perpendicular to each other then the angular momentum is $L = rp$. At that instant, the particle moves exactly as if it were on the rim of a wheel rotating about the origin in a plane defined by $\vec{r}$ and $\vec{p}$.

Example 7.1.1. *A particle of mass* 3 kg *moves in* xy *plane with a uniform velocity* $\vec{v} = 2m/s\ \hat{i} + 3m/s\ \hat{j}$. *At time* t, *the particle passes through the point* $P(3m,\ 2m)$ *as shown in Fig. 7.2. Find the magnitude and the direction of the angular momentum about the origin at time* t.

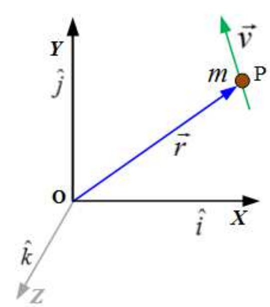

Figure 7.2: Angular momentum about origin in a plane

Solution: The given data is

$$\begin{aligned} m &= 3\ kg \\ \vec{v} &= 2m/s\ \hat{i} + 3m/s\ \hat{j} \\ P &= P(3m,\ 2m) \end{aligned}$$

The position vector of P is

$$\vec{r} \;=\; 3m\ \hat{i} + 2m\ \hat{j}$$

and linear momentum is

$$\vec{p} \;=\; m\vec{v} = 6N.s\ \hat{i} + 9N.s\ \hat{j}$$

Using (7.1.2), the angular momentum about O is

$$\begin{aligned}
\vec{L} &= \vec{r} \times \vec{p} \\
&= \left(3\ \hat{i} + 2\ \hat{j}\right) \times \left(6m/s\ \hat{i} + 9m/s\ \hat{j}\right) \\
&= 18\left(\hat{i} \times \hat{i}\right) + 27\left(\hat{i} \times \hat{j}\right) + 12\left(\hat{j} \times \hat{i}\right) + 18\left(\hat{j} \times \hat{j}\right)
\end{aligned}$$

Using $\hat{i} \times \hat{i} = 0 = \hat{j} \times \hat{j}, \quad \hat{i} \times \hat{j} = \hat{k}$ and $\hat{j} \times \hat{i} = -\hat{k}$, we have

$$\begin{aligned}
\vec{L} &= 27\hat{k} - 12\hat{k} \\
&= 15\ kg - m^2/s\ \hat{k}
\end{aligned}$$

The magnitude of the angular momentum about the origin at time t is 15 $kg - m^2/s$, and the direction is along z axis.

7.1.1 Angular Momentum and Uniform Circular Motion

Consider a particle of mass m moves with linear velocity is $v$in a circular path of radius r, in the xy plane as shown in Figure 7.3. In this motion $\vec{r}$ and $\vec{v}$ are perpendicular to each

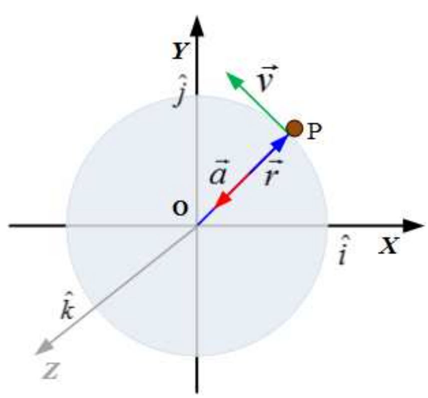

Figure 7.3: Angular momentum in uniform circular motion.

other, so $\theta = 90°$. Then the magnitude of angular momentum is

$$L = rp = rmv \tag{7.1.3}$$

Next $\vec{r}$ and $\vec{v}$ lies in xy plane. Let $\hat{i}$ be unit vector along $X - axis$ and $\hat{j}$ be unit vector along $Y - axis$. Then $\hat{k}$ be a unit vector perpendicular xy plane. Hence angular momentum has direction along $\hat{k}$.

Corollary 7.1.1. *A particle in uniform circular motion has a constant angular momentum about an axis through the center of its path.*

Proof: Since the particle is moving with uniform circular motion, so the linear momentum like velocity of the particle is always changing (in direction, not magnitude).

$$\frac{dv}{dt} = 0$$

In this case r is also fixed. Next take time derivative of (7.1.3)

$$\begin{aligned}\frac{dL}{dt} &= \frac{drp}{dt} = \frac{drmv}{dt} \\ &= 0\end{aligned}$$

Hence a particle in uniform circular motion has a constant angular momentum about an axis through the center of its path.

7.1.2 Angular Momentum of a System of n Particles

Consider a system of n particles. Let the ith particle with mass m_i is moving with linear momentum $\vec{p_i}$. At time t it is at point P_i having position vector $\vec{r_i}$, then its angular momentum is

$$\vec{L_i} = \vec{r_i} \times \vec{p_i}$$

Angular momentum of a system of particles is the sum of angular momentums all n particles. $i.e$

$$\vec{L} = \vec{L}_1 + \vec{L}_2 \ldots \vec{L}_n = \sum_{i=1}^{n} \vec{L}_i$$

7.1.3 Magnitude of Angular Momentum in Polar Coordinates

The angular momentum L of the particle is

$$\begin{aligned}\vec{L} &= \vec{r} \times \vec{p} \\ &= \vec{r} \times m\vec{v} = m(\vec{r} \times \vec{v})\end{aligned}$$

The velocity in polar coordinates is

$$v = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

The angular momentum in polar coordinates is

$$\begin{aligned}
\vec{L} &= m\left(r\hat{r} \times (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})\right) \\
&= m\left(r\dot{r}(\hat{r} \times \hat{r}) + r^2\dot{\theta}(\hat{r} \times \hat{\theta})\right) \\
&= m\left(0 + r^2\dot{\theta}\hat{k}\right) \\
&= m\left(r^2\dot{\theta}\right)\hat{k} \qquad (7.1.4)
\end{aligned}$$

The magnitude of angular momentum is

$$|\vec{L}| = m\left(r^2\dot{\theta}\right) \qquad (7.1.5)$$

The unit mass magnitude of angular momentum is

$$\frac{|\vec{L}|}{m} = r^2\dot{\theta} \qquad (7.1.6)$$

7.1.4 Law of Conservation of Angular Momentum

Theorem 7.1.2. *The total angular momentum of a system is constant in both magnitude and direction if the resultant external torque acting on the system is zero, that is, if the system is isolated.*

Proof: First we show that time rate of change of the angular momentum $\vec{L}$ equals the net torque.

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$$

From (7.1.2), the angular momentum and then its time derivative are

$$\begin{aligned}
\vec{L} &= \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \\
\frac{d\vec{L}}{dt} &= \dot{\vec{r}} \times m\vec{v} + \vec{r} \times m\dot{\vec{v}} \\
&= \dot{\vec{r}} \times m\dot{\vec{r}} + \vec{r} \times m\ddot{\vec{r}} \\
&= 0 + \vec{r} \times \vec{F} \\
&= \vec{\tau}
\end{aligned}$$

Hence time rate of change of the angular momentum $\vec{L}$ equals the net torque.
if the resultant external torque acting on the system is zero

$$\frac{d\vec{L}}{dt} = 0$$

Hence, the total angular momentum of a system is constant in both magnitude and direction if the resultant external torque acting on the system is zero.

Exercises

1. A particle of mass 2 kg moves in xy plane with a uniform velocity $\vec{v} = 2.4 m/s\ \hat{i} + 3.3 m/s\ \hat{j}$. At time t, the particle passes through the point $P(3m,\ 4m)$. Find the magnitude and the direction of the angular momentum about the origin at time t.

Chapter 8

Work Energy and Conservative Force

Work and energy are the same thing. Energy cant be created or destroyed, it can only be changed from one type into another type. When a force is applied on an object and it moves a distance we say that work has been done and energy has been transformed (changed from one type to another type).

8.1 Work

Consider a regular trihedral system with O as origin. Let a particle of mass m is moving under a force $\vec{F}$ along a curve C. Let at time t it be at point P, with position vector $\vec{r}$. After a very small time interval Δt it moved an infinitesimal displacement $\vec{dr}$ and is at point Q as shown in Fig. 8.1. Then the work done by a force $\vec{F}$ in taking the particle from point P to point Q along the curve C in an infinitesimal displacement $\vec{dr}$ is the dot product of $\vec{F}$ and $\vec{dr}$. Hence

$$dW = \vec{F} \cdot \vec{dr} \tag{8.1.1}$$

Also the total work done in moving from A to B is

$$W = \int_A^B \vec{F} \cdot \vec{dr} \tag{8.1.2}$$

If θ is an angle between $\vec{F}$ and $\vec{dr}$, then (8.1.1) can be written as

$$dW = F dr \cos\theta$$

is the general expression for work done by a force. The expression may be rearranged as

$$dW = F \cos\theta dr \tag{8.1.3}$$

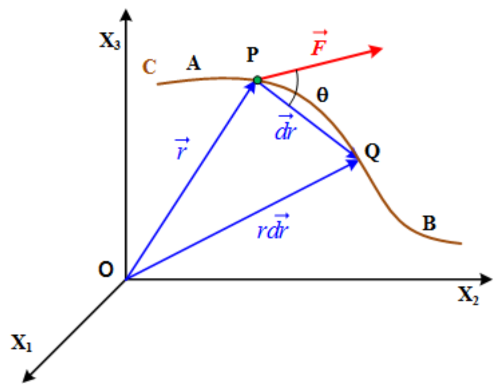

Figure 8.1: Work done

gives a new definition of work done. The work done by a force is defined as the product of component of force in the direction of motion and the distance moved.

8.1.1 Work done by a Constant Force

If $\vec{F}$ is constant and $\vec{AB} = \vec{r}_B - \vec{r}_A = \vec{r}$

$$\begin{aligned} W &= \vec{F} \cdot \int_{r_A}^{r_B} \vec{dr} \\ &= \vec{F} \cdot (\vec{r}_A - \vec{r}_B) \\ &= \vec{F}.\vec{r} \\ Net\,Work &= Net\,Force \cdot displacement \end{aligned}$$

If θ is an angle between $\vec{F}$ and $\vec{r}$ then the work done is

$$W = Fr\cos\theta$$

or

$$W = F\cos\theta r$$

Its unit in SI is Joule (J) or $N.m$. Note the work is done only if an object moves in the direction of F.

Example 8.1.1. *A crate is pulled for a distance of* 6 *m along a floor with a horizontal force of* 5 *N. Find the work done by the force.*

Solution The given data is:

$$\begin{aligned} F &= 5\ N \\ d &= 6\ m \end{aligned}$$

Here the force and the displacement are in the same direction, so the angle between them is $\theta = 0$. Hence the work done is just the product of force and distance. *i.e*

$$\begin{aligned} W &= Fd \\ &= 5(6) = 30\ N \cdot m \end{aligned}$$

8.2 Energy

Energy is defined as the capacity to do work. It is non-material property capable of causing changes in matter. In dynamics, we deal with mechanical energy which is of two types, namely kinetic and potential energy.

8.2.1 Kinetic Energy

Energy of an object due to its motion is called kinetic energy. It is the amount of work done by a force in bringing a moving particle to rest from its existing position. It is denoted by T.

Consider a regular trihedral system with O as origin. Let a particle of mass m is moving with velocity $\vec{v}$, under the a force $\vec{F}$ along a curve C. Let at time t it be at point P, with position vector $\vec{r}$. Then the work done by a force $\vec{F}$ in taking the particle from point P to Q (rest) along the curve C is:

$$T = W = \int_P^Q \vec{F} \cdot \vec{dr}$$

If the applied force $\vec{F} = m\vec{a}$, then

$$T = \int_P^Q m\vec{a} \cdot d\vec{r} \tag{8.2.1}$$

The acceleration is

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt} \tag{8.2.2}$$

and the velocity of the particle is

$$\vec{v} = \frac{d\vec{r}}{dt}$$

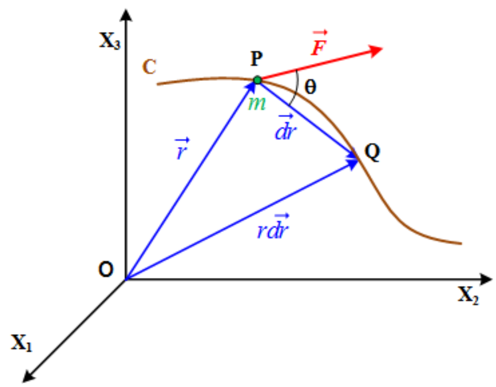

Figure 8.2: Work done

or

$$d\vec{r} = \vec{v}dt \tag{8.2.3}$$

Using (8.2.2) and (8.2.3), (8.2.1) becomes

$$\begin{aligned} T &= \int_P^Q m\frac{d\vec{v}}{dt}\cdot\vec{v}dt \\ &= \int_P^Q \frac{1}{2}m\frac{dv^2}{dt}dt \\ &= \int_P^Q \frac{d}{dt}\left(\frac{1}{2}mv^2\right)dt \\ &= \frac{1}{2}mv^2 \end{aligned} \tag{8.2.4}$$

(8.2.4) is an expression for kinetic energy. In SI, it is measured in Joules (J).

8.2.2 Kinetic Energy in terms of Work

Kinetic energy is the amount of work done by a force in bringing a moving particle to rest from its existing position. If m is the mass of the particle, then by Newton's second law of motion, the applied force is $\vec{F} = m\vec{a}$. Using (8.2.2) and (8.2.3), (8.1.1) can be written as

$$dW = m\frac{dv}{dt}\cdot vdt$$

$$\begin{aligned} dW &= mvdv \\ &= d\left(\frac{1}{2}mv^2\right) \end{aligned} \quad (8.2.5)$$

Now the total work done from A to B is

$$\begin{aligned} W_{AB} &= \int_A^B Fdr \\ &= \int_A^B d\left(\frac{1}{2}mv^2\right) \\ &= \left(\frac{1}{2}mv^2\right)_A^B \\ &= \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 \end{aligned}$$

The quantity $T = \frac{1}{2}mv^2$ is the kinetic energy. Hence the work done is

$$\begin{aligned} W_{AB} &= T_B - T_A \\ &= \Delta T \end{aligned} \quad (8.2.6)$$

We can postulate some results as under:

(a) If $T_A > T_B$ then $W_{AB} < 0$
The work is done by the particle against the force and its kinetic energy has decreased.

(b) If $T_A < T_B$ then $W_{AB} > 0$
The work is done by the force on the particle and its kinetic energy has increased.

In any case the work done depends upon the difference in kinetic energies of the particle in the two positions. The work done against the dissipative force like the fractional force is always negative.

8.2.3 Potential Energy

Potential energy is energy of position. The amount of potential energy possessed by an object is proportional to how far it was displaced from its original position. If the displacement occurs vertically, raising an object off of the ground, is known as gravitational potential energy. It is denoted by U. If m is the mass of the object raised a height h from the ground as shown in Fig. 8.3, then gravitational potential energy of the object is

$$\begin{aligned} \text{gravitational potential energy} &= \text{weight} \times \text{height} \\ U &= mgh \end{aligned}$$

The concept of potential energy can be used when dealing with conservative force that will be discussed later.

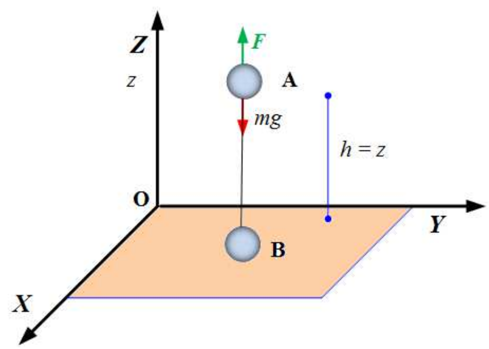

Figure 8.3: Potential energy

8.2.4 Potential Energy is converted to Kinetic Energy and vice-versa

Consider Fig. 8.1. Let $\vec{v}_i$ be the velocity of the particle at P and $\vec{v}_f$ be at P. Considering xy plane as the zero level and hight above it is the distance, that is ($d = h$ height). Its equation of motion can be written as

$$\begin{aligned} v_f^2 - v_i^2 &= 2gh \\ \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 &= mgh \\ \Delta T &= \Delta U \end{aligned}$$

Note only changes in potential energy can be measured. Total amount of energy at any instant cannot be determined.
At ground level all energy is kinetic energy and at maximum height h all energy is potential energy.

8.3 Power

Rate of doing work by a force $\vec{F}$ is called power or activity.

$$\begin{aligned} dP &= \frac{dW}{dt} \\ &= \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v} \end{aligned} \tag{8.3.1}$$

Also the power is defined as the rate at which energy is transferred by a force $\vec{F}$. Consider

$$\begin{aligned}
\frac{dT}{dt} &= \frac{d}{dt}\left(\frac{1}{2}mv^2\right) \\
&= \frac{1}{2}m\frac{d}{dt}\left(\vec{v}\cdot\vec{v}\right) = m\frac{d\vec{v}}{dt}\cdot\vec{v} \\
&= \vec{F}\cdot\vec{v} \\
&= P
\end{aligned}$$

It is measured in Watts $(W)\rightarrow$ 1 Joule of energy transferred in 1 second
We usually measure it in kW (kilowatts)

8.3.1 Efficiency

Ratio of output work to input work of a machine

$$Efficiency = \frac{W_{output}}{W_{input}} \times 100$$

8.4 Work done by a Variable Force

Consider a body moves under the influence of a force $\vec{F}(t)$. Suppose that the body moves a displacement $d\vec{r}(t)$ between time t_1 and t_2. Then the work done by the force is

$$W = \int_C \vec{F}\cdot d\vec{r}$$

As F and r are functions of t, hence the work done is

$$W = \int_{t_1}^{t_2} \vec{F}\cdot d\vec{r}dt$$

That is, work is the path integral of the force along the trajectory. Work may be either positive or negative, where in the latter case we will say that it is the body that has performed work.

8.5 Conservative Force

If the force field acting on a physical body is such that the work done along a closed path is zero, the force is called conservative. In other words we can say that a vector field is

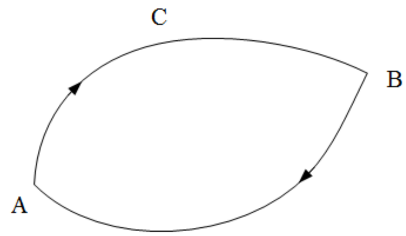

Figure 8.4: Work done

called conservative if integrals along paths only depend on the end points and not on the trajectory.

Consider a particle is moving along a curve C under the action of a variable force F. Let the particle moves from A to B and then from B to A, forming a closed path. If the total work done is zero, the acting force is conservative. *i.e.*

$$W = \int_C F.dr = 0 \tag{8.5.1}$$

If the force F is uniquely defined at every point of a region of space, the set of all such forces is a called a force field. If at every point of the space, the force F is conservative, then the force field is said to be conservative.
It is not always true that the work done by an external force is stored as some form of potential energy. This is only true if the force is conservative.
Examples: the force of gravity and the spring force are conservative forces.
For a non-conservative (or dissipative) force, the work done in going from A to B depends on the path taken.
Examples: friction and air resistance.

8.6 Examples of conservative and Non Conservative Force Field

In this section we will give some examples of conservative and non conservative systems.

8.6.1 The Earth's Gravitational Field is Conservative

The zero level of the potential energy is arbitrary; it can be assigned to any position. If the xy plane is chosen the zero level of potential energy , the potential energy at any point A is the work done by the force when the body moves from the point A to the point B with zero potential energy. So,

$$U(A) = W_{AB}$$

Consider a particle of mass m is initially at A, the gravitational force is $F = mg$. The work done by this force along the path AB is

$$W_{AB} = mg(h - 0) = mgh$$

Near Earths surface, the work done by gravity on an object of mass m depends only on

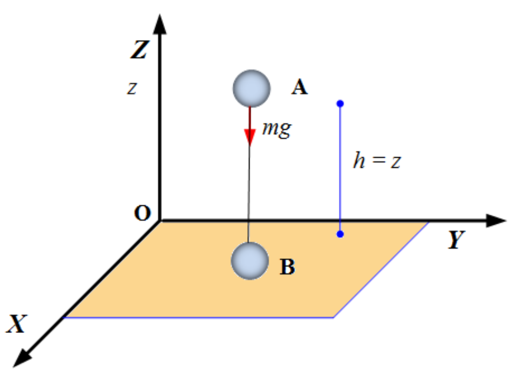

Figure 8.5: Work done

the change in the objects height h that depends on the end points of the path.
Vector Approach
Consider a particle of mass m is initially at A, the gravitational force always acts downward, having only z component, so can be written as

$$\vec{F} = mg\hat{k} = \langle 0,\ 0,\ mg \rangle$$

and dr can be written as

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k} = \langle dx,\ dy,\ dz \rangle$$

The work done by this force along the path AB is

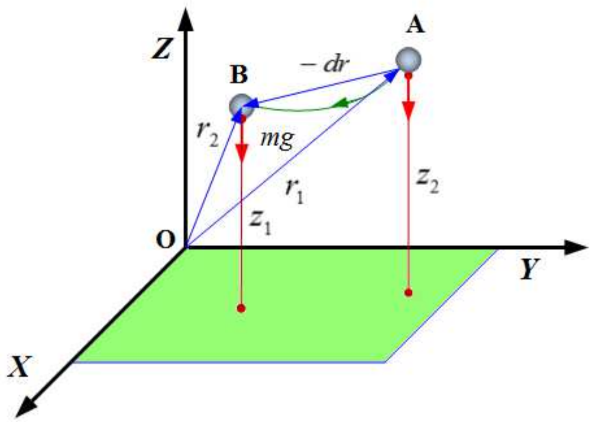

Figure 8.6: Work done

$$\begin{aligned}
W &= -\int_A^B \vec{F}.d\vec{r} \\
&= \int_B^A \langle 0,\ 0,\ mg\rangle.\langle dx,\ dy,\ dz\rangle \\
&= mg\int_B^A dz \\
&= mg.z\Big|_B^A \\
&= mg.\left(z_A - z_B\right) \\
&= mg.\left(z_2 - z_1\right)
\end{aligned}$$

Here the work done depends upon the initial and final positions of the particle, and is independent of the path. Hence the force is a conservative force and the earth's gradational field is conservative.

2.

$$\vec{F} = k\hat{s}$$

where k is some constant and $\hat{s}$ unit arc length. Also we can take $dr \equiv ds$. The work done by this force along the path AB is

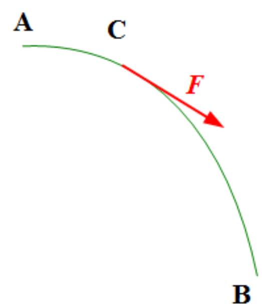

Figure 8.7: Work done

$$\begin{aligned} W &= \int_A^B k\hat{s}.dr = k\int_A^B \hat{s}.ds \\ &= k\int_A^B 1ds\cos\theta \\ &= k\cos\theta s\Big|_A^B \\ &= k\cos\theta\,(s_B - s_A) \end{aligned}$$

Here the work done depends upon the arc length of the path. Hence the force is not conservative force.

8.6.2 Potential Energy and Conservative Force

The potential energy of a particle in a field of force F is defined as the total work done in moving a particle from its existing position to its standard position (zero level of the potential energy) along the curve.
Let O be the origin of an inertia frame of reference fixed in space. Let P_0 be the position (standard) of a particle on a curve C and $P(t)$ be an arbitrary existing position of a particle at any time t. Let

$$\begin{aligned} \vec{OP_0} &= \vec{r_0} \\ \vec{OP} &= \vec{r} \end{aligned}$$

Analytically we can write, the expression for the potential energy is

$$\begin{aligned} U_{(P)} &= \int_r^{r_0} \vec{F}.\vec{dr} \\ &= -\int_{r_0}^{r} \vec{F}.\vec{dr} \end{aligned}$$

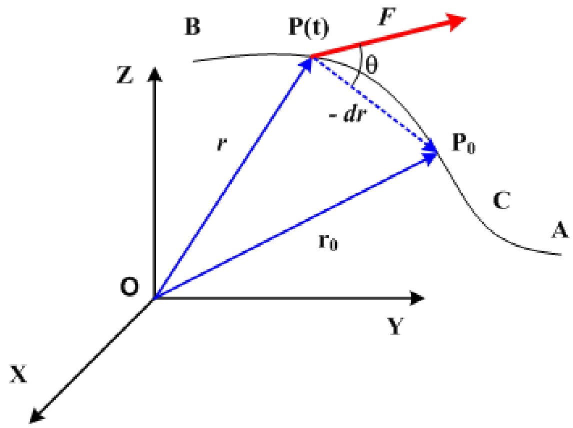

Figure 8.8: Work done

Theorem 8.6.1. *A vector field is conservative if and only if it is the gradient of a scalar field.*

$$\vec{F} = -\vec{\nabla}U \tag{8.6.1}$$

where $U(r)$ is called the potential field, or the potential energy; the negative sign is a convention whereby the force is directed in the direction of decreasing potential.

Proof Consider Fig. , we can write

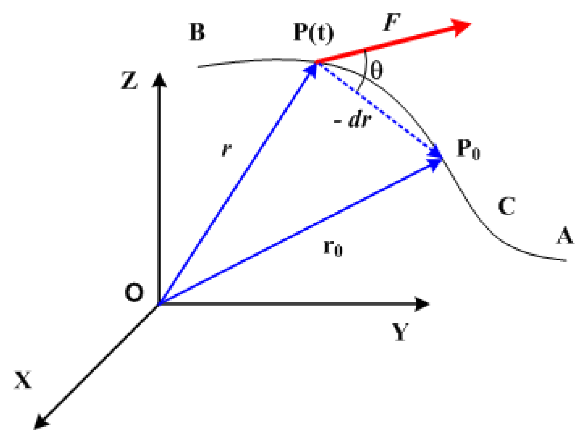

Figure 8.9: Work done

$$\begin{aligned}\vec{r} &= \langle x, y, z\rangle \\ \vec{r_0} &= \langle x_0, y_0, z_0\rangle \\ \vec{dr} &= \langle dx, dy, dz\rangle \\ \vec{F} &= \langle F_x, F_y, F_z\rangle \\ U &= U(x, y, z)\end{aligned}$$

Let P_0 be the zero level of potential energy, then U at P is

$$\begin{aligned}U_{(P)} &= \int_{(x,y,z)}^{(x_0,y_0,z_0)} \langle F_x, F_y, F_z\rangle.\langle dx, dy, dz\rangle \\ &= -\int_{(x_0,y_0,z_0)}^{(x,y,z)} (F_x dx + F_y dy + F_z dz) \\ &= -\int_{P_0}^{P} \vec{F}.\vec{dr} \qquad (8.6.2)\end{aligned}$$

differentiating we have

$$\begin{aligned}dU_{(P)} &= -d\left[\int_{(x_0,y_0,z_0)}^{(x,y,z)} (F_x dx + F_y dy + F_z dz)\right] \\ &= -(F_x dx + F_y dy + F_z dz) \\ dU(x, y, z) &= (-F_x dx - F_y dy - F_z dz) \\ \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz &= -F_x dx - F_y dy - F_z dz)\end{aligned}$$

$$\left(\frac{\partial U}{\partial x} + F_x\right) dx + \left(\frac{\partial U}{\partial y} + F_y\right) dy + \left(\frac{\partial U}{\partial z} + F_z\right) dz = 0 \tag{8.6.3}$$

Since x, y, and z are linearly independent, so dx, dy, and dz are also linearly independent. This implies that the coefficient of dx, dy, and dz must be equal to zero. *i.e.*

$$\begin{aligned}\frac{\partial U}{\partial x} + F_x &= 0\\ \frac{\partial U}{\partial y} + F_y &= 0\\ \frac{\partial U}{\partial z} + F_z &= 0\end{aligned}$$

or we have

$$\begin{aligned}F_x &= -\frac{\partial U}{\partial x}\\ F_y &= -\frac{\partial U}{\partial y}\\ F_z &= -\frac{\partial U}{\partial z}\end{aligned}$$

Hence $\vec{F}$ can be written as

$$\begin{aligned}\vec{F} &= \langle F_x, F_y, F_z\rangle = \left\langle -\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial z}\right\rangle\\ &= -\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle U(x, y, z)\\ &= -\nabla U\end{aligned}$$

where

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle$$

is an operator.

Conversely suppose that

$$\vec{F} = -\nabla U$$

The work done is

$$\begin{aligned} W &= \int_P^{P_0} F.dr \\ &= -\int_P^{P_0} \nabla U.dr \\ &= -\int_P^{P_0} \left\langle \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right\rangle .\langle dx, dy, dz\rangle \\ &= -\int_P^{P_0} \left(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz \right) \\ &= -\int_P^{P_0} dU = -U\Big|_P^{P_0} \qquad (8.6.4) \\ &= -U(P_0) + U(P) \\ &= \Delta U \end{aligned}$$

Hence the work done along a trajectory r connecting the points P and P_0, is independent of path, so the vector field of force F is conservative.

Theorem 8.6.2. *A necessary and sufficient condition for a vector field to be conservative is*

$$curl\vec{F} = \vec{O} \tag{8.6.5}$$

Proof Let $\vec{F}$ is conservative, Then there exist a function $U(x, y, z)$ of class C^2 (second order partial derivatives of U exist and are continuous) and $\vec{F}$ can be expressed as

$$\vec{F} = -\nabla U$$

Apply curl on both sides

$$\begin{aligned} curl\vec{F} &= -curl\ \nabla U \\ &= -\nabla \times \nabla U \end{aligned}$$

This cross product can be written as

$$\begin{aligned} curl\vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \end{vmatrix} \\ &= \left(\frac{\partial^2 U}{\partial y \partial Z} - \frac{\partial^2 U}{\partial z \partial y}\right)\hat{i} - \left(\frac{\partial^2 U}{\partial z \partial x} - \frac{\partial^2 U}{\partial x \partial z}\right)\hat{j} + \left(\frac{\partial^2 U}{\partial x \partial y} - \frac{\partial^2 U}{\partial y \partial x}\right)\hat{k} \end{aligned}$$

Since U is of class C^2, then $\frac{\partial^2 U}{\partial y \partial Z} = \frac{\partial^2 U}{\partial z \partial y}$ and all other pairs are so. Hence we have

$$\begin{aligned} curl\vec{F} &= \langle 0, 0, 0 \rangle \\ &= \vec{O} \end{aligned}$$

Conversely suppose that

$$curl\vec{F} = \vec{O}$$

Let

$$\vec{F} = \langle P, Q, R \rangle$$

be a force to do the work. Then

$$\begin{aligned} curl\vec{F} &= \vec{O} \\ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} &= \langle 0, 0, 0 \rangle \\ \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k} &= 0\hat{i} + 0\hat{j} + 0\hat{k} \end{aligned}$$

Since the two vectors are equal, this mean that their corresponding elements are equal. *i.e*

$$\begin{aligned}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) &= 0\\ \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) &= 0\\ \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) &= 0\end{aligned}$$

or we can write

$$\begin{aligned}\frac{\partial R}{\partial y} &= \frac{\partial Q}{\partial z}\\ \frac{\partial P}{\partial z} &= \frac{\partial R}{\partial x}\\ \frac{\partial Q}{\partial x} &= \frac{\partial P}{\partial y}\end{aligned}$$

which is possible only if there exist a function U of class C^2 such that

$$\vec{F} = -\vec{\nabla}U$$

Hence $\vec{F}$ is conservative.

Example 8.6.1. *The force*

$$\vec{F} = \langle x, y, z\rangle$$

is conservative. Also find the corresponding potential function.

Solution For a conservative force we need to show only

$$curl\vec{F} = \vec{O}$$

Next

$$\begin{aligned}curl\vec{F} &= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix}\\ &= \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\hat{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\hat{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\hat{k}\end{aligned}$$

Since $x,\ y,\ z$ are linearly independent, that means

$$\begin{aligned}\frac{\partial z}{\partial y} &= 0 = \frac{\partial y}{\partial z}\\ \frac{\partial x}{\partial z} &= 0 = \frac{\partial z}{\partial x}\\ \frac{\partial y}{\partial x} &= 0 = \frac{\partial x}{\partial y}\end{aligned}$$

Then

$$\begin{aligned}curl\vec{F} &= \langle 0,0,0\rangle\\ &= \vec{O}\end{aligned}$$

Hence the given force $\vec{F}$ is conservative and there exist a function U of class C^2 such that

$$\begin{aligned}\vec{F} &= -\vec{\nabla}U\\ \langle x,y,z\rangle &= \langle -\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial z}\rangle\end{aligned}$$

The two vectors are equal, so there corresponding entries are equal.

$$x = -\frac{\partial U}{\partial x} \quad \text{or} \quad -\frac{\partial U}{\partial x} = x \tag{8.6.6}$$

$$y = -\frac{\partial U}{\partial y} \quad \text{or} \quad -\frac{\partial U}{\partial y} = y \tag{8.6.7}$$

$$z = -\frac{\partial U}{\partial z} \quad \text{or} \quad -\frac{\partial U}{\partial z} = z \tag{8.6.8}$$

Partially integrate (8.6.6) with respect to x

$$\begin{aligned}-U &= \int x dx + f(y,z)\\ &= \frac{x^2}{2} + f(y,z)\end{aligned} \tag{8.6.9}$$

Partially differentiate (8.6.9) with respect to y

$$-\frac{\partial U}{\partial y} = \frac{\partial f}{\partial y} \tag{8.6.10}$$

Using (8.6.10) in (8.6.7)

$$\frac{\partial f}{\partial y} = y \tag{8.6.11}$$

Partially integrate (8.6.11) with respect to y

$$f = \frac{x^2}{2} + g(z) \tag{8.6.12}$$

Then (8.6.9) becomes

$$-U = \frac{x^2}{2} + \frac{y^2}{2} + g(z)$$

Partially differentiate (8.6.13) with respect to z

$$-\frac{\partial U}{\partial z} = \frac{dg}{dz} \tag{8.6.13}$$

From (8.6.8) and (8.6.13), we can write

$$\frac{dg}{dz} = z \tag{8.6.14}$$

Integrating (8.6.14)

$$g = \frac{z^2}{2} + c \tag{8.6.15}$$

Using (8.6.15) in (8.6.13) we have

$$-U = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + C$$

Ignoring C, the corresponding potential function is

$$U = -\frac{1}{2}\left(x^2 + y^2 + z^2\right) \tag{8.6.16}$$

From this potential function, the corresponding conservative force can be calculated as Let the force is

$$\vec{F} = \langle P, Q, R \rangle$$

$$\begin{aligned} \vec{F} &= -\nabla U \\ \langle P, Q, R \rangle &= \langle -\frac{\partial U}{\partial X}, -\frac{\partial U}{\partial Y}, -\frac{\partial U}{\partial z} \rangle \end{aligned}$$

From (8.6.16), we have

$$\begin{aligned} -\frac{\partial U}{\partial x} &= x \\ -\frac{\partial U}{\partial y} &= y \\ -\frac{\partial U}{\partial z} &= z \end{aligned}$$

Hence the corresponding conservative force is

$$\vec{F} = \langle x, y, z \rangle$$

Example 8.6.2. *A particle moves under the action of a force*

$$\vec{F} = \langle 3x^2 + 6xy, 3x^2 - 3y^2, 0 \rangle$$

from $A(1,1,0)$ *to* $B(2,3,0)$. *Then determine*

(a) *Is the force conservative?*

(b) *If yes, find the corresponding potential energy function.*

(c) *The work done from* A *to* B

Solution For a conservative force we need to show only

$$curl\vec{F} = \vec{O}$$

Next

$$\begin{aligned} curl\vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2+6xy & 3x^2-3y^2 & 0 \end{vmatrix} \\ &= \left(\frac{\partial(0)}{\partial y} - \frac{\partial\left(3x^2-3y^2\right)}{\partial z}\right)\hat{i} + \left(\frac{\partial\left(3x^2+6xy\right)}{\partial z} - \frac{\partial(0)}{\partial x}\right)\hat{j} \\ &+ \left(\frac{\partial\left(3x^2-3y^2\right)}{\partial x} - \frac{\partial\left(3x^2+6xy\right)}{\partial y}\right)\hat{k} \end{aligned}$$

Since $x,\ y,\ z$ are linearly independent, then

$$\begin{aligned} curl\vec{F} &= (0)\,\hat{i} + (0)\,\hat{j} + (6x-6x)\,\hat{k} \\ &= \langle 0,0,0 \rangle \\ &= \vec{O} \end{aligned}$$

Hence the given force $\vec{F}$ is conservative.

(b) Since the given force is conservative, then there exist a function U of class C^2 such that

$$\begin{aligned} \vec{F} &= -\vec{\nabla}U \\ \langle 3x^2+6xy, 3x^2-3y^2, 0 \rangle &= \langle -\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial z} \rangle \end{aligned}$$

The two vectors are equal, so there corresponding entries are equal.

$$3x^2 + 6xy = -\frac{\partial U}{\partial x} \quad \text{or} \quad -\frac{\partial U}{\partial x} = 3x^2 + 6xy \tag{8.6.17}$$

$$3x^2 - 3y^2 = -\frac{\partial U}{\partial y} \quad \text{or} \quad -\frac{\partial U}{\partial y} = 3x^2 - 3y^2 \tag{8.6.18}$$

$$0 = -\frac{\partial U}{\partial z} \quad \text{or} \quad -\frac{\partial U}{\partial z} = 0 \tag{8.6.19}$$

The given force is a two dimensional force, so we can ignore (8.6.19) Partially integrate (8.6.17) with respect to x

$$\begin{aligned} -U &= \int \left(3x^2 + 6xy\right) dx + f(y) \\ &= x^3 + 3x^2y + f(y) \end{aligned} \tag{8.6.20}$$

Partially differentiate (8.6.20) with respect to y

$$-\frac{\partial U}{\partial y} = 3x^2 + \frac{df}{dy} \tag{8.6.21}$$

Using (8.6.21) in (8.6.18)

$$\frac{df}{dy} = -3y^2 \tag{8.6.22}$$

Partially integrate (8.6.22) with respect to y

$$f = -y^3 + C \tag{8.6.23}$$

Then (8.6.20) becomes

$$-U = x^3 + 3x^2y - y^3 + C \tag{8.6.24}$$

Ignoring C, the corresponding potential function is

$$U = -x^3 - 3x^2y + y^3 \tag{8.6.25}$$

(c) Work done from A to B can be calculated by using (8.6.4)

$$W = -U\Big|_A^B$$

Using (8.6.25) the work done is

$$\begin{aligned} W &= \left|x^3 + 3x^2y - y^3\right|_{(1,1,0)}^{(2,3,0)} \\ &= [(8 + 36 - 27) - (1 + 3 - 1)] = [17 - 3] \\ &= 14\ J \end{aligned}$$

Example 8.6.3. *Examples of potential energy functions.*

1. For a mass under the influence of earth gravity
$$U(r) = U(z) = mgz$$
2. For a mass suspended on a spring,
$$U(z) = \frac{1}{2}kz^2$$
3. For a planet under the influence of a stars gravity,
$$U(r) = -\frac{GMm}{r^2}$$

8.7 Law of Conservation of Energy

StatementWithin a closed, isolated system, energy can change form, but the total amount of energy is constant

$$T_{initial} + U_{initial} = T_{final} + U_{final} \tag{8.7.1}$$

The sum of kinetic energy and potential energy represents the total mechanical energy.
Proof Consider a particle of mass m is moving under the influence of a conservative force field F. If the particle performs a trajectory $r(t)$ connecting the points $r(t_1)$ and $r(t_2)$ then from (8.6.4), we can write

$$\begin{aligned} -U(rt_2) + U(rt_1) &= \int_P^{P_0} \vec{F}.d\vec{r} \\ &= m\int_P^{P_0} \vec{a}.d\vec{r} \end{aligned} \tag{8.7.2}$$

Using (8.2.2) and (8.2.3), (8.7.2) becomes

$$\begin{aligned} -U(rt_2) + U(rt_1) &= \int_{t_1}^{t_2} \frac{d\vec{v}}{dt} \cdot \vec{v}dt \\ &= \int_{t_1}^{t_2} \frac{1}{2}m\frac{dv^2}{dt}dt \\ &= \int_{t_1}^{t_2} \frac{d}{dt}Kdt \\ &= T(v(t_2)) - T(v(t_1)) \\ \Delta U &= \Delta T \end{aligned} \tag{8.7.3}$$

It can also be written as

$$T(v(t_1)) + U(rt_1) \quad = \quad T(v(t_2)) + U(rt_2) \tag{8.7.4}$$

8.7.4 is known as the law of conservation of energy.
Defining the total mechanical energy, or simply the energy,

$$E(r, v) \quad = \quad U(r) + T(v), \tag{8.7.5}$$

we conclude that it assumes the same value at time t_1 and t_2, *i.e.*, it is conserved (it is a function of the trajectory whose value remains constant in time).

Theorem 8.7.1. *In a conservative vector field, the total energy (mechanical) is constant throughout the motion.*

Proof It is another to proof the law of conservation of energy.
It can be proved by showing

$$\frac{d}{dt}E(r, v) \quad = \quad 0$$

Taking time derivative of (8.7.5)

$$\begin{aligned} \frac{d}{dt}E(r, v) &= \frac{d}{dt}\big[U(r) + T(v)\big] \\ &= \frac{d}{dt}\big[U(r(t)) + T(v(t))\big] \end{aligned} \tag{8.7.6}$$

Now by chain rule, the first term on right hand side can be written as

$$\begin{aligned} \frac{d}{dt}\big[U(r(t))\big] &= \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt} \\ &= \vec{\nabla} U \cdot \vec{\dot{r}} \end{aligned}$$

Since $\vec{F}$ is conservative, then by (8.6.1), we can write

$$\frac{d}{dt}\big[U(r(t))\big] \quad = \quad -\vec{F} \cdot \vec{v} \tag{8.7.7}$$

Again by chain rule, the second term on right hand side can be written as

$$\begin{aligned} \frac{d}{dt}\big[T(v(t))\big] &= \frac{d}{dt}\big[\frac{1}{2}mv^2\big] \\ &= \frac{1}{2}m\frac{d}{dt}v^2 \\ &= m\frac{d\vec{v}}{dt} \cdot \vec{v} \\ &= \vec{F} \cdot \vec{v} \end{aligned} \tag{8.7.8}$$

Using (8.7.7) and (8.7.8), (8.7.6) becomes

$$\frac{d}{dt}E(r,v) = -\vec{F}\cdot\vec{v}+\vec{F}\cdot\vec{v}=0 \tag{8.7.9}$$

(8.7.4) and (8.7.9) represent the principle of conservation of energy.
Second Method We will show that the sum of kinetic and potential energies is constant. By Newton's second law of motion its equation of motion is

$$F = m\ddot{r} \tag{8.7.10}$$

Multiply (8.7.10) with $\dot{r}=\frac{dr}{dt}$,

$$m\dot{r}\ddot{r} = F\frac{dr}{dt} \tag{8.7.11}$$

Integrating (8.7.11) with respect to t

$$\frac{1}{2}m\dot{r}^2 = \int F\frac{dr}{dt}dt + constant$$

or we can write

$$\frac{1}{2}m\dot{r}^2 - \int F\cdot dr = constant \tag{8.7.12}$$

Since the force is conservative, so we have

$$\vec{F} = -\vec{\nabla U}$$

where $U(r)$ is the potential energy and can be written as

$$U = -\int F\cdot dr$$

and the term $\frac{1}{2}m\dot{r}^2$ is the kinetic energy of the system. Using these results, (8.7.12) becomes

$$T+U = constant \tag{8.7.13}$$

Hence the total energy of the system is conserved. The conservation of total mechanical energy when forces are conservative is useful as shows in the following examples.

Example 8.7.1. *A body is dropped (at rest) from a height of h meters. If the motion is free fall, show that the energy of the system is conserved.*

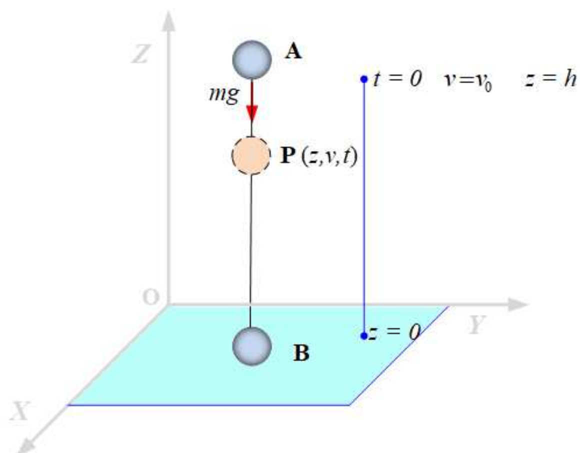

Figure 8.10: Downward motion

Solution We will show that the sum of kinetic and potential energies is constant. The particle is executing one dimensional motion and its motion is along $z-axis$. Let any time t the particle is at P as shown in the Fig 8.10. At P the kinetic energy is

$$T = \frac{1}{2}m\dot{z}^2$$

Taking ground (xy plane) as zero level for potential energy, then potential energy is

$$U = mgz$$

By Newton's second law of motion its equation of motion is

$$\begin{aligned} F &= -W \\ m\ddot{z} &= -mg \end{aligned} \tag{8.7.14}$$

Multiply (8.7.14) with $\dot{z}$,

$$m\dot{z}\ddot{z} + mg\dot{z} = 0 \tag{8.7.15}$$

(8.7.15) can be written as

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{z}^2 + mgz\right) = 0 \tag{8.7.16}$$

The term $\frac{1}{2}m\dot{z}^2$ is the kinetic energy and mgz is the potential energy of the system. Using these results, (8.7.16) becomes

$$\begin{aligned} \frac{d}{dt}(T+U) &= 0 \\ \frac{dE}{dt} &= 0 \end{aligned} \tag{8.7.17}$$

Integrating (8.7.17), we have

$$E = constant \tag{8.7.18}$$

Hence the total energy of the system is conserved.

Example 8.7.2. *A body is dropped (at rest) from a height of h meters. If the motion is free fall, use energy approach to find speed with which it will hit the ground.*

Solution As the body starts from rest, so the initial data is

$$\begin{aligned} t_0 &= 0 \\ v_0 &= 0 \\ z_0 &= h \end{aligned}$$

One way to solve it is via the equations of motion:
The other way of solving this exercise is with energies. Taking xy plane as zero level for potential energy. At P the potential energy is

$$U(z) = mgz$$

the conservation of energy implies that

$$U(z(0)) + \frac{mv^2(0)}{2} = U(z(t_1)) + \frac{mv^2(t_1)}{2}$$

i.e.,

$$\begin{aligned} mgh + 0 &= 0 + \frac{mv^2(t_1)}{2} \\ v(t_1) &= \sqrt{2hg}. \end{aligned}$$

which gives the exact same answer.

Example 8.7.3. *Earth does not perform any work on the moon because the trajectory of the moon is perpendicular to the vector that connects the moon to the earth, i.e., it is (approximately) perpendicular to the force of gravity.*

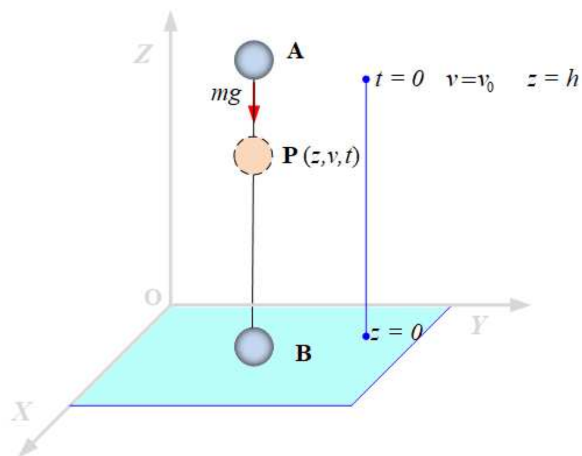

Figure 8.11: Downward motion

Remark 8.7.1. For a single particle with constant mass m, Newtons equations for the kinetic energy is given by (8.7.8)

$$\frac{d}{dt}\big(T(v(t))\big) \quad = \quad \vec{F} \cdot \vec{v}$$

and more generally, if the mass can vary, then

$$\frac{d}{dt}\big(mT(v(t))\big) \quad = \quad \vec{F} \cdot \vec{p}$$

Exercises

1. A force of magnitude $20N$ is applied on cart and it moves a distance of 10 m Find the work done if the angle between force and distance is as following: (a) $0°$, (b) $30°$ (c) $45°$ (d) $60°$ (e) $90°$, (f) $120°$ and (g) $180°$.

2. A force of 400 N is applied on a block to reach the top of 2 m long smooth ramp that is inclined at an angle of $30°$ with the ground (Figure 8.12). Let there are no frictional forces. Find the work done if the angle between force and ramp is as following: (a) $0°$, (b) $30°$ (c) $45°$ (d) $60°$ (e) $90°$, (f) $120°$ and (g) $135°$.

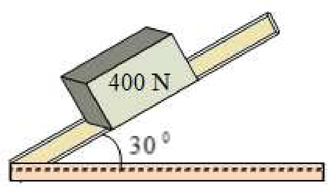

Figure 8.12: Block on inclined plane

3. A wagon is pulled horizontally by exerting a constant force of $50N$ on the handle at an angle of $60°$ with the horizontal. How much work is done in moving the wagon 10 m?

4. A force of $\vec{F} = \langle 3, -1, 2 \rangle$ N is applied to a point that moves on a line from $A(0, 1, -1)$ to $B(4, 1, 2)$. If distance is measured in feet, how much work is done?

5. Determine which of the following forces are conservative. If a force is conservative find the corresponding potential energy function.

   (a) $\vec{F} = x^2yz^3\hat{i} + 2xy^2 \sin z\hat{j} + 3ze^y\hat{k}$

   (b) $\vec{F} = x^2e^y\hat{i} + 2xe^y\hat{j} + 3\cos ye^x\hat{k}$

   (c) $\vec{F} = x^2y\hat{i} + 2xy^2z^3\hat{j} + 3ze^x\hat{k}$

6. A particle moves under the action of a force

$$\vec{F} = \langle y^2 \cos x + z^3, 2y \sin x - 4, 3xz^2 + 2 \rangle$$

from $A(0, 1, -1)$ to $B(\frac{\pi}{2}, -1, 2)$. Then determine

   (a) The force is conservative.

   (b) Find the corresponding potential energy function.

   (c) The work done from A to B

7. The escape velocity of a particle on earth is the minimum velocity required in order for a particle to escape from earths gravitational field. Use the conservation of energy to calculate the escape velocity. Could you obtain this result by directly solving Newtons equations, which for motion along the radial direction take the form,

$$m\frac{d^2r}{dt^2} = \frac{-GMm}{r^2}$$

with initial data $r(0) = R$ and $v(0) = v_0$, where R is the radius of earth.

Chapter 9

Virtual Displacement and Virtual Work

The concepts of virtual displacement and virtual work are very useful and are given next.

9.0.1 Virtual Displacement

A hypothetical displacement of a system in which the forces and constraints remain unchanged and which takes place during infinitesimal time interval is called virtual displacement. It is denoted by δr_i for the ith particle.
Note: During this displacement, the forces of constraints do not do work.

9.0.2 Real and Virtual Displacement

Let $\vec{r_i}$ be the position vector of the ith particle having generalized coordinates q_i at time t. Then

$$r_i = r_i(q_i, t) \tag{9.0.1}$$

and the quantity

$$dr_i = \frac{\partial r_i}{\partial q_i} dq_i + \frac{\partial r_i}{\partial t} dt \tag{9.0.2}$$

is called the real displacement. If t is fixed then $dt = 0$ and the quantity

$$\delta r_i = \frac{\partial r_i}{\partial q_i} \delta q_i \tag{9.0.3}$$

is called the virtual displacement.

9.0.3 Virtual Work

The work done by a force in virtual displacement.

Example 9.0.1. *A particle of mass m moves under the central force* $F = -\mu\frac{m}{r^2}$*, where* μ *is some constant. Find virtual work done.*

The particle moves in polar coordinates, so r and θ are the generalized coordinates. Then $r,\ \theta,\ \dot{r}$ and $\dot{\theta}$ are linearly independent. The force acting on the particle is

$$F = -\mu\frac{m}{r^2}$$

The generalized force F can be written in polar components as

$$\begin{aligned} F_r &= -\mu\frac{m}{r^2} \\ F_\theta &= 0 \end{aligned}$$

The virtual work done is

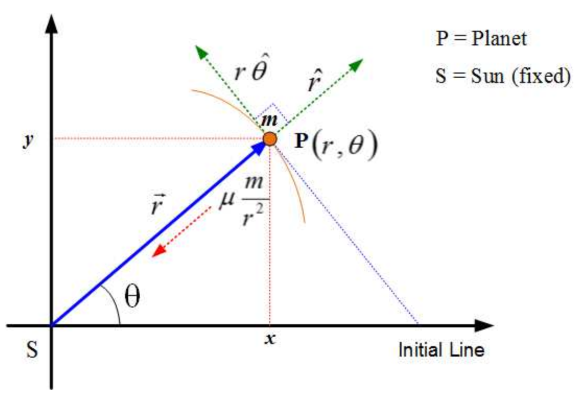

Figure 9.1: Polar motion

$$\begin{aligned} \delta W &= \sum_{i=1}^{2} F_i.\delta q_i \\ &= F_1.\delta q_1 + F_2.\delta q_2 \\ &= F_r.\delta r + F_\theta.\delta\theta \\ &= F_r.\delta r \end{aligned}$$

9.1 Workless Constraints

In the case of a displacement consistent with the constraints of a system, the forces of constraints generally do no work. Thus, for example, if a particle in contact with a smooth plane, is displaced along the plane, the reactive force, being perpendicular to the displacement, does zero work. Some of the usual workless constraints are described below:

1. Reaction of a smooth with which the body is in contact. For, in such a case, the reaction of the surface at a point of contact is entirely normal to the surface and, therefore, does zero work.

2. Reaction at a fixed point or fixed axis of a body.
   For in this case, displacement of the point of application of any reactive force is zero, and so the work done by it vanishes

3. Reaction at the point of contact of a fixed surface on which a body rolls without sliding. For, the point of contact is instantaneously at rest, so its displacement in the direction of the reaction is zero.

4. The mutual action and reaction of two bodies which roll over each other. The bodies, when regarded as a system, will exert equal and opposite forces on each other. The algebraic sum of the works done by such forces in any displacement, therefore, vanishes.

5. Tension of an inextensible string.

9.1.1 Principle of Virtual Work

The necessary and sufficient condition for a system of N particles to be in equilibrium is the total virtual work done by applied forces is zero.
Proof: Consider a system of N particles. Let Q_i be the force acting on the ith particle. Then

$$Q_i = F_i + f_i \tag{9.1.1}$$

Where F_i are external applied forces and f_i are constraint forces. If δr_i is the virtual displacement of the ith particle. Then the virtual work is

$$\delta W_i = (F_i + f_i) . \delta r_i \tag{9.1.2}$$

Let the system be in equilibrium, then

$$Q_i = 0 \quad \forall i \tag{9.1.3}$$
$$\Rightarrow \delta W_i = 0 \quad \forall i \tag{9.1.4}$$

And for the whole system

$$\begin{aligned}\sum_{i=1}^{N}\delta W_i &= \sum_{i=1}^{N} Q_i.\delta r_i = 0\\ &= \sum_{i=1}^{N}(F_i + f_i).\delta r_i = 0\\ &= \sum_{i=1}^{N} F_i.\delta r_i + \sum_{i=1}^{N} f_i.\delta r_i = 0\end{aligned}$$

Since the work done by the constraint forces is zero, we have

$$\sum_{i=1}^{N}\delta W_i = \sum_{i=1}^{N}(F_i.\delta r_i) = 0 \tag{9.1.5}$$

Conversely suppose that the total work done by applied forces is zero. Then

$$\sum_{i=1}^{N}\delta W_i = 0$$

If δr_i is the virtual displacement for the applied force F_i, then we have

$$\sum_{i=1}^{N} F_i.\delta r_i = 0$$

or

$$\sum_{i=1}^{N} F_i = 0$$

Hence the system is in equilibrium.

Chapter 10

Centers of Mass and Gravity

The concepts of centre of mass and center of gravity of a system are very useful in mechanics. Many of the important quantities are similarly simplified using the center of mass. In this chapter we shall state some basic definitions and some usual methods to compute them.
Homogeneous object: A material or a body is called homogeneous if its composition is uniform throughout and inhomogeneous otherwise. It means a homogeneous material has the same properties at every point of the space. For example, an object of uniform density is sometimes described as homogeneous.

10.1 Density of Homogeneous Material

Density is a very important measurement of a material. Objects of same material regardless of how big or small an object is, their density will be same. For example, a watch made of gold and a brick made of gold have different masses and volumes, but they have the same density. It is denoted by ρ. We can define it, keeping in mind the dimensions of the object.

10.1.1 Density of One Dimensional Object

The density of one dimensional homogeneous material is its mass per unit length. If a homogeneous body has mass m and length l, then its density is given by

$$\rho = \frac{m}{l} \qquad (10.1.1)$$

The mass m of a homogeneous body can be expressed as

$$m = \rho l \qquad (10.1.2)$$

10.1.2 Density of Two Dimensional Object

The density of two dimensional homogeneous material is its mass per unit area. If a homogeneous body has mass m and area A, then its density is given by

$$\rho = \frac{m}{A} \tag{10.1.3}$$

The mass m of a two dimensional homogeneous object can be expressed as

$$m = \rho A \tag{10.1.4}$$

The above definitions are just to understand the concept. For physical bodies, the following definition is applicable.

10.1.3 Density of Three Dimensional Object

The density of three dimensional homogeneous material is its mass per unit volume. If a homogeneous body has mass m and volume V, then its density is given by

$$\rho = \frac{m}{V} \tag{10.1.5}$$

The mass m of a three dimensional homogeneous object can be expressed as

$$m = \rho V \tag{10.1.6}$$

Example 10.1.1. *A table and a chair are made up of red cedar. If the mass and volume of table are* $49.9\ kg$ *and* $0.13\ m^3$ *respectively, and the mass and volume of chair are* $19.2\ kg$ *and* $0.05\ m^3$ *respectively. Is the density of both objects same?*

Solution: The given data is

$$\begin{aligned} m_t &= 49.9\ kg \\ V_t &= 0.13\ m^3 \\ m_c &= 19.2\ kg \\ V_c &= 0.05\ m^3 \end{aligned}$$

Using (10.1.5), the density of table is

$$\begin{aligned} \rho_t &= \frac{m}{V} = \frac{49.9}{0.13} \\ &= 384\ kg/m^3 \end{aligned}$$

and the density of chair is

$$\begin{aligned}\rho_c &= \frac{19.2}{0.05} \\ &= 384\ kg/m^3\end{aligned}$$

The table and chair has same density. It means, density is the property of a material. **Lamina**: A lamina is an idealized flat object that has negligible thickness so that it is viewed as a two-dimensional plane region (see Fig. 10.1).

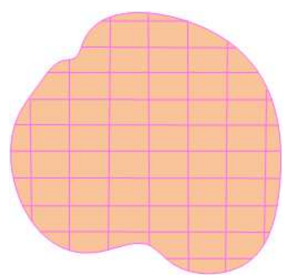

Figure 10.1: lamina

Example 10.1.2. *A triangular lamina with vertices (0, 0), (0, 1), and (1, 0) has mass 3 kg. If the length is measured in meters, find its density.*

Solution: The given triangle is right angle triangle with base =1 = perpendicular, its area is

$$A = \frac{1}{2}(1)(1) = \frac{1}{2}m^2$$

(10.1.3), its density is given by

$$\begin{aligned}\rho &= \frac{m}{A} = \frac{3}{1/2} \\ &= 6\ kg/m^2\end{aligned}$$

10.2 Moment of Mass

Consider a regular trihedral system with O as origin. Let a mass m is located at P at a distance r from O as shown in Fig. 10.2, then its moment about O is

$$M = mr \qquad (10.2.1)$$

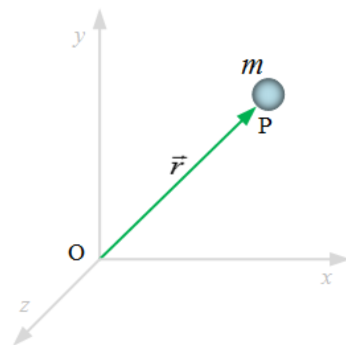

Figure 10.2: Moment of mass

In words, the moment of the mass about O, is the mass multiplied by its distance from O. The units of moment in SI will be $kg.m$.
Note: A large moment corresponds to a large turning effect.
Example Let a mass of 10 kg is placed at a distance 8 m from origin, then its moment is 80 $kg.m$. If the same mass is placed at a distance 10 m from origin, then its moment is 100 $kg.m$.

10.3 Center of Gravity

Earth attracts every body by a constant gravitational force (weight of the body). As a body is composed of many particles so each particle is affected by gravity, hence a large number of forces are acting on the entire body. The point at which the resultant of these forces acts is called center of gravity of the body. It can be defined as an imaginary point in a body of matter where the total weight of the body may be thought to be located.

10.4 Center of Mass

For every system of mass m, there is a unique location in space, where all the mass can be assumed to be located. This place is called the center of mass, and is defined as point with respect to which the linear moment of mass m is zero. It is commonly designated by $c.m$ or C.
Note: Center of mass is independent of gravitational field while center of gravity is affected by gravitational field.
When the gravitational field is uniform, the center of mass is also its center of gravity but if the body is lying in varying gravitational fields, the center of gravity will be shifted from

center of mass towards stronger gravitational field. For example if a stronger gravitational field is found towards right and a weaker gravitational field is found towards left of a body, the center of mass is unmoved but the center of gravity will be shifted towards stronger gravitational field.
Here we will consider only earth's gravitational field that is uniform, hence the center of mass will be the center of gravity of the body.

10.4.1 Center of Mass of a System of Two Particles

Consider a regular trihedral system and two particle of mass m_1 and m_2, situated at point P_1 and P_2, whose position vectors relative to origin O are $\vec{r}_1$ and $\vec{r}_2$. At center of mass, the linear moment of mass is zero. Mathematically

$$m_1\vec{r}_1 + m_2\vec{r}_2 = 0 \tag{10.4.1}$$

10.4.2 Center of Mass of a Set of n Particles

Consider a regular trihedral system and a set of n particles of masses $m_1, m_2, ..., m_n$, situated at point $P_1, P_2, ..., P_n$, whose position vectors relative to origin O are $\vec{r}_1, \vec{r}_2, ..., \vec{r}_n$. At center of mass, the sum of linear moments of all masses is zero. Mathematically

$$\sum_{i=1}^{n} m_i\vec{r}_i = 0 \tag{10.4.2}$$

Theorem 10.4.1. *Every set of particles has one and only center of mass.*

Proof Consider a regular trihedral system and a system of n particles of masses $m_1, m_2, ..., m_n$, situated at point $P_1, P_2, ..., P_n$, whose position vectors relative to origin O are $\vec{r}_1, \vec{r}_2, ..., \vec{r}_n$. Suppose C is a center of mass of the system and $\vec{r}$ be its position vector relative to O. Then the position vector of P_i relative to C is $\vec{r}_i - \vec{r}$. Then by definition, at C the sum of moments of all masses is zero.

$$\sum_{i=1}^{n} m_i\left(\vec{r}_i - \vec{r}\right) = 0$$

$$\sum_{i=1}^{n} m_i\vec{r}_i - \vec{r}\sum_{i=1}^{n} m_i = 0$$

or we can write

$$\vec{r}\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} m_i\vec{r}_i$$

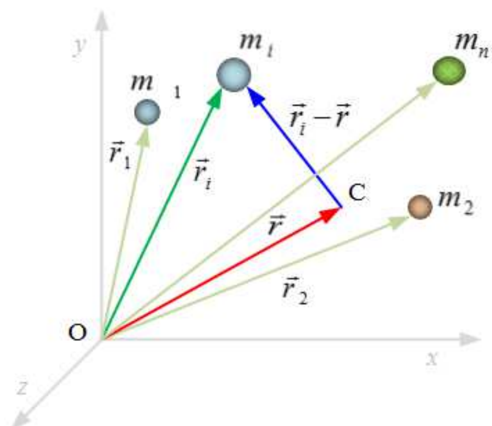

Figure 10.3: center of mass

therefore

$$\vec{r}_m = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} \tag{10.4.3}$$

(10.4.3) gives the position vector of C relative to O. Let C' be an other center of mass of the system and $\vec{r'}$ be its position vector relative to O. Then the position vector of P_i relative to C is $\vec{r}_i - \vec{r'}$. Then by definition, at C' the sum of moments of all masses is zero.

$$\sum_{i=1}^{n} m_i \left(\vec{r}_i - \vec{r'}\right) = 0$$

with the same above reasoning, we can write

$$\vec{r'}_m = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} \tag{10.4.4}$$

From (10.4.3) and (10.4.4),we can write

$$\vec{r'}_m = \vec{r}_m$$

Hence the system has one and only center of mass.

Example 10.4.1. *Find center of mass of the system consisting of two particles connected by a massless rod given in the following cases:*

(a) Both masses are 1 *kg and length of rod is* 2 *m.*

(b) The mass on right from O is 2 *kg and the mass on left is* 1 *kg. The length of rod is* 2 *m.*

(c) Both masses are 1 *kg. The mass on right is* 1.5 *m away from O and mass on left from O is* 1 *m away from O.*

Solution: The rod is considered to be 1 *dimensional* object just to understand the concept. And for one dimensional motion $+$, $-$ signs are enough to represent the direction of a vector.

(a) Both masses are same and length of rod is 2 m.

Let the center of the rod be at the origin. Let one mass is at A with position vector 1 m and the other mass is at B with position vector -1 m. The system is shown the Fig. 10.4. Here $n = 2$. Using (10.4.3), the center of mass of the system is

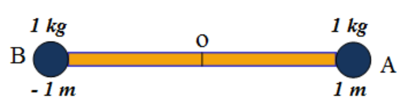

Figure 10.4: System of 2 particles with same masses and same distances

$$
\begin{aligned}
\vec{r}_m &= \frac{\sum_{i=1}^{2} m_i \vec{r}_i}{\sum_{i=1}^{2} m_i} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \\
&= \frac{1(1) + 1(-1)}{1+1} = \frac{0}{2} \\
&= 0\ m
\end{aligned}
$$

In this case origin is the center of mass.

(b) The mass on right from O is 2 kg and the mass on left from O is 1 kg. The length of rod is 2 m.

Let the center of the rod be at the origin. Let the mass of 2 kg is at A with position vector 1 and mass of 1 kg is at B with position vector -1. The system is shown the Fig. 10.5. Here $n = 2$. Using (10.4.3), the position vector of center of mass of the system is

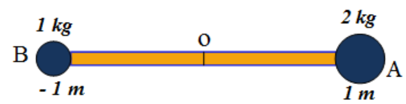

Figure 10.5: System of 2 particles with different masses and same distances

$$\begin{aligned} \vec{r}_m &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \\ &= \frac{2(1) + 1(-1)}{1 + 1} = \frac{1}{2} \\ &= 0.5\ m \end{aligned}$$

In this case the center of mass is shifted towards right from origin.

(c) Both masses are 1 kg. The mass on right is 1.5 m away from O and mass on left is 1 m away from O.

Let one mass 1 kg is at A with position vector 1.5 and mass 1 kg is at B with position vector -1. The system is shown the Fig. 10.6. Here $n = 2$. Using (10.4.3), the center of

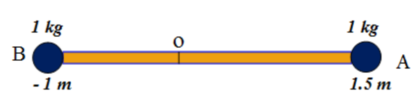

Figure 10.6: System of 2 particles with same masses and different distances

mass of the system is

$$\begin{aligned} \vec{r}_m &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \\ &= \frac{1(1.5) + 1(-1)}{1 + 1} = \frac{0.5}{2} \\ &= 0.25\ m \end{aligned}$$

In this case the center of mass is shifted towards right from origin.

10.4.3 Cartesian Coordinates of the Center of Mass

Consider a regular trihedral system and a system of n particles of masses $m_1, m_2, ..., m_n$, situated at point $P_1, P_2, ..., P_n$, whose position vectors relative to origin O are $\vec{r}_1, \vec{r}_2, ..., \vec{r}_n$. Then the position vector of P_i is

$$\vec{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k} \quad (10.4.5)$$

Suppose C is a center of mass of the system and $\vec{r} = (\bar{x}, \bar{y}, \bar{z})$ be its position vector relative to O. Then

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \quad (10.4.6)$$

$$\bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i} \quad (10.4.7)$$

$$\bar{z} = \frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i} \quad (10.4.8)$$

In case of plane coordinate system, $z-coordinate$ can be ignored considering xy plane and in case of collinear coordinate system only one coordinate will be sufficient.

Example 10.4.2. *A mass of* 3 *kg is located at* $(0, 0)$*, a mass of* 4 *kg is located at* $(5, 4)$ *and a mass of* 8 *kg is located at* $(-3, 3)$*. Find the coordinates of their centre of mass.*

Solution: The given system has three masses and is two dimensional. Here $n = 3$ Let

$$\begin{aligned} m_1 &= 3\ kg \\ m_2 &= 5\ kg \\ m_1 &= 8\ kg \end{aligned}$$

Then

$$\begin{aligned} \sum_{i=1}^{3} m_i &= m_1 + m_2 + m_3 = 3 + 5 + 8 \\ &= 16\ kg \end{aligned}$$

Sum of moments of all masses about $x\ axis$ is

$$\begin{aligned} \sum_{i=1}^{3} m_i x_i &= m_1x_1 + m_2x_2 + m_3x_3 = 3(0) + 4(5) + 8(-3) \\ &= 0 + 20 - 24 = -4\ kg.m \end{aligned}$$

Sum of moments of all masses about y $axis$ is

$$\begin{aligned}\sum_{i=1}^{3} m_i y_i &= m_1y_1 + m_2y_2 + m_3y_3 = 3(0) + 4(4) + 8(3) \\ &= 0 + 16 + 24 = 40\ kg.m\end{aligned}$$

using (10.4.6), the x coordinate of center of mass are

$$\begin{aligned}\bar{x} &= \frac{\sum\limits_{i=1}^{3} m_i x_i}{\sum\limits_{i=1}^{3} m_i} \\ &= \frac{-4}{16} = -0.25\ m\end{aligned}$$

using (10.4.7), the y coordinate of center of mass are

$$\begin{aligned}\bar{y} &= \frac{\sum\limits_{i=1}^{3} m_i y_i}{\sum\limits_{i=1}^{3} m_i} \\ &= \frac{40}{16} = 2.5\ m\end{aligned}$$

Hence the centre of mass of the system is located at the point (-0.25, 2.5).

10.5 Centroid of a Body or System

For a uniform body, the center of mass is called centroid. In this case, if body or system has n particles of equal masses $m_1 = m_2 = ... = m_n$, situated at point $P_1, P_2, ..., P_n$, then its center of mass or centroid is

$$\begin{aligned}\vec{r} &= \frac{\sum\limits_{i=1}^{n} \vec{r}_i}{n} \\ (\bar{x}, \bar{y}, \bar{z}) &= \left(\frac{\sum\limits_{i=1}^{n} x_i}{n}, \frac{\sum\limits_{i=1}^{n} y_i}{n}, \frac{\sum\limits_{i=1}^{n} z_i}{n} \right) \qquad (10.5.1)\end{aligned}$$

10.5.1 Center of Mass of a System of n Particles in Plane or Space

If n mass points are not necessarily on a line but are in a plane or in space with position vectors $\vec{r}_1, \vec{r}_2, ...\vec{r}_n$, then its center of mass is

$$\vec{r}_m = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ...m_n\vec{r}_n}{m_1 + m_2 + ...m_n} \qquad (10.5.2)$$

The center of mass of a system of two particles of masses m_1, m_2 is

$$\vec{r}_m = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \tag{10.5.3}$$

But (10.5.3) gives the position vector of the point dividing the directed line segment from $\vec{r}_1$ to $\vec{r}_2$ in the ratio $m_1 : m_2$. Hence the center of mass of two particles of masses m_1, m_2 divides the directed line segment from $\vec{r}_1$ to $\vec{r}_2$ in the ratio $m_1 : m_2$.
In case of two equal masses $m_1 = m_2 = m$, the centroid is

$$\vec{r}_m = \frac{\vec{r}_1 + \vec{r}_2}{2} \tag{10.5.4}$$

Example 10.5.1. *Consider three masses* $2, 3, 4$ *kg are situated at* P_1, P_2, P_3 *having position vectors* $\hat{i}$, $2\hat{i} - \hat{j}$ *and* $3\hat{i} + \hat{j} - 4\hat{k}$. *What will be their centroid and center of mass?*

Solution The position vector of the centroid is

$$\begin{aligned}\vec{r}_m &= \frac{\hat{i} + 2\hat{i} - \hat{j} + 3\hat{i} + \hat{j} - 4\hat{k}}{3} \\ &= \frac{6\hat{i} - 4\hat{k}}{3}\end{aligned}$$

And the position vector of the center of mass is

$$\begin{aligned}\vec{r}_m &= \frac{2\left(\hat{i}\right) + 3\left(2\hat{i} - \hat{j}\right) + 4\left(3\hat{i} + \hat{j} - 4\hat{k}\right)}{2 + 3 + 4} \\ &= \frac{19\hat{i} - \hat{j} - 12\hat{k}}{9}\end{aligned}$$

Hence the coordinates of the center of mass is $\left(\frac{19}{9}, -\frac{1}{9}, -\frac{4}{3}\right)$.

10.6 Center of Mass of a Continuous Distribution of Matter

The formulae obtained in the preceding article are applicable in the case of discrete systems only. If we are to find the center of mass of a continuous distribution of matter forming a body, integration methods explained below are to be employed. First of all consider one dimensional object.

10.6.1 Center of Mass of One Dimensional Object

Consider a body (a line or curve) of mass m and length l in one dimension. We subdivide the object into n parts. Take a small element of length ds with r be its position vector (see Fig. 10.7), then mass of small element is

$$dm = \rho ds$$

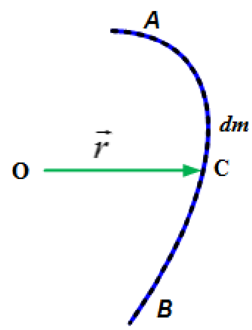

Figure 10.7: Center of mass of 1 dimensional system

Where ρ is the density of the body. Then the center of mass of the body is

$$\begin{aligned} \bar{r} &= \frac{\int\limits_s \vec{r}dm}{\int\limits_s dm} \\ &= \frac{\int\limits_s \vec{r}dm}{m} \end{aligned} \tag{10.6.1}$$

Where the integration has to be performed over the entire body.
More clearly if one dimensional system is x axis and the mass m is from x_1 to x_2, then its total length is $l = x_2 - x_1$. Consider small element dm having length dx, having position vector $\vec{x}$ from the orion. Let ρ be the its density at $\vec{r}$, then

$$\rho = \frac{dm}{dx}$$

or small element is

$$dm = \rho dx$$

then the center of mass of is

$$\begin{aligned} \bar{x} &= \frac{\int\limits_{x_1}^{x_2} \vec{x}\rho dx}{\int\limits_{x_1}^{x_2} \rho dx} \\ &= \frac{1}{m}\int\limits_{x_1}^{x_2} \vec{x}\rho dx \end{aligned} \tag{10.6.2}$$

Where $m = \int_{x_1}^{x_2} \rho dx$ is the total mass of the body.
If the body is homogenous (has uniform distribution of mass), then the center of mass of is

$$\begin{aligned}\bar{x} &= \frac{\int_{x_1}^{x_2} \vec{x} dx}{\int_{x_1}^{x_2} dx} \\ &= \frac{1}{l} \int_{x_1}^{x_2} \vec{x} dx \end{aligned} \tag{10.6.3}$$

Example 10.6.1. *Find center of mass of a uniform rod of mass m kg of length a m.*

Solution: A uniform rod of mass m kg of length a m is shown in Fig. 10.8 Consider a small element mass dm of width dx at a distance x from origin O. Here

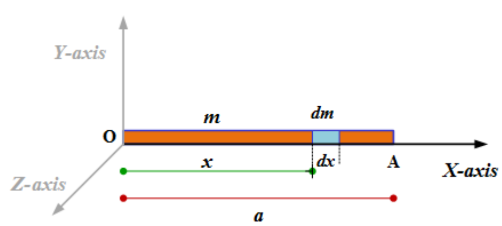

Figure 10.8: rod of length a

$$\begin{aligned}\vec{x} &= x \\ m &= m\ kg \\ l &= a \\ x_1 &= 0 \\ x_2 &= a\end{aligned}$$

Since the rod is uniform, using (10.6.3) the center of mass of the rod is

$$\begin{aligned}\bar{x} &= \frac{1}{a}\int_0^a x dx \\ &= \frac{1}{a}\left[\frac{x^2}{2}\right]_0^a \\ &= \frac{1}{a}\frac{(a)^2}{2} \\ &= \frac{1}{2}a\ m\end{aligned}$$

Hence the center of mass of the rod is its mid point.

10.6.2 Center of Mass of Two Dimensional Object

Consider a lamina of mass m and area A in cartesian coordinate system. We subdivide the lamina into n rectangles by drawing lines parallel to coordinate axes. Take a small rectangle of area ds (see Fig. 10.9), then mass of small element is

$$dm = \rho ds$$

Let $r_i = (x_i, y_i)$ be any point in it. Then the center of mass of the lamina is

$$\begin{aligned}\vec{r}_m &= \frac{\int_s \vec{r} dm}{\int_s dm} \\ &= \frac{\int_s \vec{r} dm}{m} \qquad (10.6.4)\end{aligned}$$

Where the integration has to be performed over the entire body.

More clearly if two dimensional system is in xy plane and the mass m has dimensions $x_1 \leq x \leq x_2$ and $y_1 \leq y \leq y_2$, then its total area is $A = (x_2 - x_1)(y_2 - y_1)$. Consider small element dm having area $dA = dxdy$. Let $r_i = (x_i, y_i)$ be any point with density ρ in it. Then

$$\rho = \frac{dm}{dA}$$

or small element is

$$dm = \rho dA$$

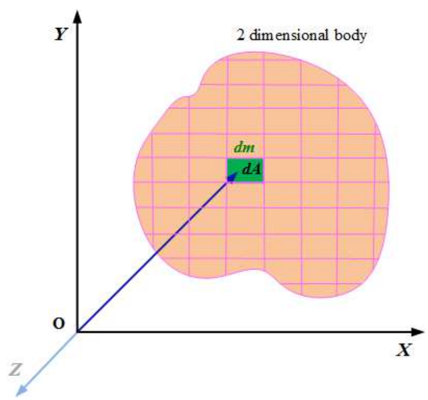

Figure 10.9: Center of mass of 2 dimensional system

Then the center of mass of the lamina is

$$\bar{r} = \frac{\int_{x_1}^{x_2}\int_{y_1}^{y_2} \vec{r}\rho dA}{\int_{x_1}^{x_2}\int_{y_1}^{y_2} \rho dA} = \frac{\int_{x_1}^{x_2}\int_{y_1}^{y_2} \vec{r}\rho dA}{m} \tag{10.6.5}$$

Where $m = \int_s \rho dA$ is the total mass of the lamina.
If the body homogenous, then the center of mass of the lamina is

$$\vec{r}_m = \frac{\int_{x_1}^{x_2}\int_{y_1}^{y_2} \vec{r} dA}{\int_{x_1}^{x_2}\int_{y_1}^{y_2} dA} = \frac{\int_{x_1}^{x_2}\int_{y_1}^{y_2} \vec{r} dA}{A} \tag{10.6.6}$$

Where $A = \int_s dA$ is the total area of the lamina.

Example 10.6.2. *Find the center of mass of a uniform rectangular lamina.*

Solution
Let $OABC$ be a rectangular lamina of mass m and OA (along x $axis$) and OB along y $axis$

Let $OA = 2a$ and $OC = 2b$ Area of lamina is

$$A = 4ab$$

Consider a small element of surface area $dA = dxdy$ at a distance y from x $axis$ Here

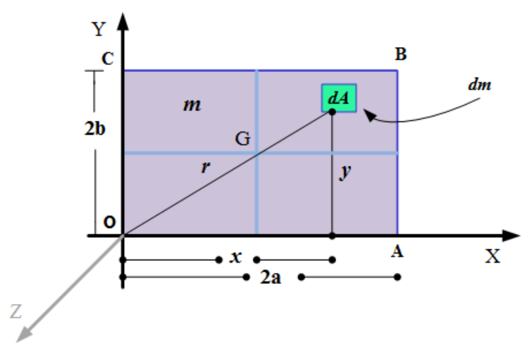

Figure 10.10: rectangular lamina

$$\begin{aligned}
\vec{r} &= \langle x, y \rangle \\
m &= m\ kg \\
A &= 4ab \\
x_1 &= 0 \\
x_2 &= 2a \\
y_1 &= 0 \\
y_2 &= 2b
\end{aligned}$$

Since lamina is uniform, using (10.6.6) the center of mass of the rod is

$$\begin{aligned}
\vec{r}_m &= \frac{\int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2} \vec{r} dA}{A} \\
\langle \bar{x}, \bar{y} \rangle &= \frac{\int\limits_{0}^{2a}\int\limits_{0}^{2b} \langle x, y \rangle dx dy}{4ab} \\
&= \frac{1}{4ab} \left\langle \left[\frac{x^2}{2}\right]_0^{2a} \int\limits_0^{2b} dy, \left[x\right]_0^{2a} \int\limits_0^{2b} y dy \right\rangle \\
&= \frac{1}{4ab} \left\langle \left[2a^2\right] \left[y\right]_0^{2b}, (2a) \left[\frac{y^2}{2}\right]_0^{2b} \right\rangle \\
&= \frac{1}{4ab} \left\langle \left[2a^2\right](2b), (2a)\left[2b^2\right] \right\rangle \\
&= \langle a, b \rangle
\end{aligned}$$

Hence the center of mass or centroid of rectangular lamina is (a, b)

10.6.3 Center of Mass of Three Dimensional Object

Consider a three dimensional rigid body of mass m and volume V in a regular trihedral system. We subdivide the lamina into n rectangular parallelepipeds by drawing planes parallel to coordinate axes. One such parallelepiped of volume ds is shown in Fig. 10.11. The mass of small element is

$$dm = \rho ds$$

Let $r_i = (x_i, y_i, z_i)$ be any point within it. Then the center of mass of the body is

$$\begin{aligned}
\bar{r} &= \frac{\int\limits_s \vec{r} dm}{\int\limits_s dm} \\
&= \frac{\int\limits_s \vec{r} dm}{m} \qquad (10.6.7)
\end{aligned}$$

Where the integration has to be performed over the entire body, and $m = \int\limits_s dm$ is the total mass of the body.

More clearly if three dimensional system is in xyz space and the mass m has dimensions $x_1 \leq x \leq x_2$, $y_1 \leq y \leq y_2$ and $z_1 \leq z \leq z_2$, then its total volume is $V =$

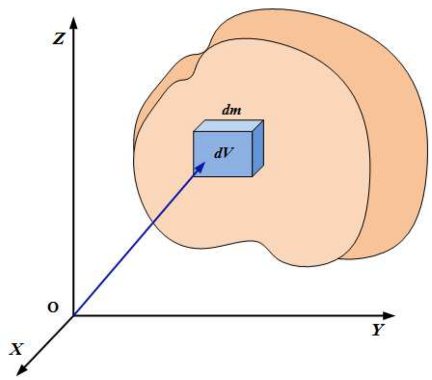

Figure 10.11: Center of mass of 3 dimensional system

$(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)$. Consider small element dm having volume $dV = dxdydz$, Let $r_i = (x_i, y_i, z_i)$ be any point with density ρ in it. Then

$$\rho = \frac{dm}{dV}$$

or small element is

$$dm = \rho dV$$

Then the center of mass of the body is

$$\begin{aligned}\vec{r}_m &= \frac{\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \vec{r}\rho dV}{\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \rho dV} \\ &= \frac{\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \vec{r}\rho dV}{m} \qquad (10.6.8)\end{aligned}$$

Where $m = \int_s \rho dV$ is the total mass of the body.

If the body homogenous, then the center of mass of the body is

$$\begin{aligned}\vec{r}_m &= \frac{\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \vec{r} dV}{\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} dV} \\ &= \frac{\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \vec{r} dV}{V} \qquad (10.6.9)\end{aligned}$$

Where $V = \int_s dm$ is the total volume of the body.

Example 10.6.3. *Find the center of mass of a uniform cube.*

Solution

Consider a uniform cube of mass m with OA along x $axis$, OB along y $axis$ and OC along z $axis$. Let $OA = a$, $OB = a$ and $OC = a$ Volume of the cube is

$$V = a^3$$

Consider a small volume $dV = dxdydz$ at a distance r from origin O. Here

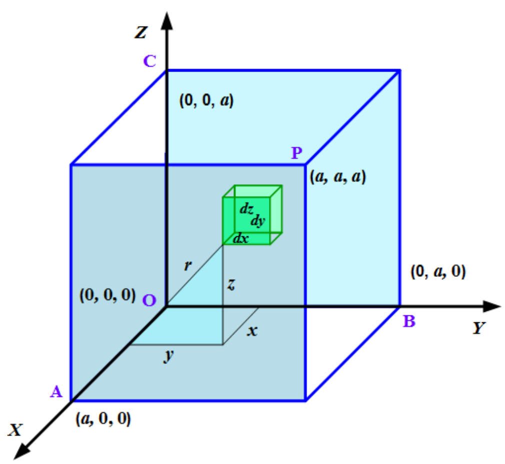

Figure 10.12: A cube with edges along coordinate axis.

$$\begin{aligned}
\vec{r} &= \langle x, y, z \rangle \\
m &= m\ kg \\
V &= a^3 \\
x_1 &= 0 \\
x_2 &= a \\
y_1 &= 0 \\
y_2 &= a \\
z_1 &= 0 \\
z_2 &= a
\end{aligned}$$

Since cube is uniform, using (10.6.9) the center of mass of the rod is

$$\begin{aligned}
\vec{r}_m &= \frac{\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \vec{r}dV}{V} \\
\langle \bar{x}, \bar{y} \rangle &= \frac{\int_0^a\int_0^a\int_0^a \langle x, y, z\rangle dxdydz}{a^3} \\
&= \frac{1}{a^3}\left\langle \left[\frac{x^2}{2}\right]_0^a \int_0^a\int_0^a dydz, \left[x\right]_0^a \int_0^a\int_0^a ydydz, \left[x\right]_0^a \int_0^a\int_0^a zdydz \right\rangle \\
&= \frac{1}{a^3}\left\langle \left[\frac{a^2}{2}\right] \left[y\right]_0^a \int_0^a dz, (a) \left[\frac{y^2}{2}\right]_0^a \int_0^a dz, a\left[y\right]_0^a \int_0^a zdz \right\rangle \\
&= \frac{1}{a^3}\left\langle \frac{1}{2}a^3\left[z\right]_0^a, \frac{1}{2}a^3\left[z\right]_0^a, a^2\left[\frac{z^2}{2}\right]_0^a \right\rangle \\
&= \frac{1}{a^3}\left\langle \frac{1}{2}a^4, \frac{1}{2}a^4, \frac{1}{2}a^4 \right\rangle \\
&= \langle \frac{a}{2}, \frac{a}{2}, \frac{a}{2} \rangle
\end{aligned}$$

Hence the center of mass or centroid of a uniform cube is $(\frac{a}{2}, \frac{a}{2}, \frac{a}{2})$

10.7 Symmetry and Center of Mass

If a body possesses some sort of symmetry, then it is too much easy to compute the position of its center of mass. We first explain the concept of symmetry and shall therefore show how to use this concept in determining the center of mass of a body.

10.7.1 Symmetry with respect to a Point

A body is said to be symmetric with respect to a point O if and only if corresponding to every point P of the body there exist a point $P^{'}$ in the body such that O is the middle point of the line segment $PP^{'}$ and $\rho(P) = \rho(P^{'})$, *i.e.*, the density of the body at the points P and $P^{'}$ is the same. Such symmetry is called central symmetry and the point O is called the center of symmetry.
It follows that a uniform body is symmetric with respect to origin O if and only if for every point $P(x, y, z)$ of the body there exist a point $P^{'}(-x, -y, -z)$ in the body such that O is the middle point of the line segment $PP^{'}$ and $\rho(P) = \rho(P^{'})$.
Examples
1. A uniform rod is symmetric with respect to its mid point, hence its mid point is center

of mass as shown in example 10.6.1.
2. A uniform circular lamina is symmetric with respect to its geometric center.
3. A uniform solid sphere or spherical shell is symmetric with respect to its geometric center.

10.7.2 Symmetry with respect to a Line

A body is said to be symmetric with respect to a line l if and only if corresponding to every point P of the body there exist a point $P^{'}$ in the body such that l bisects the line segment $PP^{'}$ perpendicularly and $\rho(P) = \rho(P^{'})$. Such symmetry is called axial symmetry and the line l is called the axis of symmetry.
In particular, a uniform body is symmetric with respect to the z axis if and only if for every point $P(x, y, z)$ of the body there exist a point $P^{'}(-x, -y, z)$ in the body such that z axis is the right bisector of the line segment $PP^{'}$.
Examples
1. A uniform circular cylinder is symmetric with respect to its axis.
2. A uniform solid sphere or spherical shell is symmetric with respect to its axis.
A uniform lamina is symmetric with respect to x axis if and only if for every point $P(x, y)$ of the body there exist a point $P^{'}(x, -y)$ of the lamina. In this case its center is the center of mass as shown in example 10.6.3.

10.7.3 Symmetry with respect to a Plane

A body is said to be symmetric with respect to a plane p if and only if corresponding to every point P of the body there exist a point $P^{'}$ in the body such that p bisects the line segment $PP^{'}$ perpendicularly and $\rho(P) = \rho(P^{'})$. Such symmetry is called axial symmetry and the line l is called the axis of symmetry.
In particular, a uniform body is symmetric with respect to the xy plane if and only if for every point $P(x, y, z)$ of the body there exist a point $P^{'}(x, y, -z)$ in the body such that xy plane bisects of the line segment $PP^{'}$.
Examples
1. A uniform solid or hollow ellipsoid is symmetric with respect to each of its principal planes.
2. A uniform solid sphere or spherical shell is symmetric with respect to each of its diametral plane (planes passing through the center).

10.8 Centroid of a Plane Region

The centroid C is a point which defines the geometric center of an object. The centroid coincides with the center of mass or the center of gravity only if the material of the body is

homogenous (density or specific weight is constant throughout the body). If an object has an axis of symmetry, then the centroid of object lies on that axis.
Consider the region bounded by the curve $y = f(x)$, the $x - axis$, the line $x = a$ and the line $x = b$ as shown in Fig. 10.13. Let the density of the region is 1. Then by (10.1.4) the

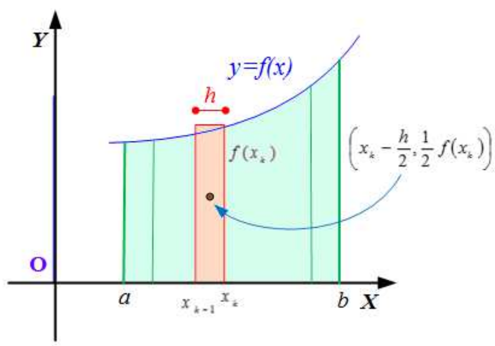

Figure 10.13: Plane region.

total mass of the system is the total area of the region

$$m = A$$

For uniform distribution of mass the area under the curve is

$$A = \int_a^b f(x)dx$$

Hence the mass of the region is

$$m = \int_a^b f(x)dx$$

As area under the curve is obtained by using approximation method. In this method the interval $[a, b]$ is divided into n subintervals of length h. Then h is

$$h = \frac{b-a}{n} \tag{10.8.1}$$

Let $x_k, k = 1, 2, ...n$ be the endpoints of each subinterval. Next construct a rectangle on each subinterval and find its area to approximate the area under the curve. On interval $x_{k-1} \leq x \leq x_k$, the rectangle has height $f(x_k)$ and width $x_k - x_{k-1} = h$. Its area is its mass

$$dA = hf(x_k) = dm$$

And its center of mass is its geometric center, given by

$$C_I = \left(x_k - \frac{h}{2}, \frac{1}{2}f(x_k)\right)$$

The mass moment about $y - axix$ of this rectangle is

$$dM_y = hf(x_k)\left(x_k - \frac{h}{2}\right)$$

We can imagine that center of mass of each rectangle is its geometric center. The mass moment about $y - axix$ of all n rectangles is

$$M_x = \sum_{k=1}^{n}\left[hf(x_k)\left(x_k - \frac{h}{2}\right)\right]$$

Similarly the mass moment about $x - axix$ of all n rectangles is

$$M_y = \sum_{k=1}^{n}\left[hf(x_k)\frac{1}{2}f(x_k)\right]$$

Taking limit $n \to \infty$, the sum of areas of all rectangles approaches to true area under the curve, and in the same way the moments about $y - axix$ and $x - axix$ of the rectangles approaches to true moments of area under the curve. As $n \to \infty$, by (10.8.1) $h \to 0$. Hence $x_k - \frac{h}{2} \to x_k$. Thus for a plane region bounded by $y = f(x)$, the $x - axis$, the line $x = a$ and the line $x = b$, the moments about $x - axis$ is

$$\begin{aligned} M_x &= \lim_{n\to\infty}\sum_{k=1}^{n}\left[hf(x_k)\frac{1}{2}f(x_k)\right] \\ &= \int_a^b xf(x)dx \end{aligned} \tag{10.8.2}$$

and $y - axis$ is

$$\begin{aligned} M_y &= \lim_{n\to\infty}\sum_{k=1}^{n}\left[hf(x_k)\left(x_k - \frac{h}{2}\right)\right] \\ &= \int_a^b \frac{1}{2}\left[f(x)\right]^2 dx \end{aligned} \tag{10.8.3}$$

Hence the x coordinate of center of mass is

$$\bar{x} = \frac{M_x}{m} = \frac{\int_a^b xf(x)dx}{\int_a^b f(x)dx} \tag{10.8.4}$$

the y coordinate of center of mass is

$$\bar{y} \quad = \quad \frac{M_y}{m} = \frac{\int_a^b \frac{1}{2}\left[f(x)\right]^2 dx}{\int_a^b f(x)dx} \tag{10.8.5}$$

Example 10.8.1. *Find the center of mass of a plane region bounded by the curve $y = \sqrt{x}$, the $x - axis$, the line $x = 1$ and the line $x = 3$.*

Solution
The given data is

$$\begin{aligned} f(x) &= \sqrt{x} = x^{\frac{1}{2}} \\ a &= 1 \\ b &= 3 \end{aligned}$$

The plane region bounded by the curve $y = \sqrt{x}$, the $x - axis$, the line $x = 1$ and the line $x = 3$ is shown in Fig. 10.14. The area of the region is

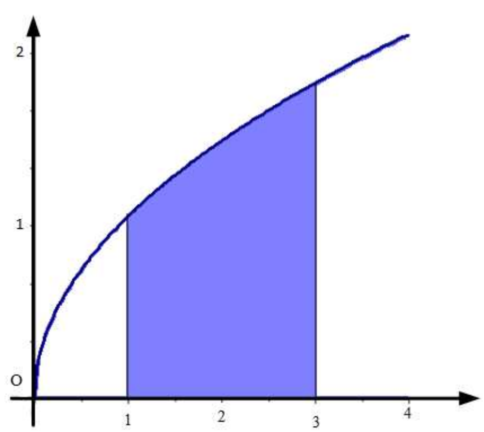

Figure 10.14: Plane region.

$$\begin{aligned} A &= \int_a^b f(x)dx \\ &= \int_1^3 (x)^{\frac{1}{2}} dx \\ &= \left[\frac{(x)^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^3 \\ &= \frac{2}{3}\left[(3)^{\frac{3}{2}} - 1 \right] \\ &= 2.8 \ units^2 \end{aligned}$$

Using (10.8.2) the mass moments about $x - axis$ is

$$\begin{aligned} M_x &= \int_a^b x f(x)dx \\ &= \int_1^3 x(x)^{\frac{1}{2}} dx = \int_1^3 (x)^{\frac{3}{2}} dx \\ &= \left[\frac{(x)^{\frac{5}{2}}}{\frac{5}{2}} \right]_1^3 \\ &= \frac{2}{5}\left[(3)^{\frac{5}{2}} - 1 \right] \\ &= 5.8 \ units^3 \end{aligned}$$

Using (10.8.3) the mass moments about $y - axis$ is

$$\begin{aligned} M_y &= \int_a^b \frac{1}{2}\left[f(x) \right]^2 dx \\ &= \int_1^3 \frac{1}{2}\left[\sqrt{x} \right]^2 dx \\ &= \int_1^3 \frac{1}{2} x dx \\ &= \frac{1}{2}\left[\frac{x^2}{2} \right]_1^3 \\ &= 2 \ units^3 \end{aligned}$$

Using (10.8.4) the x coordinate of center of mass is

$$\begin{aligned}\bar{x} &= \frac{M_x}{m} = \frac{\int\limits_a^b xf(x)dx}{\int\limits_a^b f(x)dx} \\ &= \frac{5.8}{2.8} \\ &= 2.1 \; units\end{aligned}$$

Using (10.8.5) the y coordinate of center of mass is

$$\begin{aligned}\bar{y} &= \frac{M_y}{m} = \frac{\int\limits_a^b \frac{1}{2}\left[f(x)\right]^2 dx}{\int\limits_a^b f(x)dx} \\ &= \frac{2}{2.8} \\ &= 0.7 \; units\end{aligned}$$

Hence the center of mass is $(2.1, 0.7)$ unit.

Exercises

1. The density of glass of mass $10kg$ is $3140\ kg/m^3$. Determine its volume.
2. A mass of $5\ kg$ is located at $(1, 0, -1)$, a mass of $4\ kg$ is located at $(2, 5, 4)$ and a mass of $2\ kg$ is located at $(4, -3, 1)$. Find the coordinates of their centre of mass.
3. A square of side a has particles of masses $1\ kg,\ 2\ kg,\ 3\ kg,\ 4\ kg$ at its vertices. Find the center of mass of the system.
4. Find the center of mass of a uniform rectangular lamina whose center is the origin of the coordinate system.
5. Find the center of mass of a uniform triangular lamina.
6. Find the center of mass of a uniform circular disc whose center
   (a) is the origin of the coordinate system.
   (b) lies on x axis of the coordinate system and passing through the origin.
   (c) lies on y axis of the coordinate system and passing through the origin.
7. Find the center of mass of a uniform elliptic disc whose center lies on the origin.
8. Find the center of mass of a plane region bounded by
   (a) lines $y = 2x,\ y = -2x$ and $x = 2$.
   (b) the curve $y = \sqrt{x}$, the $x - axis$ and the line $x = 4$.
   (c) the curve $y = x^2$, the $x - axis$ and the lines $x = 1, x = 2$.

Chapter 11

Moments and Products of Inertia

Inertia The tendency of a body to preserve its state of rest or uniform motion unless acted upon by an external force.

11.1 Moments of Inertia

The moment of inertia plays much the same role in rotational dynamics as mass does in linear dynamics. In classical mechanics, moment of inertia, also called mass moment of inertia, rotational inertia, polar moment of inertia of mass, or the angular mass.
It is the inertia of a rotating body with respect to its rotation. *i.e.* a quantity that measures the resistance of an object to changes to its rotation. The symbols I and sometimes J are usually used to refer it.

11.1.1 Moments of Inertia of a Particle

Moment of inertia of a particle of mass m about line l or AB axis is defined as

$$I = md^2 \tag{11.1.1}$$

Where I stands for moment of inertia and d is the perpendicular distance of the particle of mass m from the line l or AB axis (see Fig. 11.1). In SI its measuring unit is $kg \cdot m^2$.
For continuous distribution of mass the moment of inertia is

$$I = \int r^2 dm \tag{11.1.2}$$

Where r is the distance of the mass element dm from the given line or axis and the integration is to be performed for entire body.
In this chapter, the body under consideration has uniform distribution of mass.
The moment of inertia describes the relationship between angular momentum and angular

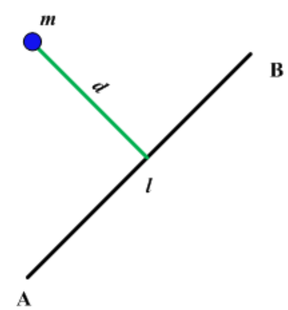

Figure 11.1: Moment of inertia of a single particle

velocity, torque and angular acceleration, and several other quantities.

11.1.2 Moments of Inertia of a System of Particles

The moment of inertia of a system of particles of masses $m_1, m_2, ..., m_n$ about line l or AB axis is defined as

$$\begin{aligned} I &= m_1 d_1^2 + m_2 d_2^2 + ... + m_n d_n^2 \\ &= \sum_{i=1}^{n} m_i d_i^2 \end{aligned} \tag{11.1.3}$$

Where d_i is the distance of the particle of mass m_i from the line l or AB axis (see Fig. 11.2).

11.2 Moment of Inertia of a Mass with Continuous Distribution

The formulae obtained in the preceding article are applicable in the case of discrete systems only. If we are to find the moment of inertia of a continuous distribution of matter forming a body, integration methods explained below are to be employed. First of all consider one dimensional object.

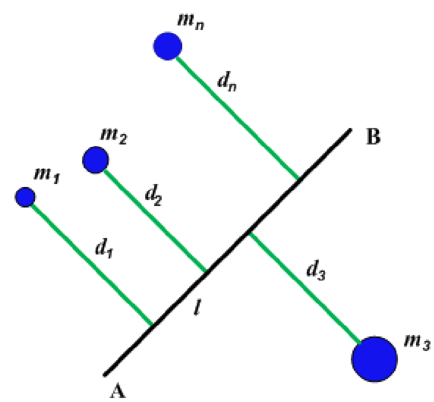

Figure 11.2: Moment of inertia of a system of particles

11.2.1 Moment of Inertia of One Dimensional Particle

Consider a body (a line or curve) of mass m in one dimension. Take a small element of length ds (see Fig. 11.3), then mass of small element,

$$dm = \rho ds$$

Let AB be the axis of rotation, then moment of inertia of small element dm is

$$dI = dmd^2$$

M.I. of body about AB axis is

$$\begin{aligned} I &= \int_s dmd^2 \\ &= \int_s \rho d^2 ds \end{aligned} \tag{11.2.1}$$

More clearly if one dimensional system is x axis and the mass m is from x_1 to x_2, then its total length is $l = x_2 - x_1$. Consider small element dm having length dx, at a distance $d = x$ from the axis of rotation. Since the body has uniform distribution of mass, then

$$\rho = \frac{mass_{total}}{length_{total}} = \frac{dm}{dx}$$

or small element is

$$dm = \frac{m}{x_2 - x_1} dx$$

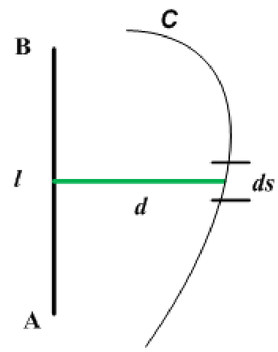

Figure 11.3: Moment of inertia of 1 dimensional system

then moment of inertia of small element dm is

$$\begin{aligned} dI &= dmd^2 \\ &= \frac{m}{x_2 - x_1} x^2 dx \end{aligned} \tag{11.2.2}$$

M.I. of body about AB axis is

$$I = \frac{m}{x_2 - x_1} \int_{x_1}^{x_2} x^2 dx \tag{11.2.3}$$

11.2.2 Moment of Inertia of Two Dimensional Particle

Consider a body (a plane curve) of mass m in two dimension. Take a small element area ds (see Fig. 11.4), then mass of small element,

$$dm = \rho ds$$

Let AB be the axis of rotation, then moment of inertia of small element dm is

$$dI = dmd^2$$

M.I. of body about AB axis is

$$\begin{aligned} I &= \iint_s dmd^2 \\ &= \iint_s \rho d^2 ds \end{aligned} \tag{11.2.4}$$

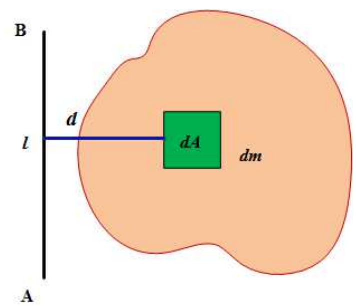

Figure 11.4: Moment of inertia of 2 dimensional system

More clearly if two dimensional system is xy plane and the mass m has dimensions $x_1 \leq x \leq x_2$ and $y_1 \leq y \leq y_2$, then its total area is $A = (x_2 - x_1)(y_2 - y_1)$. Consider small element dm having area $dA = dxdy$, at a distance d from the axis of rotation. Since the body has uniform distribution of mass, then

$$\rho = \frac{mass_{total}}{Area_{total}} = \frac{dm}{dA}$$

or small element is

$$dm = \frac{m}{A}dA$$

then moment of inertia of small element dm about the axis of rotation is

$$\begin{aligned} dI &= dmd^2 \\ &= \frac{m}{A}d^2dA \end{aligned} \tag{11.2.5}$$

M.I. of body about AB axis is

$$I = \frac{m}{A}\int_{x_1}^{x_2}\int_{y_1}^{y_2} d^2dxdy \tag{11.2.6}$$

11.3 Moment of Inertia of Three Dimensional Particle

Consider a body (a surface) of mass m in three dimension. Take a small element of volume ds (see Fig. 11.5), then mass of small element,

$$dm = \rho ds$$

Let AB be the axis of rotation, then moment of inertia of small element dm is

$$dI = dmd^2$$

M.I. of body about AB axis is

$$\begin{aligned} I &= \iiint\limits_{s} dmd^2 \\ &= \iiint\limits_{V} \rho d^2 ds \end{aligned} \tag{11.3.1}$$

More clearly if three dimensional system is xyz space and the mass m has dimensions $x_1 \leq$

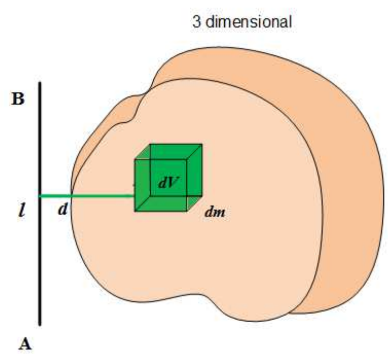

Figure 11.5: Moment of inertia of 3 dimensional system

$x \leq x_2$, $y_1 \leq y \leq y_2$ and $z_1 \leq y \leq z_2$, then its total volume is $V = (x_2 - x_1)(y_2 - y_1)(z_2 - z_1)$. Consider small element dm having volume $dV = dxdydz$, at a distance d from the axis of rotation. Since the body has uniform distribution of mass, then

$$\rho = \frac{mass_{total}}{Volume_{total}} = \frac{dm}{dV}$$

or small element is

$$dm = \frac{m}{V} dV$$

then moment of inertia of small element dm about the axis of rotation is

$$\begin{aligned} dI &= dmd^2 \\ &= \frac{m}{V} d^2 dV \end{aligned} \tag{11.3.2}$$

M.I. of body about AB axis is

$$I = \frac{m}{V}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} d^2 dxdydz \tag{11.3.3}$$

11.4 Radius of Gyration

Radius of gyration of a body about an axis, is the effective distance (perpendicular distance) of point P from its axis where whole mass can be assumed to be concentrated so that I remains the same.
For a system of mass $M = \sum_{i=1}^{n} m_i$ and moment of inertia I about an axis, the radius of Gyration denoted by K and is defined as

$$\begin{aligned} K^2 &= \frac{I}{M} \\ &= \frac{\sum_{i=1}^{n} m_i r_i^2}{\sum_{i=1}^{n} m_i} \end{aligned} \tag{11.4.1}$$

where r_i the perpendicular distances of mass m_i from axis of rotation. When the mass has

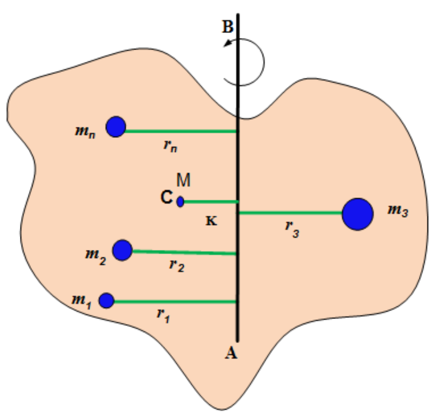

Figure 11.6: Radius of gyration

equal distribution $i.e$ $M = nm$

$$\begin{aligned} K^2 &= \frac{\sum\limits_{i=1}^{n} mr_i^2}{\sum\limits_{i=1}^{n} m_i} \\ &= \frac{\left(mr_1^2 + mr_2^2 + ... + mr_n^2\right)}{nm} \\ &= \frac{nm\left(r_1^2 + r_2^2 + ... + r_n^2\right)}{nm} \\ &= \sum_{i=1}^{n} r_i^2 \end{aligned} \tag{11.4.2}$$

and the radius of gyration is

$$K = \sqrt{\sum_{i=1}^{n} r_i^2}$$

When a system consists of a single particle then the radius of Gyration about a line or axis

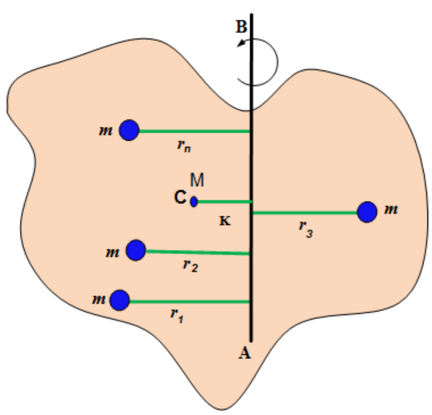

Figure 11.7: Radius of gyration

is simply the distance of the particle from the line or axis.

$$\begin{aligned} K &= \sqrt{\frac{I}{m}} \\ K^2 &= \frac{I}{m} \\ &= \frac{mr^2}{m} \end{aligned} \tag{11.4.3}$$

or

$$K = r \tag{11.4.4}$$

11.5 Moment of Inertia about Coordinate Axes

11.5.1 Moment of Inertia of a Single Particle

Consider a particle P of masses m in a regular trihedral system $OXYZ$. Its perpendicular distance from OZ axis is

$$\begin{aligned} d &= PR = OQ \\ &= \sqrt{x^2 + y^2} \end{aligned} \tag{11.5.1}$$

Then its moment of inertia about OZ axis is

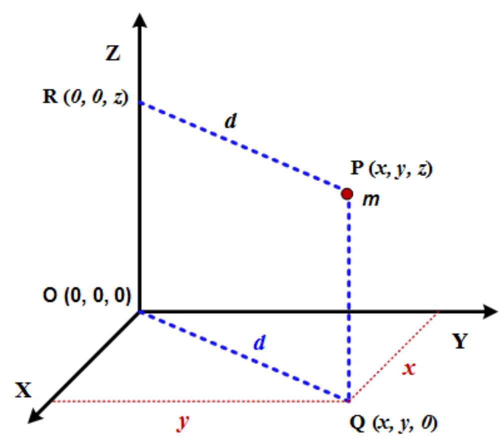

Figure 11.8: M.I of of a single particle

$$\begin{aligned} I_{OZ} &= md^2 \\ &= m(x^2 + y^2) \end{aligned} \tag{11.5.2}$$

Similarly about OX axis

$$I_{OX} = m(y^2 + z^2) \tag{11.5.3}$$

and OY axis

$$I_{OY} = m(x^2 + z^2) \tag{11.5.4}$$

11.5.2 Moment of Inertia of a System of n Particles

Consider a system of particles $P_1(x_1, y_1, z_2)$, $P_2(x_2, y_2, z_2)$, ... $P_n(x_n, y_n, z_n)$ with masses $m_1, m_2, ..., m_n$ respectively in a regular trihedral system $OXYZ$. The ith particle of mass m_i is at point P_i having a distance d_i from z axis. Then its moment of inertia about the axis OZ is

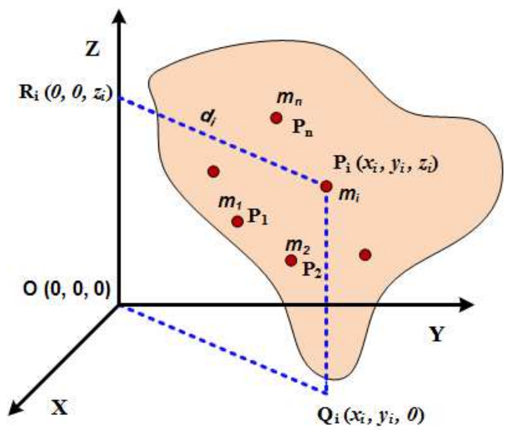

Figure 11.9: M.I of of a system of particles

$$\begin{aligned} dI_{OZ} &= m_i {d_i}^2 \\ &= m_i \left({x_i}^2 + {y_i}^2\right) \end{aligned} \quad (11.5.5)$$

and the moment of inertia of the rigid body about the z axis is

$$C = I_{OZ} = I_{zz} = \sum_{i=1}^{n} m_i \left({x_i}^2 + {y_i}^2\right) \quad (11.5.6)$$

about OX axis is

$$A = I_{OX} = I_{xx} = \sum_{i=1}^{n} m_i({y_i}^2 + {z_i}^2) \quad (11.5.7)$$

and about OY axis is

$$B = I_{OY} = I_{yy} = \sum_{i=1}^{n} m_i({x_i}^2 + {z_i}^2) \quad (11.5.8)$$

11.6 Product of Inertia

The quantities

$$D = \sum_{i=1}^{n} m_i y_i z_i \tag{11.6.1}$$

$$E = \sum_{i=1}^{n} m_i z_i x_i \tag{11.6.2}$$

$$F = \sum_{i=1}^{n} m_i x_i y_i \tag{11.6.3}$$

are called products of inertia $w.r.t.$ pair of axes $(oy, oz), (oz, ox)$ and (ox, oy) respectively. They are a measure of the imbalance in the mass distribution. They can be positive, negative, or zero.

11.6.1 Product of Inertia for a System of Continuous Distribution of Mass

The $M.I.$ about $x-$axis, $y-$axis and $z-$axis are defined as under

$$A = \iiint\limits_V \rho \left(y^2 + z^2\right) dV \tag{11.6.4}$$

$$B = \iiint\limits_V \rho \left(z^2 + x^2\right) dV \tag{11.6.5}$$

$$C = \iiint\limits_V \rho \left(x^2 + y^2\right) dV \tag{11.6.6}$$

Similarly, the products of inertia $w.r.t.$ pair of axes $(oy, oz), (oz, ox)$ and (ox, oy) respectively are as under

$$D = \iiint\limits_V \rho yz dV \tag{11.6.7}$$

$$E = \iiint\limits_V \rho zx dV \tag{11.6.8}$$

$$F = \iiint\limits_V \rho xy dV \tag{11.6.9}$$

For laminas in xy plane, we put $z = 0$, then

$$\begin{cases} A = \sum_{i=1}^{n} m_i y_i^2 \\ B = \sum_{i=1}^{n} m_i x_i^2 \\ C = \sum_{i=1}^{n} m_i \left(x_i^2 + y_i^2\right) \end{cases} \tag{11.6.10}$$

Similarly, the products of inertia *w.r.t.* pair of axes (oy, oz), (oz, ox) and (ox, oy) respectively are as under

$$\begin{cases} D = 0 \\ E = 0 \\ F = \sum_{i=1}^{n} m_i x_i y_i \end{cases} \tag{11.6.11}$$

and for continuous distribution of mass

$$\begin{aligned} A &= \iint_S \rho y^2 dA \\ B &= \iint_S \rho x^2 dA \\ C &= \iint_S \rho \left(x^2 + y^2\right) dA \end{aligned}$$

Similarly, the products of inertia *w.r.t.* pair of axes (oy, oz), (oz, ox) and (ox, oy) respectively are as under

$$\begin{aligned} D &= 0 \\ E &= 0 \\ F &= \iint_S \rho xy dx dy \end{aligned}$$

11.7 Parallel Axis Theorem

The parallel axis theorem states that the moment of inertia of a body about any axis is equal to the moment of inertia about a parallel axis through the center of mass, plus the mass of the body × the square of the distance between the two axes.

Proof: Consider a rigid body in a regular trihedral system. Let C be the center of mass and M be the total mass of the body. *i.e.*

$$M = \sum_{i=1}^{n} m_i \tag{11.7.1}$$

Let m_i be the mass of the ith particle. Then for its $M.I$ about $z-axis$, we consider another axis PC parallel to it. Let d be the distance between these two parallel axes. Let $\vec{r_i}$ and $\vec{a_i}$ be the position vectors of the particle P of mass m_i relative to O and C respectively. Let r_c be the position vector of point C relative to O. Then

$$\vec{r_i} = \vec{r}_C + \vec{a}_i \quad (11.7.2)$$

The moment of inertia of mass m_i about $z-axis$ is

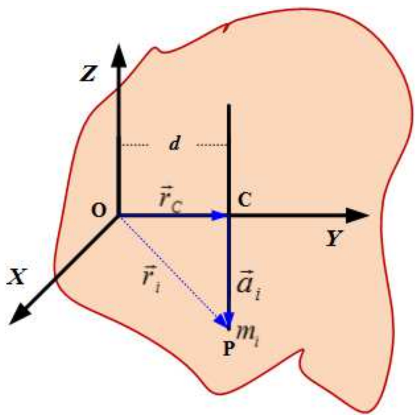

Figure 11.10: Parallel axis theorem

$$I_{zz} = m_i r_i^2$$

And the moment of inertia of the body about $z-axis$ is

$$\begin{aligned} I_{zz} &= \sum_{i=1}^{n} m_i r_i^2 \\ &= \sum_{i=1}^{n} m_i \left(\vec{r_i} \cdot \vec{r_i}\right) \end{aligned}$$

using (11.7.2) we have

$$\begin{aligned} I_{zz} &= \sum_{i=1}^{n} m_i \left(\vec{r_C} + \vec{a_i}\right) \cdot \left(\vec{r_C} + \vec{a_i}\right) \\ &= \sum_{i=1}^{n} m_i \left(r_C^2 + 2r_C a_i + a_i^2\right) \\ &= r_C^2 \sum_{i=1}^{n} m_i + 2r_C \sum_{i=1}^{n} m_i a_i + \sum_{i=1}^{n} m_i a_i^2 \quad (11.7.3) \end{aligned}$$

From Fig. 11.10, we have $r_C^2 = d^2$. Also the term $m_i a_i$ represents moment of a force about a point lying on its line of action. So the term $m_i a_i = 0$. And the term $\sum_{i=1}^{n} m_i a_i^2$ represents $M.I$ of m_i about PC axis. Using above results along with (11.7.1), (11.7.3) can be written as

$$I_{zz} = d^2 M + I_{PC} \tag{11.7.4}$$

11.8 Perpendicular Axis Theorem

For a $2D$ object (a thin plate) the moment of inertia about a perpendicular axis equals the sum of the moments of inertia about any two axes at right angles through the same point in the plane. It states that the moment of inertia of the lamina about the $z - axis$ is equal to the sum of the moments of inertia about the $x-$ and $y - axes$.

$$I_{zz} = I_{xx} + I_{yy}$$

Proof: Consider a regular trihedral system. Let Ox, Oy are the axes in the xy plane of lamina and Oz be the normal axis, Let a particle of mass m, lying at point P in xy plane. Then its distance from x axis is x and from y axis is y. Then its $M.I.$ about Ox and Oy axes are respectively:

$$I_{xx} = mx^2 \tag{11.8.1}$$
$$I_{yy} = my^2 \tag{11.8.2}$$

Let r be its distance from oz axis $i.e.$ perpendicular axis(see Fig 11.11). Then the $M.I.$ about Oz axis is

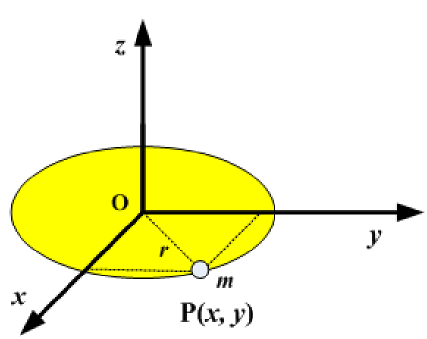

Figure 11.11: M.I. about perpendicular axis

$$\begin{aligned} I_{zz} &= mr^2 \\ &= m\left(x^2+y^2\right) \\ &= mx^2+my^2 \end{aligned} \quad (11.8.3)$$

Using (11.8.1) and (11.8.2), (11.8.3) becomes

$$I_{zz} = I_{xx} + I_{yy}$$

Similarly for a system of n particles, the $M.I.$ about perpendicular axis, i.e., oz axis is

$$\begin{aligned} I_{zz} &= \sum_i m_i r_i^2 \\ &= \sum_i m_i\left(x_i^2+y_i^2\right) \\ &= I_{xx}+I_{yy} \end{aligned}$$

11.8.1 Converse of Perpendicular Axis Theorem

It states that the moment of inertia of an object about the $z-axis$ is equal to the sum of the moments of inertia about the $x-$ and $y-axes$ $i.e.$

$$I_{zz} = I_{xx} + I_{yy} \quad (11.8.4)$$

then the object is a plane lamina.
Proof: Consider a particle P of masses m in a regular trihedral system $OXYZ$. Its perpendicular distance from OZ axis is

$$\begin{aligned} d &= PR = OQ \\ &= \sqrt{x^2+y^2} \end{aligned} \quad (11.8.5)$$

Then its moment of inertia about OZ axis is

$$\begin{aligned} I_{zz} &= md^2 \\ &= m(x^2+y^2) \end{aligned} \quad (11.8.6)$$

Similarly about OX axis

$$I_{xx} = m(y^2+z^2) \quad (11.8.7)$$

and OY axis

$$I_{yy} = m(x^2+z^2) \quad (11.8.8)$$

Using (11.8.5), (11.8.5) and (11.8.5) in (11.8.4), we have

$$\begin{aligned} mx^2+my^2 &= my^2+mz^2+mx^2+mz^2 \\ 2mz^2 &= 0 \end{aligned}$$

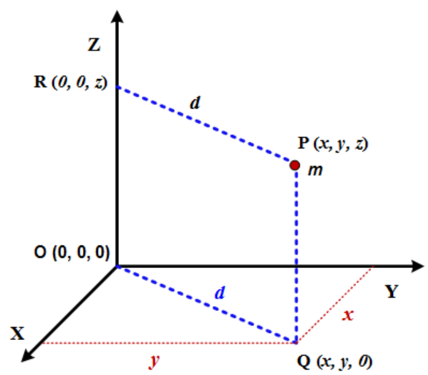

Figure 11.12: Position of a body

Since $2m \neq 0, \Rightarrow z = 0$
Hence the given object is a plane.
Similarly for a system of n particles

$$\begin{aligned}\sum_{i=1}^{n} m_i x^2 + \sum_{i=1}^{n} m_i y^2 &= \sum_{i=1}^{n} m_i y^2 + \sum_{i=1}^{n} m_i z^2 + \sum_{i=1}^{n} m_i x^2 + \sum_{i=1}^{n} m_i z^2 \\ 2\sum_{i=1}^{n} m_i {z_i}^2 &= 0\end{aligned}$$

Since $2\sum_{i=1}^{n} m_i \neq 0, \Rightarrow z_i = 0 \ \forall i$
Hence the given object is a plane.

11.9 Angular Momentum of a Rigid Body

Angular momentum can be subdivided as the body can rotate about instantaneous axis and about fixed axis.

11.9.1 Angular Momentum of a Body Rotating About an Instantaneous Axis

Consider a system of n particles rotating about an axis OC through O with angular velocity $\vec{\omega}$. Let $\hat{a}$ be a unit vector in the direction of OC axis having direction cosines λ, μ, ν. Then

$$\hat{a} = \langle \lambda, \mu, \nu \rangle \tag{11.9.1}$$

Let a particle of mass m_i lying at P having position vector $\vec{r_i}$. Then

$$\vec{OP} = \vec{r} = \langle x_i, y_i, z_i \rangle \tag{11.9.2}$$

Let d be its perpendicular distance from OC axis. Then

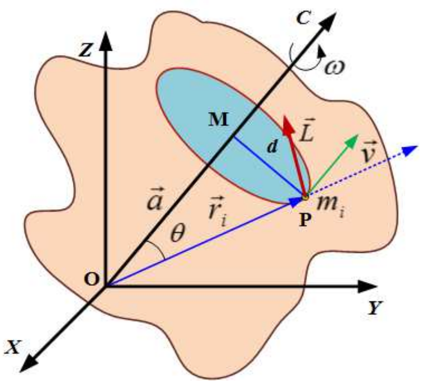

Figure 11.13: Angular Momentum of a rigid body about instantaneous axis

$$\begin{aligned} d = PM &= r\sin\theta = |\vec{r}||\hat{a}|\sin\theta \\ &= |\vec{r}\times\hat{a}| \end{aligned} \tag{11.9.3}$$

Using (11.9.1) and (11.9.2), $\vec{r}\times\hat{a}$ is

$$\begin{aligned} \vec{r}\times\hat{a} &= \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_i & y_i & z_i \\ \lambda & \mu & \nu \end{pmatrix} \\ &= \langle \nu y_i - \mu z_i, \lambda z_i - \nu x_i, \mu x_i - \lambda y_i \rangle \end{aligned}$$

(11.9.3) can be written as

$$\begin{aligned} d &= \sqrt{(\nu y_i - \mu z_i)^2 + (\lambda z_i - \nu x_i)^2 + (\mu x_i - \lambda y_i)^2} \\ &= \sqrt{\lambda^2\left(y_i^2+z_i^2\right) + \mu^2\left(z_i^2+x_i^2\right) + \nu^2\left(x_i^2+y_i^2\right) - 2\mu\nu y_i z_i - 2\lambda\nu x_i z_i - 2\lambda\mu x_i y_i} \end{aligned}$$

or

$$\begin{aligned} d^2 &= \lambda^2\left(y_i^2+z_i^2\right) + \mu^2\left(z_i^2+x_i^2\right) + \nu^2\left(x_i^2+y_i^2\right) \\ &- 2\mu\nu y_i z_i - 2\lambda\mu x_i z_i - 2\lambda\mu x_i y_i \end{aligned} \tag{11.9.4}$$

Hence $M.I.$ of mass m_i about OC axis with unit vector $\langle \lambda, \mu, \nu \rangle$ is

$$\begin{aligned} I_{OC} &= m_i \left[\lambda^2 \left(y_i^2 + z_i^2\right) + \mu^2 \left(z_i^2 + x_i^2\right) + \nu^2 \left(x_i^2 + y_i^2\right)\right] \\ &- m_i \left[2\mu\nu y_i z_i + 2\lambda\nu x_i z_i + 2\lambda\mu x_i y_i\right] \end{aligned} \tag{11.9.5}$$

and the $M.I.$ of of the body about OC axis is

$$\begin{aligned} I_{OC} &= \sum_{i=1}^{n} m_i \left[\lambda^2 \left(y_i^2 + z_i^2\right) + \mu^2 \left(z_i^2 + x_i^2\right) + \nu^2 \left(x_i^2 + y_i^2\right)\right] \\ &- \sum_{i=1}^{n} m_i \left[2\mu\nu y_i z_i + 2\lambda\nu x_i z_i + 2\lambda\mu x_i y_i\right] \end{aligned} \tag{11.9.6}$$

11.9.2 Angular Momentum of a Body Rotating About a Fixed Point and Fixed Axis

Consider a system of n particles rotating about an axis OC through O with angular velocity $\vec{\omega}$. Let one of the points of the rigid body is fixed, so translation motion is absent. Let a particle of mass m_i lying at P having position vector $\vec{r}_i$. Then its linear velocity in terms of angular velocity is

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i \tag{11.9.7}$$

The linear momentum p of the ith particle is

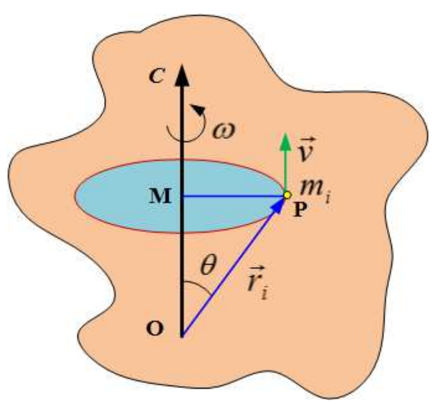

Figure 11.14: Angular momentum of a body

$$\vec{p}_i = m_i \vec{v}_i \tag{11.9.8}$$

Using (11.9.7), (11.9.8) becomes

$$\vec{p}_i = m_i (\vec{\omega} \times \vec{r}_i) \tag{11.9.9}$$

The angular momentum L_i of the ith particle is

$$\vec{L}_i = \vec{r}_i \times \vec{p}_i \tag{11.9.10}$$

Using (11.9.9), (11.9.10) becomes

$$\vec{L}_i = \vec{r}_i \times m_i (\vec{\omega} \times \vec{r}_i) \tag{11.9.11}$$

The angular momentum L of the rigid body is

$$\begin{aligned} L = \sum_{i=1}^{n} \vec{L}_i &= \sum_{i=1}^{n} m_i [\vec{r}_i \times (\vec{\omega} \times \vec{r}_i)] \\ &= \sum_{i=1}^{n} m_i [(\vec{r}_i \cdot \vec{r}_i)\, \vec{\omega} - (\vec{r}_i \cdot \vec{\omega})\, \vec{r}_i] \\ &= \sum_{i=1}^{n} m_i r_i^2 \vec{\omega} - \sum_{i=1}^{n} m_i (\vec{r}_i \cdot \vec{\omega})\, \vec{r}_i \end{aligned} \tag{11.9.12}$$

Next we consider a regular trihedral system. Then each $\vec{r}_i$ is

$$\vec{r}_i = \langle x_i, y_i, z_i \rangle \tag{11.9.13}$$

and ω can be written as

$$\vec{\omega} = \langle \omega_x, \omega_y, \omega_z \rangle$$

Then we have

$$r_i^2 = x_i^2 + y_i^2 + z_i^2 \tag{11.9.14}$$

and

$$\vec{r}_i \cdot \vec{\omega} = x_i \omega_x + y_i \omega_y + z_i \omega_z$$

Using above results (11.9.13)

$$\begin{aligned} \langle L_x, L_y, L_z \rangle &= \sum_{i=1}^{n} m_i \left(x_i^2 + y_i^2 + z_i^2 \right) \langle \omega_x, \omega_y, \omega_z \rangle \\ &- \sum_{i=1}^{n} m_i (x_i \omega_x + y_i \omega_y + z_i \omega_z) \langle x_i, y_i, z_i \rangle \end{aligned} \tag{11.9.15}$$

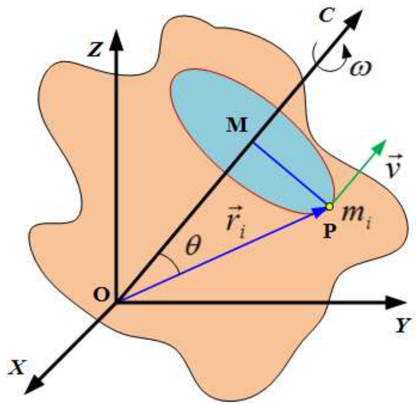

Figure 11.15: Angular momentum of a rigid body

Next x component of (11.9.15) can be written as

$$\begin{aligned} L_x &= \sum_{i=1}^{n} m_i \left(x_i^2 + y_i^2 + z_i^2\right) \omega_x - \sum_{i=1}^{n} m_i \left(x_i^2 \omega_x + x_i y_i \omega_y + x_i z_i \omega_z\right) \\ &= \sum_{i=1}^{n} m_i \left(y_i^2 + z_i^2\right) \omega_x - \sum_{i=1}^{n} m_i x_i y_i \omega_y - \sum_{i=1}^{n} m_i x_i z_i \omega_z \end{aligned} \quad (11.9.16)$$

Using (11.5.7), (11.6.1) and (11.6.2), (11.9.16) can be written as

$$L_x = I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \quad (11.9.17)$$

Using M.I. about coordinate axes (11.5.6), (11.5.7), (11.5.8) and product of inertia (11.6.1), (11.6.2) and (11.6.3), the x, y, z components of angular momentum can be written as

$$\begin{aligned} L_x &= I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ L_y &= -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \\ L_z &= -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z \end{aligned}$$

In compact form it can be written as

$$L_i = \sum_{j=1}^{3} I_{ij}\omega_j \qquad i = 1, 2, 3 \quad (11.9.18)$$

In another format it can be written as

$$\begin{aligned} L_x &= A\omega_x - F\omega_y - E\omega_z \\ L_y &= -F\omega_x + B\omega_y - D\omega_z \\ L_z &= -E\omega_x - D\omega_y + C\omega_z \end{aligned}$$

and in matrix form can be written as

$$\begin{pmatrix} L_x \\ \\ L_y \\ \\ L_z \end{pmatrix} = \begin{pmatrix} A & -F & -E \\ \\ -F & B & -D \\ \\ -E & -D & C \end{pmatrix} \begin{pmatrix} \omega_x \\ \\ \omega_y \\ \\ \omega_z \end{pmatrix} \quad (11.9.19)$$

and in symbolic form can be written as

$$\vec{L} = [I]\vec{\omega}$$

The inertia matrix $[I]$ gives us an idea about how the mass is distributed in a rigid body. (11.9.19) represents angular momentum of a rigid body about a fixed point in terms of inertia. It follows from (11.9.19), that the tensors of inertia are always symmetric. In many

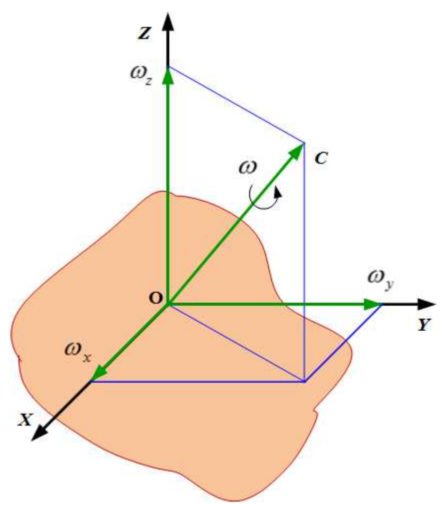

Figure 11.16: Angular momentum when rotation of a body is about arbitrary axis

situations of importance, even for bodes of some symmetry, the angular momentum vector L and the angular velocity vector ω are not parallel.

Corollary 11.9.1. *Show that*

$$L_k = I_k\omega \quad (11.9.20)$$

where L_k is the angular momentum and I_k is the moment of inertia about $\hat{k}$ axis.

Solution: Let the rotation is about $\hat{k}$ axis. Then

$$\vec{\omega} = \omega\hat{k}$$

where is a unit vector pointing along the object's axis of rotation (in the sense given by the right-hand grip rule). then (11.9.7) can be written as

$$\vec{v}_i = \omega\hat{k} \times \vec{r}_i \tag{11.9.21}$$

The angular momentum L_i of the ith particle (11.9.10)

$$\begin{aligned} \vec{L}_i &= \vec{r}_i \times \vec{p}_i = m_i\vec{r}_i \times \vec{v}_i \\ &= \vec{r}_i \times \vec{p}_i = \omega m_i\vec{r}_i \times \left(\hat{k} \times \vec{r}_i\right) \end{aligned}$$

And for the whole system

$$L = \sum_{i=1}^{n} \vec{L}_i = \omega \sum_{i=1}^{n} m_i \left[\vec{r}_i \times \left(\hat{k} \times \vec{r}_i\right)\right]$$

We can write the component of angular momentum along the axis of rotation as

$$L_k = \vec{L}.\hat{k} = \omega \sum_{i=1}^{n} m_i \left[\hat{k} \cdot \vec{r}_i \times \left(\hat{k} \times \vec{r}_i\right)\right] \tag{11.9.22}$$

since $\vec{a} \cdot \vec{b} \times \vec{c} = \vec{a} \times \vec{b} \cdot \vec{c}$, (11.9.22) can be rewritten as

$$L_k = \omega \sum_{i=1}^{n} m_i \left[\left(\hat{k} \times \vec{r}_i\right) \cdot \left(\hat{k} \times \vec{r}_i\right)\right] \tag{11.9.23}$$

$$= \omega \sum_{i=1}^{n} m_i \left|\hat{k} \times \vec{r}_i\right|^2 \tag{11.9.24}$$

Next the term $\sum_{i=1}^{n} m_i \left|\hat{k} \times \vec{r}_i\right|^2$ is the moment of inertia about $\hat{k}$ axis. Let it be I_k. Then (11.9.23) can be rewritten as

$$L_k = I_k\omega \tag{11.9.25}$$

It means that the component of a rigid body's angular momentum vector along its axis of rotation is simply the product of the body's moment of inertia about this axis and the body's angular velocity. Does this result imply that we can automatically write

$$\vec{L} = I\vec{\omega} \tag{11.9.26}$$

Unfortunately, in general, the answer to the above question is no! This conclusion follows because the body may possess non-zero angular momentum components about axes perpendicular to its axis of rotation. Thus, in general, the angular momentum vector of a rotating body is not parallel to its angular velocity vector. This is a major difference from translational motion, where linear momentum is always found to be parallel to linear velocity.

11.10 Kinetic Energy of a Body Rotating About a Fixed Point

Consider a system of n particles rotating about an axis OC through O with angular velocity $\vec{\omega}$. Let a particle of mass m_i lying at P having position vector $\vec{r}_i$. Then its linear velocity in terms of angular velocity is

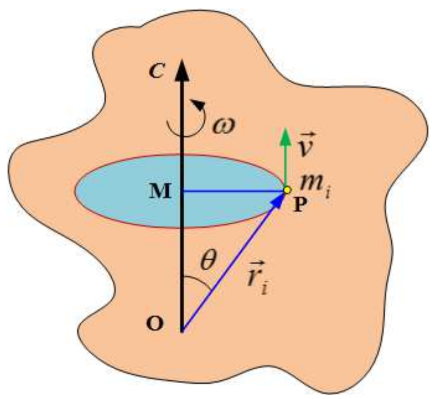

Figure 11.17: Rotational Kinetic energy of a body

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i \tag{11.10.1}$$

And the kinetic energy of the ith particle is

$$\begin{aligned} K_i &= \frac{1}{2} m_i v_i^2 \\ &= \frac{1}{2} m_i |\vec{v}_i|^2 \\ &= \frac{1}{2} m_i (\vec{v}_i \cdot \vec{v}_i) \end{aligned}$$

Next the kinetic energy of the whole system is

$$K = \sum_{i=1}^{n} K_i = \frac{1}{2} \sum_{i=1}^{n} m_i (\vec{v}_i \cdot \vec{v}_i) \tag{11.10.2}$$

Using (11.10.1), (11.10.2) becomes

$$
\begin{aligned}
K = \quad &= \frac{1}{2}\sum_{i=1}^{n} m_i(\vec{\omega}\times\vec{r}_i)\cdot(\vec{\omega}\times\vec{r}_i) \\
&= \frac{1}{2}\sum_{i=1}^{n} m_i\vec{\omega}\cdot[\vec{r}_i\times(\vec{\omega}\times\vec{r}_i)] \\
&= \frac{1}{2}\vec{\omega}\cdot\sum_{i=1}^{n} m_i(\vec{r}_i\times\vec{v}_i) \\
&= \frac{1}{2}\vec{\omega}\cdot\sum_{i=1}^{n} (\vec{r}_i\times m_i\vec{v}_i) \\
&= \frac{1}{2}\vec{\omega}\cdot\sum_{i=1}^{n} (\vec{r}_i\times\vec{p}_i) \\
&= \frac{1}{2}\vec{\omega}\cdot\sum_{i=1}^{n} \vec{L}_i \qquad (11.10.3) \\
&= \frac{1}{2}\vec{\omega}\cdot\vec{L} \qquad (11.10.4)
\end{aligned}
$$

Next we consider a regular trihedral system. Its angular momentum in terms of inertia is

$$
\vec{L}_i = \sum_{i=1}^{3} I_{ij}\vec{\omega}_j, \quad i = j = 1, 2, 3 \qquad (11.10.5)
$$

and ω can be written as

$$
\vec{\omega} = \langle \omega_1, \omega_2, \omega_3 \rangle
$$

Using (11.10.5), (11.10.3) becomes

$$
K = \frac{1}{2}\sum_{i,j=1}^{3} I_{ij}\vec{\omega}_i\cdot\vec{\omega}_j, \quad i = j = 1, 2, 3 \qquad (11.10.6)
$$

Consider $OXYZ$ a regular trihedral system, then

$$
K = \frac{1}{2}\left[I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 + 2I_{xy}\omega_x\omega_y + 2I_{yz}\omega_y\omega_z + 2I_{zx}\omega_z\omega_x\right]
$$

If the body rotates about z axis with angular velocity ω then

$$
\omega_z = \omega, \ , \ \omega_x = 0 = \omega_y
$$

Then above relation becomes

$$
K = \frac{1}{2}I_{zz}\omega_z^2
$$

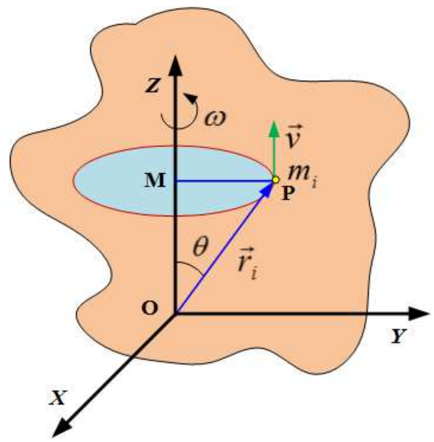

Figure 11.18: Rotational kinetic energy when rotation of a body is about z axis

In this case the components of angular momenta are

$$\begin{aligned} L_x &= I_{zx}\omega_z \\ L_y &= I_{yz}\omega_z \\ L_z &= I_{zz}\omega_z \end{aligned}$$

Which shows that in general, the direction of angular velocity and the direction of the angular momenta are different.
If the axis of rotation is coordinate axis it follows.

$$K = \frac{1}{2}I\omega^2 \qquad (11.10.7)$$

11.11 Principal Axes

Consider a regular trihedral system. A body of mass m is rotating about an axis through a point O with angular velocity $\vec{\omega}$. If the angular momentum $\vec{L}$ is parallel to angular velocity $\vec{\omega}$ then the axis of rotation is known as principal axis. In equation form we can write as

$$\vec{L} = k\vec{\omega} \qquad (11.11.1)$$

$$\langle L_x, L_y, L_z\rangle = \langle k\omega_x, k\omega_y, k\omega_z\rangle \qquad (11.11.2)$$

where k is some constant.
For a three-dimensional body, it is always possible to find three mutually orthogonal axis (a regular trihedral system), for which the products of inertia are zero, and the inertia

matrix takes a diagonal form. Then the rotation is about only one of these axis, and the angular momentum vector is parallel to the angular velocity vector. In most problems, such systems are preferred. For symmetric bodies, it may be obvious which axis are principle axis. However, for an irregular-shaped body this coordinate system may be difficult to determine by inspection.

Theorem 11.11.1. *Prove that in general, there are three principal axes through a point of rigid body.*

Proof Consider (11.9.19)

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Using (11.11.2) we can write

$$\begin{pmatrix} k\omega_x \\ k\omega_y \\ k\omega_z \end{pmatrix} = \begin{pmatrix} A\omega_x & -F\omega_y & -E\omega_z \\ -F\omega_x & B\omega_y & -D\omega_z \\ -E\omega_x & -D\omega_y & C\omega_z \end{pmatrix}$$

or can be written as

$$\begin{aligned} k\omega_x &= A\omega_x - F\omega_y - E\omega_z \\ k\omega_y &= -F\omega_x + B\omega_y - D\omega_z \\ k\omega_z &= -E\omega_x - D\omega_y + C\omega_z \end{aligned}$$

and we have

$$\begin{aligned} (A-k)\omega_x - F\omega_y - E\omega_z &= 0 \\ -F\omega_x + (B-k)\omega_y - D\omega_z &= 0 \\ -E\omega_x - D\omega_y + (C-k)\omega_z &= 0 \end{aligned} \qquad (11.11.3)$$

The above system has a non-zero solution only if

$$\begin{vmatrix} A-k & -F & -E \\ -F & B-k & -D \\ -E & -D & C-k \end{vmatrix} = 0$$

$$\left| \begin{pmatrix} A-k & -F & -E \\ -F & B-k & -D \\ -E & -D & C-k \end{pmatrix} + k \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$|I - kE| = 0 \tag{11.11.4}$$

Where E is identity matrix.
(11.11.13) is known as characteristic equation of symmetric inertia matrix. This equation has three real roots (since matrix is symmetric), known as eigen values. Let $k = (k_1, k_2, k_3)$ be the corresponding roots. Let (λ, μ, ν) has values as

$$\begin{aligned} (\lambda_1, \mu_1, \nu_1) \to k &= k_1 \\ (\lambda_2, \mu_2, \nu_2) \to k &= k_2 \\ (\lambda_3, \mu_3, \nu_3) \to k &= k_3 \end{aligned}$$

These three sets of value determine three principal axes $\hat{a}_1, \hat{a}_2, \hat{a}_3$ has the form

$$\hat{a}_j = \langle \lambda_j, \mu_j, \nu_j \rangle \quad \text{with } j = 1, 2, 3$$

Another Approach Let $\hat{a}$ be a unit vector along principal axis of body through O. Then angular velocity vector is

$$\vec{\omega} = \omega \hat{a}$$

and angular momentum is

$$\vec{L} = L\hat{a}$$

using (11.11.1)

$$\vec{L} = k\omega\hat{a}$$

Also consider (11.9.12)

$$L = \sum_{i=1}^{n} m_i r_i^2 \vec{\omega} - \sum_{i=1}^{n} m_i \left(\vec{r}_i \cdot \vec{\omega}\right) \vec{r}_i$$

using above results we have

$$\begin{aligned} k\omega\hat{a} &= \sum_{i=1}^{n} m_i r_i^2 \omega\hat{a} - \sum_{i=1}^{n} m_i \left(\vec{r}_i \cdot \omega\hat{a}\right) \vec{r}_i \\ &= \omega \left(\sum_{i=1}^{n} m_i r_i^2 \hat{a} - \sum_{i=1}^{n} m_i \left(\vec{r}_i \cdot \hat{a}\right) \vec{r}_i \right) \end{aligned}$$

Canceling ω on both sides, we have

$$k\hat{a} = \sum_{i=1}^{n} m_i r_i^2 \hat{a} - \sum_{i=1}^{n} m_i \left(\vec{r}_i \cdot \hat{a}\right) \vec{r}_i$$

or we can write

$$\sum_{i=1}^{n} m_i \left(\vec{r}_i \cdot \hat{a}\right) \vec{r}_i = \left(\sum_{i=1}^{n} m_i r_i^2 - k \right) \hat{a} \tag{11.11.5}$$

Let

$$\vec{r}_i = \langle x_i, y_i, z_i \rangle$$

and

$$\hat{a} = \langle \lambda, \mu, \nu \rangle$$

Then

$$r_i^2 = x_i^2 + y_i^2 + z_i^2$$

and

$$\vec{r}_i \cdot \hat{a} = x_i\lambda + y_i\mu + z_i\nu$$

using above results (11.11.5) becomes

$$\sum_{i=1}^{n} m_i \left(x_i\lambda + y_i\mu + z_i\nu\right) \langle x_i, y_i, z_i \rangle = \left(\sum_{i=1}^{n} m_i \left(x_i^2 + y_i^2 + z_i^2\right) - k \right) \langle \lambda, \mu, \nu \rangle$$

Since the above two vectors are equal, so their corresponding entries must be equal

$$\begin{aligned}
\sum_{i=1}^{n} m_i\left(x_i^2\lambda + x_i y_i\mu + x_i z_i\nu\right) &= \left(\sum_{i=1}^{n} m_i\left(x_i^2+y_i^2+z_i^2\right) - k\right)\lambda \\
\sum_{i=1}^{n} m_i\left(x_i y_i\lambda + y_i^2\mu + y_i z_i\nu\right) &= \left(\sum_{i=1}^{n} m_i\left(x_i^2+y_i^2+z_i^2\right) - k\right)\mu \\
\sum_{i=1}^{n} m_i\left(x_i z_i\lambda + y_i z_i\mu + z_i^2\nu\right) &= \left(\sum_{i=1}^{n} m_i\left(x_i^2+y_i^2+z_i^2\right) - k\right)\nu
\end{aligned}$$

Let

$$\vec{r}_i = \langle x_i, y_i, z_i\rangle$$

and

$$\hat{a} = \langle \lambda, \mu, \nu\rangle$$

Then

$$r_i^2 = x_i^2 + y_i^2 + z_i^2$$

and

$$\vec{r}_i \cdot \hat{a} = x_i\lambda + y_i\mu + z_i\nu$$

using above results (11.11.5) becomes

$$\sum_{i=1}^{n} m_i\left(x_i\lambda + y_i\mu + z_i\nu\right)\langle x_i, y_i, z_i\rangle = \left(\sum_{i=1}^{n} m_i\left(x_i^2+y_i^2+z_i^2\right) - k\right)\langle \lambda, \mu, \nu\rangle$$

Since the above two vectors are equal, so their corresponding entries must be equal

$$\begin{aligned}
\sum_{i=1}^{n} m_i\left(x_i^2\lambda + x_i y_i\mu + x_i z_i\nu\right) &= \left(\sum_{i=1}^{n} m_i\left(x_i^2+y_i^2+z_i^2\right) - k\right)\lambda \\
\sum_{i=1}^{n} m_i\left(x_i y_i\lambda + y_i^2\mu + y_i z_i\nu\right) &= \left(\sum_{i=1}^{n} m_i\left(x_i^2+y_i^2+z_i^2\right) - k\right)\mu \\
\sum_{i=1}^{n} m_i\left(x_i z_i\lambda + y_i z_i\mu + z_i^2\nu\right) &= \left(\sum_{i=1}^{n} m_i\left(x_i^2+y_i^2+z_i^2\right) - k\right)\nu
\end{aligned}$$

or

$$\begin{aligned}
\left(\sum_{i=1}^{n} m_i\left(y_i^2+z_i^2\right) - k\right)\lambda - \sum_{i=1}^{n} m_i x_i y_i\mu - \sum_{i=1}^{n} m_i x_i z_i\nu &= 0 \\
\left(\sum_{i=1}^{n} m_i\left(x_i^2+z_i^2\right) - k\right)\mu - \sum_{i=1}^{n} m_i x_i y_i\lambda - \sum_{i=1}^{n} m_i y_i z_i\nu &= 0 \\
\left(\sum_{i=1}^{n} m_i\left(x_i^2+y_i^2\right) - k\right)\nu - \sum_{i=1}^{n} m_i x_i z_i\lambda - \sum_{i=1}^{n} m_i y_i z_i\mu &= 0
\end{aligned}$$

Using the notations of $M.I$ about coordinate axis and products of inertia

$$\begin{aligned}(A-k)\lambda - F\mu - E\nu &= 0\\ -F\lambda + (B-k)\mu - D\nu &= 0\\ -E\lambda - D\mu + (C-k)\nu &= 0\end{aligned}$$

Next is the same as above.

Theorem 11.11.2. *Three principal axes through a point of a rigid body are mutually orthogonal.*

Proof Let $\hat{a}_1, \hat{a}_2, \hat{a}_3$ be three principal axes corresponding to eigen values k_1, k_2, k_3 of the characteristic equation

$$\begin{vmatrix} A-k & -F & -E \\ -F & B-k & -D \\ -E & -D & C-k \end{vmatrix} = 0$$

Let all the eigen values k_1, k_2, k_3 are different. Then (11.11.5)

$$\sum_{i=1}^{n} m_i \left(\vec{r}_i \cdot \hat{a}\right) \vec{r}_i = \left(\sum_{i=1}^{n} m_i r_i^2 - k\right) \hat{a}$$

takes the form

$$\sum_{i=1}^{n} m_i \left(\vec{r}_i \cdot \hat{a}_1\right) \vec{r}_i = \left(\sum_{i=1}^{n} m_i r_i^2 - k_1\right) \hat{a}_1 \qquad (11.11.6)$$

$$\sum_{i=1}^{n} m_i \left(\vec{r}_i \cdot \hat{a}_2\right) \vec{r}_i = \left(\sum_{i=1}^{n} m_i r_i^2 - k_2\right) \hat{a}_2 \qquad (11.11.7)$$

$$\sum_{i=1}^{n} m_i \left(\vec{r}_i \cdot \hat{a}_3\right) \vec{r}_i = \left(\sum_{i=1}^{n} m_i r_i^2 - k_3\right) \hat{a}_3 \qquad (11.11.8)$$

Next we eliminate the sums. First consider (11.11.6) and (11.11.7). Scalar Multiplication of (11.11.6) by $\hat{a}_2$ and (11.11.7) by $\hat{a}_1$ and then subtracting we get.

$$(k_1 - k_2)\, \hat{a}_1 \cdot \hat{a}_2 = 0$$

Since k_1 and k_2 are different. *i.e* $k_1 - k_2 \neq 0$. Then

$$\hat{a}_1 \cdot \hat{a}_2 = 0 \qquad (11.11.9)$$

Similarly

$$\hat{a}_1 \cdot \hat{a}_3 = 0 \tag{11.11.10}$$

and

$$\hat{a}_2 \cdot \hat{a}_3 = 0 \tag{11.11.11}$$

(11.11.9), (11.11.10) and (11.11.11) shows that $\hat{a}_1, \hat{a}_2, \hat{a}_3$ are mutually orthogonal.

Remark 11.11.1. For a general three-dimensional body, it is always possible to find three mutually orthogonal axis (a regular trihedral system) for which the products of inertia are zero, and the inertia matrix is a diagonal matrix. In most problems, this would be the preferred system in which to formulate a problem. For a rotation about only one of these axis, the angular momentum vector is parallel to the angular velocity vector. For symmetric bodies, it may be obvious which axis are principle axis. However, for an irregular-shaped body this coordinate system may be difficult to determine by inspection; we will present a general method to determine these axes in the next section.

1. If a body has $k_1 \neq k_2 \neq k_3$ *i.e.* all principal moments of inertia are distinct, then there are exactly three mutually perpendicular axis through O. It is termed as **asymmetric top**
2. If a body has $k_1 \neq k_2 = k_3$ (*i.e* two roots are equal) then there is one principal axis corresponding to k_1 through O, and every line through O and perpendicular to $\hat{a}_1$ axis is a principal axis. Hence we have infinite set of principal axes with one fixed principal axis along ($\hat{a}_1$). It is termed as **symmetrical top**.
3. If a body has $k_1 = k_2 = k_3$ then any three mutually perpendicular axis through O (center of a sphere) are principal axis, it is termed as **spherical top**.
4. If a body has $k_1 = 0$ and $k_2 = k_3$, as for example, two point masses connected by a weightless shaft, or a diatomic molecule, it is called a **rotor**.

Theorem 11.11.3. *If the principal axes are along coordinate axes, then the products of inertia are zero and hence write inertia matrix and angular momentum.*

Proof Consider a regular trihedral system. Let $\hat{a}_1, \hat{a}_2, \hat{a}_3$ be three principal axes with eigen values k_1, k_2, k_3. Let $\hat{a}_1$ be along $x - axis$, $\hat{a}_2$ be along $y - axis$ and $\hat{a}_3$ be along $z - axis$. The position vector r_i of mass m_i relative to principal axes is

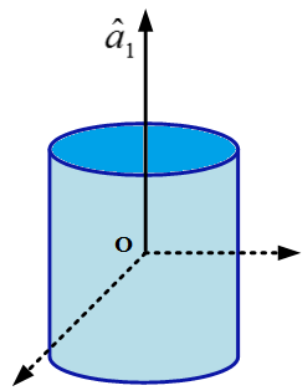

Figure 11.19: Principal axis for cylinder

$$r_i = x_i\hat{a}_1 + y_i\hat{a}_2 + z_i\hat{a}_3 \tag{11.11.12}$$

Consider (11.11.6), (11.11.7) and (11.11.8)

$$\begin{aligned}
\sum_{i=1}^{n} m_i\left(\vec{r}_i \cdot \hat{a}_1\right)\vec{r}_i &= \left(\sum_{i=1}^{n} m_i r_i^2 - k_1\right)\hat{a}_1 \\
\sum_{i=1}^{n} m_i\left(\vec{r}_i \cdot \hat{a}_2\right)\vec{r}_i &= \left(\sum_{i=1}^{n} m_i r_i^2 - k_2\right)\hat{a}_2 \\
\sum_{i=1}^{n} m_i\left(\vec{r}_i \cdot \hat{a}_3\right)\vec{r}_i &= \left(\sum_{i=1}^{n} m_i r_i^2 - k_3\right)\hat{a}_3
\end{aligned}$$

Consider the terms

$$\begin{aligned}
\vec{r}_i \cdot \hat{a}_1 &= \left(x_i\hat{a}_1 + y_i\hat{a}_2 + z_i\hat{a}_3\right)\cdot\hat{a}_1 \\
&= x_i \\
\text{and}\quad \vec{r}_i \cdot \hat{a}_2 &= y_i \\
\vec{r}_i \cdot \hat{a}_3 &= z_i
\end{aligned}$$

$$r_i^2 = x_i^2 + y_i^2 + z_i^2$$

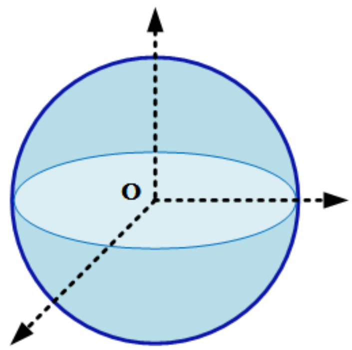

Figure 11.20: Principal axis for sphere

Using above results (11.11.6), (11.11.7) and (11.11.8) becomes.

$$\begin{aligned}\sum_{i=1}^{n} m_i x_i \left(x_i \hat{a}_1 + y_i \hat{a}_2 + z_i \hat{a}_3\right) &= \left(\sum_{i=1}^{n} m_i \left(x_i^2 + y_i^2 + z_i^2\right) - k_1\right) \hat{a}_1 \\ \sum_{i=1}^{n} m_i \left(x_i^2 \hat{a}_1 + x_i y_i \hat{a}_2 + x_i z_i \hat{a}_3\right) &= \left(\sum_{i=1}^{n} m_i \left(x_i^2 + y_i^2 + z_i^2\right) - k_1\right) \hat{a}_1\end{aligned}$$

Comparing coefficients of $\hat{a}_i;\ i = 1, 2, 3$ on both sides, we have

$$\sum_{i=1}^{n} m_i x_i^2 = \left(\sum_{i=1}^{n} m_i \left(x_i^2 + y_i^2 + z_i^2\right) - k_1\right)$$

or

$$k_1 = \sum_{i=1}^{n} m_i \left(y_i^2 + z_i^2\right) = A^*$$

and

$$\begin{aligned}\sum_{i=1}^{n} m_i x_i y_i = 0 &= F^* \\ \sum_{i=1}^{n} m_i x_i z_i = 0 &= E^*\end{aligned}$$

Similarly (11.11.7) gives

$$k_2 = \sum_{i=1}^{n} m_i \left(x_i^2 + z_i^2\right) = B^*$$

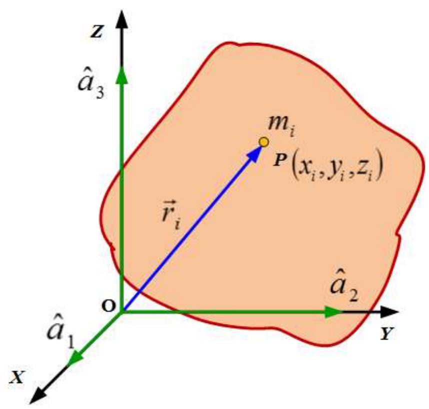

Figure 11.21: Principal axes along coordinate axes

and

$$\begin{aligned}\sum_{i=1}^{n} m_i x_i y_i = 0 &= F^* \\ \sum_{i=1}^{n} m_i y_i z_i = 0 &= D^*\end{aligned}$$

and (11.11.8) gives

$$k_3 = \sum_{i=1}^{n} m_i \left(x_i^2 + y_i^2\right) = C^*$$

and

$$\begin{aligned}\sum_{i=1}^{n} m_i x_i z_i = 0 &= E^* \\ \sum_{i=1}^{n} m_i y_i z_i = 0 &= D^*\end{aligned}$$

From above we see that products of inertia are zero. $i.e$

$$D^* = 0 = E^* = F^*$$

Hence the inertia matrix for principal axes through O is

$$\begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix} = \begin{pmatrix} A^* & 0 & 0 \\ 0 & B^* & 0 \\ 0 & 0 & C^* \end{pmatrix} \tag{11.11.13}$$

Next we define principal axes as:
Three mutually perpendicular lines through any point of a body which are such that the product of inertia about them vanishes are known as principal axes.
Let $\omega_x, \omega_y, \omega_z$ be the components of ω along $\hat{a}_1, \hat{a}_2, \hat{a}_3$ axes respectively, then angular momentum (11.9.19) becomes:

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} A^* & 0 & 0 \\ 0 & B^* & 0 \\ 0 & 0 & C^* \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \tag{11.11.14}$$

Hence A^*, B^*, C^* are also called principal moments of inertia. In this case the angular momentum (11.11.2) is

$$\langle L_x, L_y, L_z \rangle = \langle A^*\omega_x, B^*\omega_y, C^*\omega_z \rangle \tag{11.11.15}$$

11.12 Equimomental Systems

Two systems are said to be equimomental if they have equal moment of inertia about every line in space.

Theorem 11.12.1. *:- The necessary and sufficient condition for two systems to be equimomental are*

1. *They have same total mass.*
2. *They have same centroid.*
3. *They have same principal axes.*

These conditions are sufficient.

Proof:- Part A
If (1) to (3) are true, two systems are equimomental.
Consider two systems, each having mass m (see Fig. 11.23). Let $O = G$ be the common

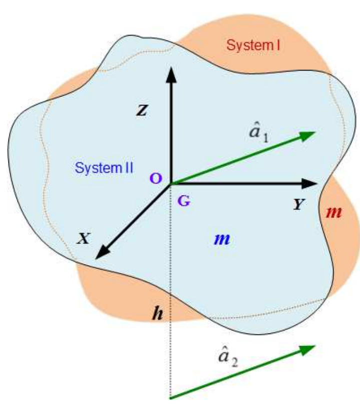

Figure 11.22: Equimomental Systems

centroid of both the system. Let A^*, B^*, C^* be the principal moment of inertia about principal axes through O for both the systems. Let l_1 be any line in the direction of $\hat{a}_1$ (unit vector) with direction cosines $\langle \lambda, \mu, \nu \rangle$ passing through O. Also l_2 be another line in the direction of $\hat{a}_2$ (unit vector) with direction cosines $\langle \lambda, \mu, \nu \rangle$. Let h be the perpendicular distance between $\hat{a}_1$ and $\hat{a}_2$. The principal moment of inertia about $\hat{a}_1$ for both the system is

$$I_{\hat{a}_1} = A^*\lambda^2 + B^*\mu^2 + C^*\nu^2 \quad (11.12.1)$$

And by parallel axes theorem, the principal moment of inertia about $\hat{a}_2$ for both the system is

$$\begin{aligned} I_{\hat{a}_2} &= I_{\hat{a}_1} + mh^2 \\ &= A^*\lambda^2 + B^*\mu^2 + C^*\nu^2 + mh^2 \end{aligned} \quad (11.12.2)$$

Hence both the system have same principal moment of inertia about any line of space. So they are equimomental.

Part B:- The conditions are necessary. Let the two systems are equimomental, then conditions (1) to (3) are true. Let m_1 be the mass of system I with centroid at G_1 and m_2 be the mass of system II with centroid at G_2 (see Fig. 11.23).

Condition (1) They have same total mass.

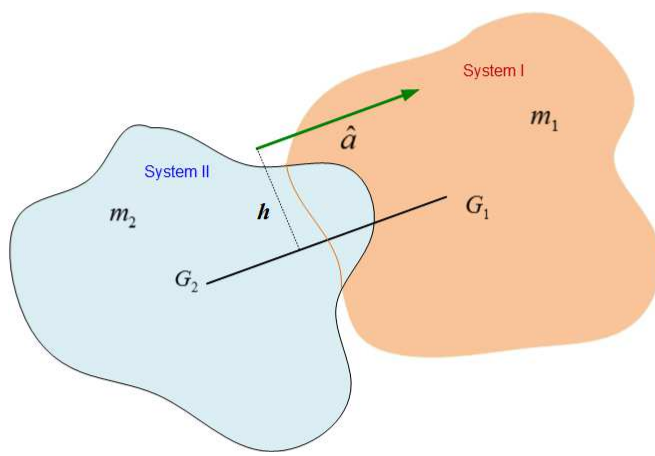

Figure 11.23: Equimomental Systems

Since the systems are equimomental *i.e.* they have same moment of inertia about any line. Let G_1G_2 be the line and I be the moment of inertia of each system about it. Also $\hat{a}$ be a unit vector parallel to G_1G_2 at a distance h. Then by parallel axes theorem, moment of inertia of system I about a line along $\hat{a}$ is

$$I_1 = I + m_1h^2 \quad (11.12.3)$$

moment of inertia of system II about a line along $\hat{a}$ is

$$I_2 = I + m_2h^2 \quad (11.12.4)$$

Since the two systems are equimomental, *i.e.* $I_1 = I_2$, then by (11.12.3) and (11.12.4), we have

$$\begin{aligned} I + m_1 h^2 &= I + m_1 h^2 \\ m_1 &= m_2 = m \end{aligned}$$

Hence both systems have same mass.
Condition (2) They have same centroid.
Since the systems are equimomental *i.e.* they have same moment of inertia about any line. Let I be the moment of inertia about line G_1G_2. (see Fig. 11.24) Let $\hat{a}_1$ and $\hat{a}_2$ be two

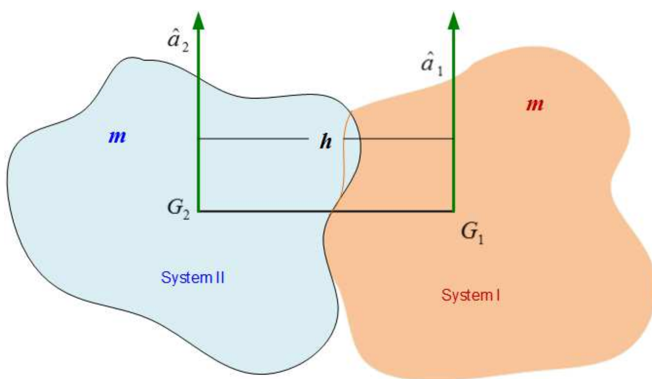

Figure 11.24: Equimomental Systems

unit vectors, perpendicular to line G_1G_2. Hence the distance between these two parallel vectors is $\overline{G_1G_2} = h$. Since the systems are equimomental *i.e.* they have same moment of inertia about any line. Let I be the moment of inertia of either system about a line along $\hat{a}_1$. Using parallel axes theorem, the moment of inertia of system I about a line along $\hat{a}_2$ is

$$I_1 = I + mh^2 \tag{11.12.5}$$

and moment of inertia of system II about about a line along $\hat{a}_2$ is

$$I_2 = I - mh^2 \tag{11.12.6}$$

Since the two systems are equimomental, *i.e.* $I_1 = I_2$, then by (11.12.5) and (11.12.6), we have

$$\begin{aligned} I + mh^2 &= I - mh^2 \\ 2mh^2 &= 0 \end{aligned}$$

since $m \neq 0$, Hence we have

$$h = 0$$

Hence both systems have same centroid. *i.e.* $G_1 = G_2 = G$

Condition (3) They have same principal axes.

Since the systems are equimomental *i.e.* they have same moment of inertia about every line through their common centroid. Hence they have same principal axes and principal moments of inertia.

11.13 Coplanar Distribution

Theorem 11.13.1. *Show that for a two dimensional mass distribution (lamina), one of the principal axes at origin O is inclined at an angle θ to the $x-axis$ through O, such that*

$$\tan 2\theta = \frac{2F}{B-A}$$

where A, B, F have their usual meanings.

Proof

Consider a fixed $xy-$cartesian system OXY. Let a particle of mass m be at $P\,(x_i, y_i)$ (see Fig. 11.25). Using (11.6.10) the moments of inertia of mass distribution (lamina) about

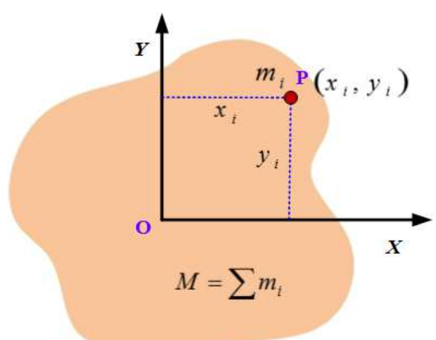

Figure 11.25: Equimomental Systems

coordinate axes $(Ox, Oy)$are

$$A = \sum_{i=1}^{n} m_i y_i^2 \tag{11.13.1}$$

$$B = \sum_{i=1}^{n} m_i x_i^2 \tag{11.13.2}$$

and using(11.6.11) the product of inertia F is

$$F = \sum_{i=1}^{n} m_i x_i y_i \tag{11.13.3}$$

Introduce another $xy-$coordinate system $Ox^{'}y^{'}$ rotatable with same origin O. Let it be rotated about the origin through an angle θ with $x^{'}y^{'}-$coordinate system. As shown in the figure 11.26. Then $P\left(x_i^{'}, y_i^{'}\right)$ in terms of $P\left(x_i, y_i\right)$ is (using (**??**) and (**??**))

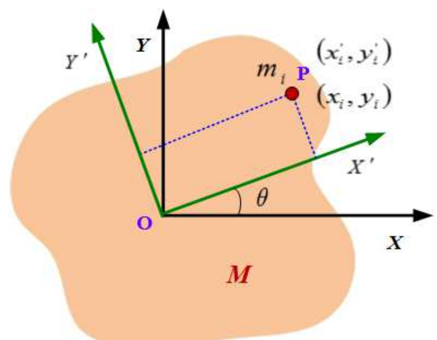

Figure 11.26: Equimomental Systems

$$\begin{aligned} x_i^{'} &= x_i \cos\theta - y_i \sin\theta \\ y_i^{'} &= x_i \sin\theta + y_i \cos\theta \end{aligned}$$

Using (11.6.10) the moments of inertia of lamina about coordinate axes $(Ox^{'}, Oy^{'})$ are

$$\begin{aligned} A_{Ox'} &= \sum_{i=1}^{n} m_i \left(y_i^{'}\right)^2 \\ B_{Oy'} &= \sum_{i=1}^{n} m_i \left(x_i^{'}\right)^2 \end{aligned}$$

and using(11.6.11) the product of inertia F is

$$F_{x'y'} = \sum_{i=1}^{n} m_i x_i^{'} y_i^{'}$$

Using above transformations, we have

$$\begin{aligned} A_{Ox'} &= \sum_{i=1}^{n} m_i \left(x_i \sin\theta + y_i \cos\theta\right)^2 \\ &= \sin^2\theta \sum_{i=1}^{n} m_i x_i^2 + \cos^2\theta \sum_{i=1}^{n} m_i y_i^2 + 2\sin\theta\cos\theta \sum_{i=1}^{n} m_i x_i y_i \end{aligned}$$

Using (11.13.1), (11.13.2) and (11.13.3), we have

$$A_{Ox'} = A\sin^2\theta + B\cos^2\theta + F\sin 2\theta \qquad (11.13.4)$$

Similarly

$$\begin{aligned} B_{Oy'} &= \sum_{i=1}^{n} m_i \left(x_i \cos\theta - y_i \sin\theta\right)^2 \\ &= A\cos^2\theta + B\sin^2\theta - F\sin 2\theta \qquad (11.13.5) \end{aligned}$$

and

$$\begin{aligned} F_{x'y'} &= \sum_{i=1}^{n} m_i \left(x_i \cos\theta - y_i \sin\theta\right)\left(x_i \sin\theta + y_i \cos\theta\right) \\ &= \sin\theta\cos\theta \left(\sum_{i=1}^{n} m_i x_i^2 - \sum_{i=1}^{n} m_i y_i^2\right) + \left(\cos^2\theta - \sin^2\theta\right) \sum_{i=1}^{n} m_i x_i y_i \\ &= (A - B)\frac{1}{2}\sin 2\theta + F\cos 2\theta \end{aligned}$$

The axes $Ox^{'}, Oy^{'}$ will be principal axes if

$$\begin{aligned} F_{x'y'} &= 0 \\ (A - B)\frac{1}{2}\sin 2\theta + F\cos 2\theta &= 0 \end{aligned}$$

Then we have

$$\tan 2\theta = \frac{2F}{B - A} \qquad (11.13.6)$$

or

$$\theta = \frac{1}{2}\arctan\left(\frac{2F}{B - A}\right) \qquad (11.13.7)$$

(11.13.7) is the direction of principal axes relative to co-ordinates axes.

Theorem 11.13.2. *For a $2-$ Dimensional mass distribution (lamina), the value of maximum and minimum moment of inertia about lines passing through a point O are attained through principal axes at O.*

Proof

The maximum/minimum (extreme) values of $A_{Ox'}, B_{Oy'}$ can also be obtained.
Consider (11.13.4)

$$\begin{aligned}
A_{Ox'} &= A\sin^2\theta + B\cos^2\theta + F\sin 2\theta \\
&= \frac{1}{2}\left[A\sin^2\theta + B\cos^2\theta\right] + \frac{1}{2}\left[A\sin^2\theta + B\cos^2\theta\right] + F\sin 2\theta \\
&= \frac{1}{2}\left[A\left(1-\cos^2\theta\right) + B\cos^2\theta\right] + \frac{1}{2}\left[A\sin^2\theta + B\left(1-\sin^2\theta\right)\right] + F\sin 2\theta \\
&= \frac{1}{2}(A+B) + \frac{1}{2}\left[(B-A)\left(\cos^2\theta - \sin^2\theta\right)\right] + F\sin 2\theta \\
&= \frac{1}{2}(A+B) + \frac{1}{2}(B-A)\cos 2\theta + F\sin 2\theta \qquad (11.13.8)
\end{aligned}$$

Similarly from (11.13.5) we have

$$B_{Oy'} = \frac{1}{2}(A+B) - \frac{1}{2}(B-A)\cos 2\theta - F\sin 2\theta \qquad (11.13.9)$$

From (11.13.6), we can write

$$\sin 2\theta = \frac{2F}{\sqrt{4F^2 + (B-A)^2}} \qquad (11.13.10)$$

and

$$\cos 2\theta = \frac{(B-A)}{\sqrt{4F^2 + (B-A)^2}} \qquad (11.13.11)$$

Using (11.13.10) and (11.13.11) in (11.13.8), we have

$$\begin{aligned}
A_{Ox'} &= \frac{1}{2}(A+B) + \frac{1}{2}\frac{(B-A)^2}{\sqrt{4F^2+(B-A)^2}} + \frac{2F^2}{\sqrt{4F^2+(B-A)^2}} \\
&= \frac{1}{2}(A+B) + \frac{1}{2}\left[\frac{(B-A)^2 + 4F^2}{\sqrt{4F^2+(B-A)^2}}\right] \qquad (11.13.12) \\
&= \frac{1}{2}\left[(A+B) + \sqrt{4F^2+(B-A)^2}\right] \qquad (11.13.13)
\end{aligned}$$

Similarly

$$B_{Oy'} = \frac{1}{2}\left[(A+B) - \sqrt{4F^2+(B-A)^2}\right] \qquad (11.13.14)$$

Also

$$C_{Oz'} = A_{Ox'} + B_{Oy'} \tag{11.13.15}$$

For critical points

$$\frac{dA_{Ox'}}{d\theta} = 0$$
$$\frac{dB_{Oy'}}{d\theta} = 0$$

From (11.13.8), we have

$$\frac{dA_{Ox'}}{d\theta} = -(B-A)\sin 2\theta + 2F\cos 2\theta = 0$$

$$\tan 2\theta = \frac{2F}{B-A}$$

same calculated in (11.13.6)
Similarly $\frac{dB_{Oy'}}{d\theta} = 0$ gives (11.13.6). So extreme values of $A_{Ox'}, B_{Oy'}$ are already attained in (11.13.12) and (11.13.14) for θ given by (11.13.7).

11.14 Euler's Dynamical Equations for the Motion of a Rigid Body About a Fixed Point

Consider a rigid body of mass m rotating about OC axis through O, with angular velocity $\vec{\omega}$. Let the axes are principal axes. Using angular momentum vector (11.11.15), in the

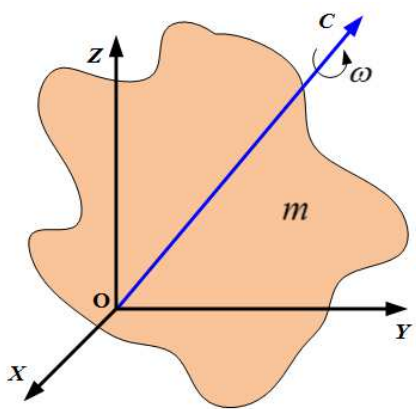

Figure 11.27: A body is rotating about OC axis

operator equation (**??**)

$$\frac{d\vec{L}}{dt} = \frac{\partial\vec{L}}{\partial t} + \vec{\omega}\times\vec{L} \tag{11.14.1}$$

Since $\frac{d\vec{L}}{dt} = \vec{\tau}$, is the torque acting on the body about O. And

$$\begin{aligned}\vec{\omega}\times\vec{L} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ A^*\omega_x & B^*\omega_y & C^*\omega_z \end{vmatrix} \\ &= \langle -\left(B^*-C^*\right)\omega_y\omega_z, -\left(C^*-A^*\right)\omega_x\omega_y, -\left(A^*-B^*\right)\omega_x\omega_y\rangle\end{aligned}$$

Also

$$\frac{\partial}{\partial t}\langle L_x, L_y, L_z\rangle = \langle A^*\dot{\omega}_x, B^*\dot{\omega}_y, C^*\dot{\omega}_z\rangle$$

Then (11.14.1) becomes

$$\begin{pmatrix} \tau_x \\ \\ \tau_y \\ \\ \tau_z \end{pmatrix} = \begin{pmatrix} A^*\dot{\omega}_x - (B^* - C^*)\,\omega_y\omega_z \\ \\ B^*\dot{\omega}_y - (C^* - A^*)\,\omega_x\omega_y \\ \\ C^*\dot{\omega}_z - (A^* - B^*)\,\omega_x\omega_y \end{pmatrix} \tag{11.14.2}$$

(11.14.2) are known as Euler's dynamical equations of motion.
If **no force** is acting the rigid body, then there is no torque and hence

$$\vec{\tau} = \langle \tau_x, \tau_y, \tau_z \rangle = 0 \tag{11.14.3}$$

then Euler's dynamical equations become

$$\left.\begin{aligned} A^*\dot{\omega}_x - (B^* - C^*)\,\omega_y\omega_z &= 0 \\ B^*\dot{\omega}_y - (C^* - A^*)\,\omega_x\omega_y &= 0 \\ C^*\dot{\omega}_z - (A^* - B^*)\,\omega_x\omega_y &= 0 \end{aligned}\right\} \tag{11.14.4}$$

11.15 Principle of Gyroscopic Compass

If a rigid body rotates about a fixed point under no force (torque free body) with constant angular velocity, once the instantaneous axis of rotation coincides with a principal axis and rotation about this axis is continuous .
Since no force is acting, then (11.14.1) becomes

$$\frac{\partial \vec{L}}{\partial t} + \vec{\omega} \times \vec{L} = 0 \tag{11.15.1}$$

Also the rotation is about a principal axis ($\vec{\omega}$ is parallel to $\vec{L}$)

$$\vec{\omega} \times \vec{L} = 0$$

then (11.15.1) becomes

$$\frac{\partial \vec{L}}{\partial t} = 0 \tag{11.15.2}$$

Integrating (11.15.2)

$$\vec{L} = C \text{ (constant)}$$

Hence $\vec{L}$ is fixed relative to the frame of principal axis through the point of rotation. Thus a wheel rotating about its axle, tends to continue its motion. This is the principle of

Gyroscopic Compass.
When the angular velocity is constant, then

$$\dot{\vec{\omega}} = \langle \dot{\omega}_x, \dot{\omega}_y, \dot{\omega}_z \rangle = 0 \tag{11.15.3}$$

Then 11.15.4 becomes

$$\begin{aligned}(B^* - C^*)\,\omega_y\omega_z &= 0\\ (C^* - A^*)\,\omega_x\omega_y &= 0\\ (A^* - B^*)\,\omega_x\omega_y &= 0\end{aligned}$$

or

$$\begin{aligned}\omega_y\omega_z &= 0\\ \omega_x\omega_y &= 0\\ \omega_x\omega_y &= 0\end{aligned}$$

It means, any two components are zero, so the instantaneous axis of rotation coincides with a principal axis.
Conversely when the instantaneous axis of rotation coincides with a principal axis, then

$$A^* = B^* = C^* = I$$

and 11.15.4 becomes

$$\begin{aligned}I\dot{\omega}_x &= 0\\ I\dot{\omega}_y &= 0\\ I\dot{\omega}_z &= 0\end{aligned}$$

or

$$\begin{aligned}\omega_x &= C_1 \text{ (constant)}\\ \omega_y &= C_2 \text{ (constant)}\\ \omega_z &= C_3 \text{ (constant)}\end{aligned}$$

Hence $\vec{\omega} =$ constant, and the body rotates about a fixed point under no force with constant angular velocity.

11.16 Momental Ellipsoid

Consider a system of n particles rotating about an axis OC through O with angular velocity $\vec{\omega}$. Let $\hat{a}$ be a unit vector in the direction of OC axis having direction cosines λ, μ, ν. Then

$$\hat{a} = \langle \lambda, \mu, \nu \rangle \tag{11.16.1}$$

Let a particle of mass m_i lying at P on OC axis, having position vector $\vec{r_i}$. Then

$$\vec{OP} = \vec{r_i} = \langle x_i, y_i, z_i \rangle \tag{11.16.2}$$

Also $\vec{OP} = \vec{r_i}$ is in the direction of $\hat{a}$, using (11.16.1) and $|\vec{OP}| = r_i$, $\vec{r_i}$ can also be written as

$$\vec{OP} = \vec{r_i} = r_i \langle \lambda, \mu, \nu \rangle \tag{11.16.3}$$

From (11.16.2) and (11.16.3), we can write

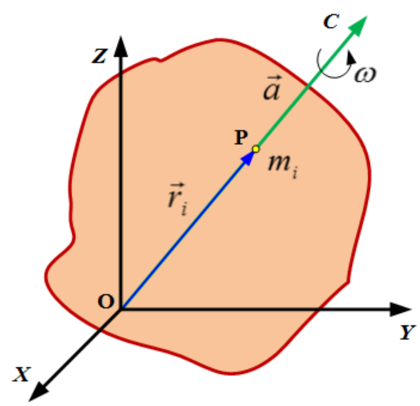

Figure 11.28: Momental Ellipsoid

$$\lambda = \frac{x_i}{r_i}, \ \mu = \frac{y_i}{r_i} \text{ and } \nu = \frac{z_i}{r_i} \tag{11.16.4}$$

Then by (11.9.6) the moment of inertia of the body about OC axis ($M.I.$ about an instantaneous axis) is

$$I_{OC} = I = A\lambda^2 + B\mu^2 + C\nu^2 - 2D\mu\nu - 2E\lambda\nu - 2F\lambda\mu \tag{11.16.5}$$

Now let P moves in such a way that Ir_i^2 remains constant, then from (11.16.4), (11.16.5) we can write as

$$Ax_i^2 + By_i^2 + Cz_i^2 - 2Dy_iz_i - 2Ez_ix_i - 2Fx_iy_i = Ir_i^2 \text{ (constant)}$$

Let P has coordinates (x, y, z), then

$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = Ir^2 \tag{11.16.6}$$

Since A, B, C are positive quantities, so (11.16.6) represents an ellipsoid known as momental ellipsoid.

11.17 Examples

In this section we will present some examples about above concepts. we can divide it into 3 categories depending upon the dimensions of the systems.

11.18 One Dimensional Systems

Example 11.18.1. *A rigid body consisting of two particles of mass m connected by a massless rod of length $2a$. Find the moment of inertia about an axis through the center of mass and perpendicular to the rod.*

Solution: The system is shown the fig. Let AB be the axis passing through the center of mass. Then each particle is at a distance a from the line AB
The moment of inertia of the system about AB axis is

$$\begin{aligned} I &= I_1 + I_2 \\ &= ma^2 + ma^2 \\ &= 2ma^2 \end{aligned}$$

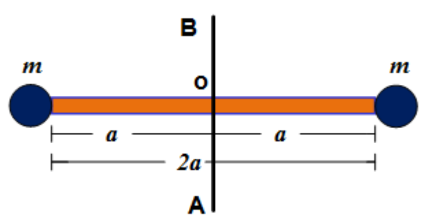

Figure 11.29: M.I of a system of particles

Example 11.18.2. *Find moment of inertia of a uniform rod of mass m lying along $x-axis$ with one end at origin having length a about*

(a) an axis passing through one end and perpendicular to the rod.

(b) Coordinate axes.

(c) *finding products of inertia, hence complete inertia matrix.*

Solution: Let m be the mass of rod of length a and LM (parallel to y axis) be the axis of rotation passing through one end and origin O and perpendicular to the rod.
Consider a small element mass dm of width dx at a distance x from LM axis. Since the

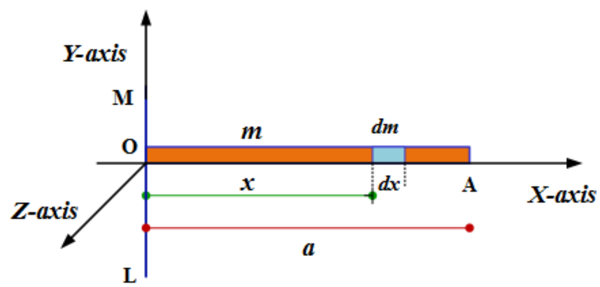

Figure 11.30: rod of length a

rod is uniform, the mass per unit length is a constant,

$$\begin{aligned}\rho &= \frac{dm}{dl} = \frac{m_{total}}{l} \\ &= \frac{dm}{dx} = \frac{m}{a} \end{aligned} \tag{11.18.1}$$

or the small element mass is

$$dm = \frac{m}{a}dx \tag{11.18.2}$$

Hence the moment of inertia ($M.I.$) of small element about LM axis is

$$dI_{LM} = \frac{m}{a}dx\ x^2$$

(a) an axis passing through one end and perpendicular to the rod.

The $M.I.$ of rod about LM axis is

$$I_{LM} = \int_0^a \frac{m}{a}\ x^2 dx$$

$$\begin{aligned} I &= \frac{m}{a}\left[\frac{x^3}{3}\right]_0^a \\ &= \frac{m}{a}\frac{(a)^3}{3} = \frac{1}{3}ma^2 \end{aligned}$$

Hence moment of inertia about an axis passing through one end and perpendicular to the rod is

$$I_{LM} = \frac{1}{3}ma^2 \tag{11.18.3}$$

(b) Coordinate axes.

In (a) x axis is along the rod, see Fig 11.30. Then the distance of mass element dm from x axis is zero, so its moment of inertia about x axis is zero. Hence the moment of inertia of whole mass about x axis is

$$A = I_{xx} = m(0)^2 = 0 \tag{11.18.4}$$

Here y and z axes are perpendicular to the rod, so mass m has same moment of inertia about both axes. In (a) LM axis, the axis of rotation is taken along y axis, so moment of inertia about y axis

$$B = I_{yy} = \frac{1}{3}ma^2$$

and about z axis

$$C = I_{zz} = \frac{1}{3}ma^2$$

(c) Products of inertia and inertia matrix.

Since the body is one dimensional, so $y = 0 = z$. Hence the products of inertia $w.r.t.$ pair of axes $(oy, oz), (oz, ox)$ and (ox, oy) respectively are as under

$$D = \iiint_V \rho yz dV = 0$$

$$E = \iiint_V \rho zx dV = 0$$

$$F = \iiint_V \rho xy dV = 0$$

Hence the inertia matrix for (a) is

$$[I] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{3}ma^2 & 0 \\ 0 & 0 & \frac{1}{3}ma^2 \end{pmatrix}$$

The radius of gyration about axis of rotation can be calculated by using (11.4.3). First about x axis is

$$K_x = \sqrt{\frac{I_{xx}}{m}} \quad = \quad 0$$

and about y axis is

$$K_y = \sqrt{\frac{I_{yy}}{m}} \quad = \quad \frac{1}{\sqrt{3}}a$$

finally about z axis is

$$K_z = \sqrt{\frac{I_{zz}}{m}} \quad = \quad \frac{1}{\sqrt{3}}a$$

Example 11.18.3. *Find moment of inertia of a uniform rod of mass m lying along $x-$axis with center at origin having length a about*

(a) an axis passing through center and perpendicular to the rod.

(b) Coordinate axes.

(c) finding products of inertia, hence complete inertia matrix.

Solution: Let m be the mass of rod of length a and LM (parallel to y axis) be the axis of rotation passing through one end and origin O and perpendicular to the rod.
Consider a small element mass dm of width dx at a distance x from LM axis. Since the

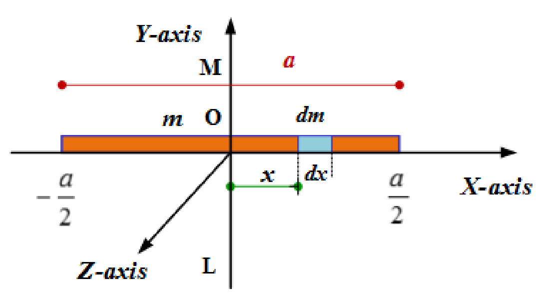

Figure 11.31: Axis of rotation passing through center and perpendicular to rod

rod is uniform, the mass per unit length is a constant,

$$\rho = \frac{dm}{dl} = \frac{m_{total}}{l} = \frac{dm}{dx} = \frac{m}{a} \tag{11.18.5}$$

or the small element mass is

$$dm = \frac{m}{a}dx \tag{11.18.6}$$

Hence the moment of inertia ($M.I.$) of small element about LM axis is

$$dI_{LM} = \frac{m}{a}dx\ x^2$$

(a) and the moment of inertia of rod about LM axis is

$$I_{LM} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{m}{a}\ x^2 dx$$

$$\begin{aligned} I_{LM} &= \frac{m}{a}\left[\frac{x^3}{3}\right]_{-\frac{a}{2}}^{\frac{a}{2}} \\ &= \frac{m}{a}\,\frac{1}{3}\left[\left(\frac{a}{2}\right)^3 + \left(\frac{a}{2}\right)^3\right] \\ &= \frac{1}{12}ma^2 \end{aligned}$$

Hence moment of inertia about an axis passing through center and perpendicular to the rod is

$$I_{LM} = \frac{1}{12}ma^2 \tag{11.18.7}$$

(b) In (a) x axis is along the rod, see Fig 11.31. Then the distance of mass element dm from x axis is zero, so its moment of inertia about x axis is zero. Hence the moment of inertia of whole mass about x axis is

$$A = I_{xx} = m(0)^2 = 0 \tag{11.18.8}$$

Here y and z axes are perpendicular to the rod, so mass m has same moment of inertia about both axes. In (a) LM axis, the axis of rotation is taken along y axis, so moment of inertia about y axis

$$B = I_{yy} = \frac{1}{12}ma^2$$

and about z axis

$$C = I_{zz} = \frac{1}{12}ma^2$$

(c) Since the body is one dimensional, so $y = 0 = z$. Hence the products of inertia *w.r.t.* pair of axes $(oy, oz), (oz, ox)$ and (ox, oy) respectively are as under

$$D = \int_l \rho yz dl = 0$$
$$E = \int_l \rho zx dl = 0$$
$$F = \int_l \rho xy dl = 0$$

Hence the inertia matrix for (a) is

$$[I] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}ma^2 & 0 \\ 0 & 0 & \frac{1}{12}ma^2 \end{pmatrix}$$

Example 11.18.4. *Show that a uniform rod of mass m is equimomental to three particles situated one at each end of the rod and one at its middle point, the masses of the particle being $\frac{1}{6}m$, $\frac{1}{6}m$ and $\frac{2}{3}m$ respectively.*

Solution: Let m be the mass of rod AB of length $2a$ and CD be the perpendicular to AB passing through middle and centroid G of it. This is the first system.
Let $\frac{1}{6}m$, $\frac{1}{6}m$ and $\frac{2}{3}m$ are the masses at A, G, B respectively. This is the second system. Its total mass is

$$\frac{1}{6}m + \frac{1}{6}m + \frac{2}{3}m = m$$

1. Both systems have same masses.

2. Both systems have same centroid G.
   Both systems are one dimensional, so we consider moment of inertia about AB axis. For this see above example.

3. Both systems have the same moment of inertia (i.e. each zero) about AB axis, passing through common centroid G.

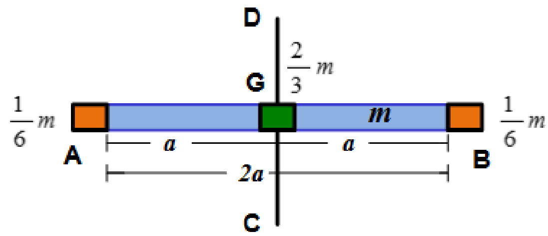

Figure 11.32: Equimomental systems

Hence all the three conditions are satisfied and the systems are equimomental Two systems are said to be equimomental if they have equal moment of inertia about every line in space.

Example 11.18.5. *Moment of inertia of a uniform rod of mass m and length* $2a$ *about an axis passing through one end and making an angle* θ *with the rod.*

Solution: Let m be the mass of rod of length $2a$ and OA be the axis of rotation passing through one end and origin O, making an angle θ with the rod. Consider a small element mass dm of width dx at a distance x from O axis. Then its distance from the axis OA is $d = x\sin\theta$ (see Fig. 11.33). Since the rod is uniform, the mass per unit length is a constant,

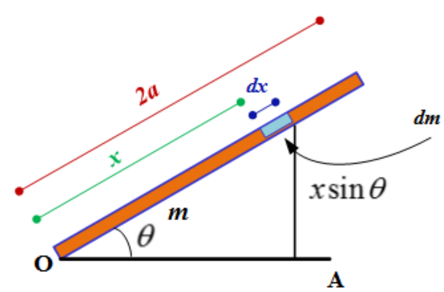

Figure 11.33: Axis of rotation passing through center and perpendicular to rod

$$\begin{aligned} \rho &= \frac{dm}{dl} = \frac{m_{total}}{l} \\ &= \frac{dm}{dx} = \frac{m}{a} \end{aligned}$$

or the small element mass is

$$dm = \frac{m}{2a}dx$$

Hence the moment of inertia $(M.I.)$ of small element about OA axis is

$$\begin{aligned} dI_{OA} &= \frac{m}{2a}dx \; d^2 \\ &= \frac{m}{2a}dx \; (x \sin\theta)^2 \end{aligned}$$

and the $M.I.$ of rod about OA is

$$\begin{aligned} I &= \int_0^{2a} \frac{m}{2a} \; (x\sin\theta)^2 \, dx \\ &= \frac{m}{2a} \left[\frac{x^3}{3}\right]_0^{2a} \sin^2\theta \\ &= \frac{m}{2a} \frac{(2a)^3}{3} \sin^2\theta = \frac{4}{3} m a^2 \sin^2\theta \end{aligned}$$

Hence

$$I_{OA} = \frac{4}{3} m a^2 \sin^2\theta \qquad (11.18.9)$$

11.19 Two Dimensional Systems

In this section we will discuss the above concepts for two dimensional systems. We can divide it two parts.

- System in Cartesian coordinate system.
- System in polar coordinate system.

11.19.1 System in Cartesian Coordinate System

Example 11.19.1. *A square of side a has particles of masses m, $2m$, $3m$, $4m$ at its vertices. Show that the principal moment of inertia at center of the square are $2ma^2$, $3ma^2$, $5ma^2$. Also find the directions of principal axes.*

Solution
Let us take an $xy-$coordinate system Oxy and $ABCD$ a square of side a with center at origin O. The coordinates of points are $O\,(0,0)$, $A\left(-\frac{a}{2},-\frac{a}{2}\right)$, $B\left(\frac{a}{2},-\frac{a}{2}\right)$, $C\left(-\frac{a}{2},\frac{a}{2}\right)$ and $D\left(\frac{a}{2},\frac{a}{2}\right)$. The particles of masses m, $2m$, $3m$, $4m$ are at A, B, C and D respectively. See

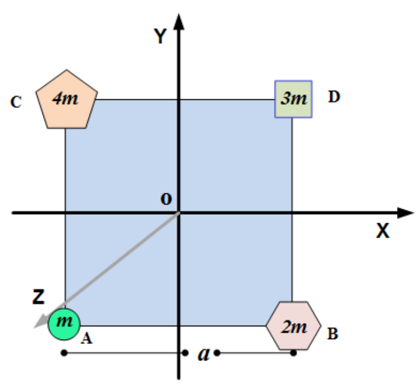

Figure 11.34: Four particles system

Fig. 11.35, the square is laminas in xy plane, with $z = 0$.

(a) **Moments of Inertia about coordinate axes**

Using (11.6.10) the moment of inertia about coordinate axes are calculated. First about

x *axis* is

$$\begin{aligned}
I_{xx} = A &= \sum_{i=1}^{n} m_i y_i^2 \\
&= m_1 y_1^2 + m_2 y_2^2 + m_3 y_3^2 + m_4 y_4^2 \\
&= m\left(-\frac{a}{2}\right)^2 + 2m\left(-\frac{a}{2}\right)^2 + 3m\left(\frac{a}{2}\right)^2 + 4m\left(\frac{a}{2}\right)^2 \\
&= m\frac{a^2}{4}\left(1+2+3+4\right) \\
&= \frac{5}{2}ma^2
\end{aligned}$$

Similarly about y *axis* is

$$I_{yy} = B = \frac{5}{2}ma^2$$

Finally about z *axis* can be calculated by using perpendicular axis theorem

$$\begin{aligned}
I_{zz} = C &= I_{xx} + I_{yy} \\
&= \frac{5}{2}ma^2 + \frac{5}{2}ma^2 \\
&= 5ma^2
\end{aligned}$$

(b) **Products of Inertia**

The products of inertia *w.r.t.* pair of axes $(oy, oz), (oz, ox)$ and (ox, oy) are calculated by using (11.6.11) respectively as under

$$\begin{aligned}
D &= 0 \\
E &= 0 \\
F &= \sum_{i=1}^{n} m_i x_i y_i \\
&= m_1 x_1 y_1 + m_2 x_2 y_2 + m_3 x_3 y_3 + m_4 x_4 y_4 \\
&= m\left(-\frac{a}{2}\right)\left(-\frac{a}{2}\right) + 2m\left(\frac{a}{2}\right)\left(-\frac{a}{2}\right) + 3m\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) + 4m\left(\frac{a}{2}\right)\left(-\frac{a}{2}\right) \\
&= m\frac{a^2}{4}\left(1-2+3-4\right) \\
&= -\frac{1}{2}ma^2
\end{aligned}$$

(c) **Inertia matrix**

The inertia matrix is

$$[I] = \begin{pmatrix} \frac{5}{2}ma^2 & \frac{1}{2}ma^2 & 0 \\ \frac{1}{2}ma^2 & \frac{5}{2}ma^2 & 0 \\ 0 & 0 & 5ma^2 \end{pmatrix}$$

(d) **Principal axes**

The first principal axis can be determined by using (11.13.12)

$$\begin{aligned} A_{Ox'} &= \frac{1}{2}\left[(A+B)+\sqrt{4F^2+(B-A)^2}\right] \\ &= \frac{1}{2}\left[\left(\frac{5}{2}ma^2+\frac{5}{2}ma^2\right)+\sqrt{4\left(\frac{1}{2}ma^2\right)^2+\left(\frac{5}{2}ma^2-\frac{5}{2}ma^2\right)^2}\right] \\ &= \frac{1}{2}\left[5ma^2+ma^2\right] \\ &= 3ma^2 \end{aligned}$$

The second principal axis can be determined by using (11.13.14)

$$\begin{aligned} B_{Oy'} &= \frac{1}{2}\left[(A+B)-\sqrt{4F^2+(B-A)^2}\right] \\ &= \frac{1}{2}\left[\left(\frac{5}{2}ma^2+\frac{5}{2}ma^2\right)-\sqrt{4\left(\frac{1}{2}ma^2\right)^2+\left(\frac{5}{2}ma^2-\frac{5}{2}ma^2\right)^2}\right] \\ &= \frac{1}{2}\left[5ma^2-ma^2\right] \\ &= 2ma^2 \end{aligned}$$

The third principal axis can be determined by using (11.13.15)

$$\begin{aligned} C_{Oz'} &= A_{Ox'}+B_{Oy'} \\ &= 3ma^2+2ma^2 \\ &= 5ma^2 \end{aligned}$$

is the moment of inertia about $zaxis$

(f) **Directions of Principle axes**

Directions of Principle axes can be calculated by using (11.13.7)

$$\begin{aligned}\theta &= \frac{1}{2}\arctan\left(\frac{2F}{B-A}\right)\\ &= \frac{1}{2}\arctan\left(\frac{2\left(\frac{1}{2}ma^2\right)}{\left(\frac{5}{2}ma^2-\frac{5}{2}ma^2\right)}\right)\\ &= \frac{1}{2}\arctan(\infty)\\ &= \frac{1}{2}\left(\frac{\pi}{2}\right)\\ &= \frac{\pi}{4}\end{aligned} \tag{11.19.1}$$

(11.19.1) is the direction of principal axes relative to co-ordinates axes.

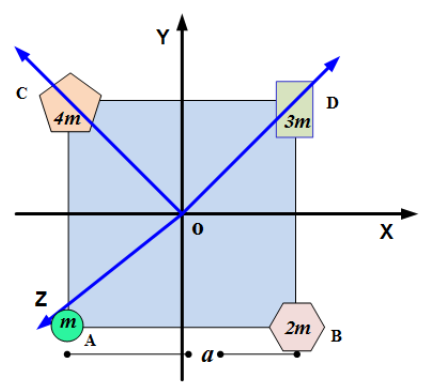

Figure 11.35: rectangular lamina

Example 11.19.2. *Moment of inertia of a rectangular lamina about an axis (line) passing through center and parallel to one side.*

Solution

(a) **Moment of inertia about** *x axis*

Let $ABCD$ be a rectangular lamina of mass m and LM parallel to AB be the line about which moment of inertia is to be calculated. Let $AB = 2a$ and $AC = 2b$, then area of lamina is $4ab$

Consider a small element of surface area $dA = dxdy$ at a distance y from LM axis (x axis). Since the lamina is uniform, the density of the lamina (the mass per unit area) is a constant

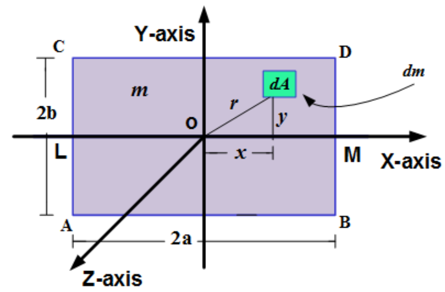

Figure 11.36: rectangular lamina

$$\begin{aligned}\rho &= \frac{mass_{total}}{area_{total}} = \frac{dm}{dA} \\ &= \frac{m}{A} = \frac{dm}{dA}\end{aligned}$$

Mass of small element (small rectangle) is

$$dm = \frac{m}{A} dA$$

Hence the moment of inertia of small element about LM axis is

$$\begin{aligned}dI &= dmy^2 \\ &= y^2 \frac{m}{4ab} dydx\end{aligned}$$

Using (11.2.6), the moment of inertia of rectangular lamina about LM is

$$\begin{aligned}I_{LM} &= \int_{-a}^{a} \int_{-b}^{b} y^2 \frac{m}{4ab} dydx \\ &= \frac{m}{4ab} \int_{-a}^{a} \left[\frac{y^3}{3} \right]_{-b}^{b} dx \\ &= \frac{m}{4ab} \frac{2b^3}{3} \left[x \right]_{-a}^{a} \\ &= \frac{mb^2}{3}\end{aligned}$$

Since LM axis is along x axis, so moment of inertia about x axis is

$$I_{xx} = \frac{mb^2}{3}$$

Second Approach

(b) **Moment of inertia about** y *axis*

Let LM parallel to AC be the line about which moment of inertia is to be calculated. Consider an elementary strip PQ of length $(BC = 2b)$ and breadth dx at a distance x from $LM - axis$. Then area of the strip is $dA = 2bdx$

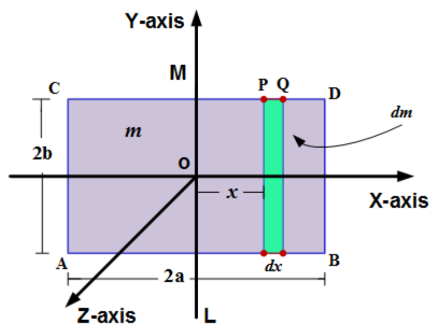

Figure 11.37: rectangular lamina

Mass of the strip $= \frac{m}{4ab}2bdx = \frac{m}{2a}dx$
$M.I.$ of small strip about LM is $dI = x^2$(mass of the strip)

$$dI = \frac{m}{2a}x^2dx$$

$M.I.$ of rectangular lamina about LM

$$\begin{aligned} I &= \frac{m}{2a}\int_{-a}^{a} x^2dx \\ &= \frac{m}{2a}\frac{2a^3}{3} \\ &= \frac{ma^2}{3} \end{aligned}$$

Since LM axis is along y axis, so moment of inertia about y axis is

$$I_{yy} = \frac{ma^2}{3}$$

(c) **Moment of inertia about** z *axis*

Here z *axis* is perpendicular to lamina with center at origin. Hence moment of inertia of rectangular lamina about a line perpendicular to lamina and passing through center is same.
Let OL be the axis of rotation passing through center O and perpendicular to lamina $ABCD$. Let $AB = 2a$ and $AC = 2b$ area of lamina is $4ab$
Mass per unit area of lamina $= \frac{m}{4ab}$ Consider a small element of surface area $dA = dxdy$ at a distance d from OL *axis*
Also the density of the lamina is $\rho = \frac{m}{4ab}$
Mass of small element (rectangle) $= \rho dA = \frac{m}{4ab}dydx$

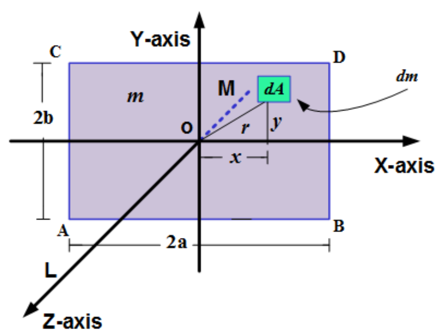

Figure 11.38: rectangular lamina

Here perpendicular distance of small element dm from OL axis is

$$d = \sqrt{x^2 + y^2}$$

$M.I.$ of small rectangle about OL is

$$\begin{aligned} dI &= dmd^2 \\ &= \left(x^2 + y^2\right) \frac{m}{4ab} dydx \end{aligned}$$

$M.I.$ of rectangular lamina about OL is

$$\begin{aligned} I &= \frac{m}{4ab} \int_{-a}^{a} \int_{-b}^{b} \left(x^2 + y^2\right) dydx \\ &= \frac{m}{4ab} 4 \int_{0}^{a} \int_{0}^{b} \left(x^2 + y^2\right) dydx \\ &= \frac{m\left(a^2 + b^2\right)}{3} \end{aligned}$$

Since LM axis is along z axis, so moment of inertia about z axis is

$$I_{zz} = \frac{m\left(a^2 + b^2\right)}{3}$$

Second Approach - Perpendicular axes theorem
Since x, y and z axes are mutually perpendicular axes. I_{zz} can be calculated as

$$\begin{aligned} I_{zz} &= I_{xx} + I_{yy} \\ &= \frac{mb^2}{3} + \frac{ma^2}{3} \\ &= \frac{1}{3} m \left(a^2 + b^2\right) \end{aligned}$$

(d) **Products of Inertia**

Since the lamina is two dimensional body,in xy plane, so $z = 0$. Hence the products of inertia *w.r.t.* pair of axes $(oy, oz), (oz, ox)$ and (ox, oy) respectively are as under

$$\begin{aligned} D &= \iint_A \rho yz dA = 0 \\ E &= \iint_A \rho zx dA = 0 \\ F &= \iint_A \rho xy dA \\ &= \frac{m}{4ab} \int_{-a}^{a} \int_{-b}^{b} xy dy dx \\ &= \frac{m}{4ab} \left| \frac{x^2}{2} \right|_{-a}^{a} \int_{-b}^{b} y dy \\ &= 0 \end{aligned}$$

(e) **Inertia Matrix**

Hence the inertia matrix is

$$[I] = \begin{pmatrix} \frac{1}{3}mb^2 & 0 & 0 \\ 0 & \frac{1}{3}ma^2 & 0 \\ 0 & 0 & \frac{1}{3}m\left(a^2 + b^2\right) \end{pmatrix}$$

(f) **Directions Principle axes**

Since the products of inertia are zero, hence coordinate axes are the principle axes.

(g) **Radius of gyration**

The radius of gyration about axis of rotation can be calculated by using (11.4.3). First about x axis is

$$K_x = \sqrt{\frac{I_{xx}}{m}} \quad = \quad \frac{1}{\sqrt{3}}b$$

and about y axis is

$$K_y = \sqrt{\frac{I_{yy}}{m}} \quad = \quad \frac{1}{\sqrt{3}}a$$

finally about z axis is

$$K_z = \sqrt{\frac{I_{zz}}{m}} \quad = \quad \frac{\sqrt{a^2+b^2}}{\sqrt{3}}$$

Example 11.19.3. *Moment of inertia of a rectangular lamina about an axis (line) passing through one end.*

Solution

(a) **Moment of inertia about** x *axis*

Let $OABC$ be a rectangular lamina of mass m and OA (along x *axis*) be the line about which moment of inertia is to be calculated. Let $OA = 2a$ and $BC = 2b$ Area of lamina is

$$A \quad = \quad 4ab$$

Mass per unit area of lamina $= \frac{m}{4ab}$ Consider a small element of surface area $dA = dxdy$ at a distance y from x *axis* Also the density of the lamina is $\rho = \frac{m}{4ab}$

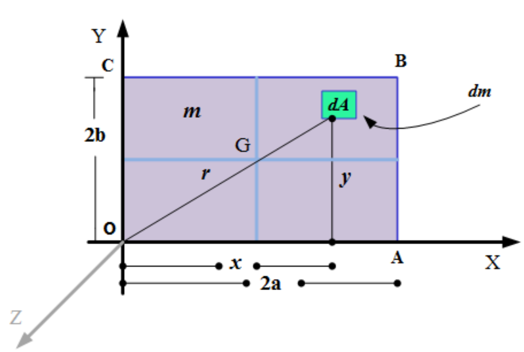

Figure 11.39: rectangular lamina

Mass of small element (rectangle) $= \rho dA = \frac{m}{4ab} dydx$ $M.I.$ of small rectangle about $LM dI =$ y^2(mass of the small element)

$$dI = y^2 \frac{m}{4ab} dydx$$

Using (11.2.6) the $M.I.$ of rectangular lamina about x $axis$

$$\begin{aligned} I_{xx} &= \int_0^{2a}\int_0^{2b} y^2 \frac{m}{4ab} dydx \\ &= \frac{m}{4ab} \int_0^{2a} \left[\frac{y^3}{3}\right]_0^{2b} dx \\ &= \frac{m}{4ab} \frac{8b^3}{3} \Big[x\Big]_0^{2a} \\ &= \frac{4}{3} mb^2 \end{aligned} \qquad (11.19.2)$$

Second Approach - Parallel axes theorem
Since the lamina has uniform distribution of mass, center of mass coincides with origin. Hence x axis is the axis passing through center of mass and LM axis is parallel to it. The distance between axes is $d = b$. Using Parallel axes theorem, I_{LM} can be calculated as

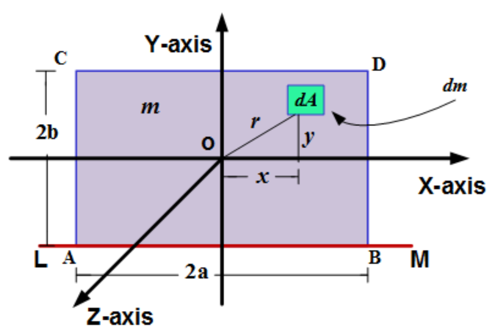

Figure 11.40: rectangular lamina

$$\begin{aligned} I_{LM} &= I_{xx} + md^2 \\ &= \frac{mb^2}{3} + mb^2 \\ &= \frac{4}{3} mb^2 \end{aligned}$$

(b) **Moment of inertia about** y *axis*

$$I_{yy} = \frac{4}{3}ma^2 \tag{11.19.3}$$

(c) **Moment of inertia about** z *axis*

To find moment of inertia about z *axis*, we can use perpendicular axis theorem.

$$\begin{aligned} I_{zz} &= I_{xx} + I_{yy} \\ &= \frac{4}{3}mb^2 + \frac{4}{3}ma^2 \\ &= \frac{4}{3}m\left(a^2 + b^2\right) \end{aligned} \tag{11.19.4}$$

(d) **Products of inertia**

Since the lamina is two dimensional body,in xy plane, so $z = 0$. Hence the products of inertia *w.r.t.* pair of axes $(oy, oz), (oz, ox)$ and (ox, oy) respectively are as under

$$\begin{aligned} D &= \iint_A \rho yz dA = 0 \\ E &= \iint_A \rho zx dA = 0 \\ F &= \iint_A \rho xy dA \\ &= \frac{m}{4ab}\int_0^{2a}\int_0^{2b} xydydx \\ &= \frac{m}{4ab}\left|\frac{x^2}{2}\right|_0^{2a}\int_0^{2b} ydy \\ &= \frac{m}{4ab}\frac{4a^2}{2}\frac{4b^2}{2} \\ &= mab \end{aligned}$$

(e) **Inertia matrix**

The inertia matrix is

$$[I] = \begin{pmatrix} \frac{4}{3}mb^2 & -mab & 0 \\ -mab & \frac{4}{3}ma^2 & 0 \\ 0 & 0 & \frac{4}{3}m\left(a^2+b^2\right) \end{pmatrix}$$

(f) **Directions of Principle axes**

Directions of Principle axes can be calculated by using (11.13.7)

$$\begin{aligned}\theta &= \frac{1}{2}\arctan\left(\frac{2F}{B-A}\right)\\ &= \frac{1}{2}\arctan\left(\frac{2mab}{\left(\frac{4}{3}ma^2\right)-\left(\frac{4}{3}mb^2\right)}\right)\\ &= \frac{1}{2}\arctan\left(\frac{3ab}{a^2-b^2}\right)\end{aligned} \quad (11.19.5)$$

(11.19.5) is the direction of principal axes relative to co-ordinates axes. **Note** For a square

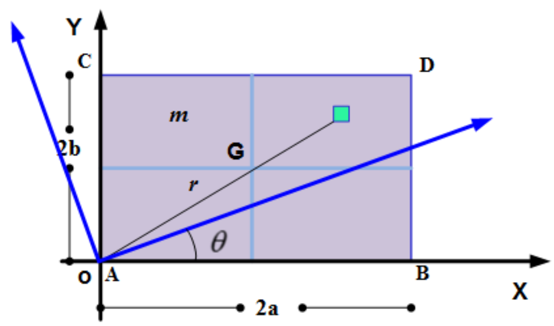

Figure 11.41: rectangular lamina

of side a,

$$\begin{aligned}\theta &= \frac{1}{2}\arctan\left(\frac{3aa}{a^2-a^2}\right)\\ &= \frac{1}{2}\arctan(\infty) = \frac{1}{2}\frac{\pi}{2}\\ &= \frac{\pi}{4}\end{aligned} \quad (11.19.6)$$

is the direction of principal axes relative to co-ordinates axes.

11.19.2 Moment of Inertia of a Uniform Triangular Disc (Lamina)

Consider ABC a thin uniform triangular disc of mass m in xy plane with base along x axis of length a. Consider a strip EF of mass dm, length l and width dy at a distance y from x axis, see Fig. 11.44. In similar triangles AEF and ABC

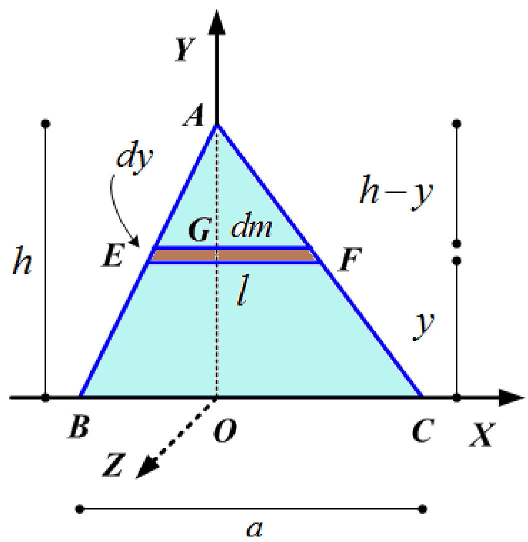

Figure 11.42: Uniform triangular disc

$$\frac{EF}{BC} = \frac{AG}{AO}$$
$$\frac{l}{a} = \frac{h-y}{h}$$
$$l = \frac{a(h-y)}{h} \tag{11.19.7}$$

Consider a strip of infinitesimal width dy and mass element dm at a distance y from x axis show in the figure . The length element of mass element dm is

$$dA = ldy \tag{11.19.8}$$

using (11.19.7) in (11.19.8)

$$dA = \frac{a(h-y)}{h}dy \tag{11.19.9}$$

Since the disc is uniform, the mass per unit area is a constant,

$$\rho = \frac{dm}{dA} = \frac{m_{total}}{A}$$
$$= \frac{dm}{\frac{a(h-y)}{h}dy} = \frac{m}{\frac{ah}{2}} \tag{11.19.10}$$

From (11.19.10), we can write

$$dm = \rho dA = \frac{2m(h-y)}{h^2}dy \tag{11.19.11}$$

Now moment of inertia of mass element dm about base (x axis) is

$$\begin{aligned} dI_{xx} &= dmy^2 \\ &= \frac{2m(h-y)}{h^2}dy\ y^2 \\ &= \frac{2m}{h^2}(h-y)y^2dy \end{aligned}$$

The moment of inertia of the disc about x axis is now an integral from $y=0$ to $y=h$

$$\begin{aligned} I_{xx} &= \frac{2m}{h^2}\int_0^h \left(hy^2 - y^3\right) dy \\ &= \frac{2m}{h^2}\left| h\frac{y^3}{3} - \frac{y^4}{4}\right|_0^h \\ &= \frac{mh^2}{6} \end{aligned} \tag{11.19.12}$$

11.19.3 Moment of Inertia of a Uniform Isosceles Triangular Disc (Lamina)

Consider ABC an isosceles triangular disc of mass m in xy plane with its symmetric axis along the positive x direction and one of its vertex at origin. Let $2b$ its base and a be its height (along x axis). If the z axis is the axis of rotation, passing through the origin (one vertex). The isosceles triangle can be regarded as the combination of the differential rod with the length $2y$ and the differential width dx (one dimensional) and consider a rod (strip) EF of mass dm, at a distance x from z axis, see Fig. 11.44.
The moment of inertia of a rod of mass m and length a about an axis perpendicular to the rod and passing through the center of mass is given (see example 1 b)

$$I = \frac{1}{12}ma^2$$

Here

$$y = \frac{b}{a}x$$

Since the disc is uniform, the mass per unit area is a constant,

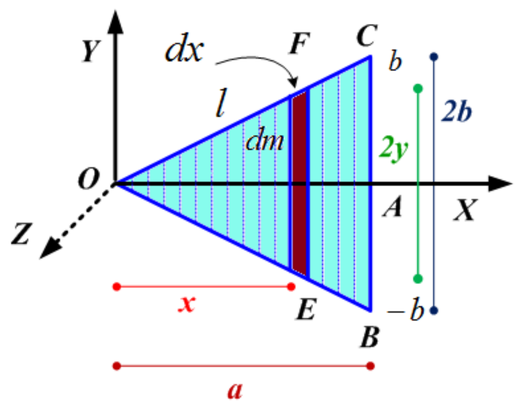

Figure 11.43: Uniform triangular disc

$$\begin{aligned} \rho &= \frac{dm}{dA} = \frac{m_{total}}{A} \\ &= \frac{dm}{2ydx} = \frac{m}{ab} \end{aligned} \tag{11.19.13}$$

From (11.19.13), we can write

$$dm = \rho dA = \frac{2m}{a^2} x dx \tag{11.19.14}$$

(a) Moment of inertia about z axis

Now moment of inertia of mass element dm about an axis perpendicular to the rod and passing through the center of mass is

$$\begin{aligned} dI &= \frac{1}{12} dm (2y)^2 \\ &= \frac{m}{6a^2} x dx \; 4 \frac{b^2}{a^2} x^2 \\ &= \frac{2mb^2}{3a^4} x^3 dx \end{aligned}$$

Using parallel axis theorem, the moment of inertia of mass element dm about z axis is

$$\begin{aligned} dI_{zz} &= dI + dm x^2 \\ &= \frac{2mb^2}{3a^4} x^3 dx + \frac{2m}{a^2} x dx \; x^2 \\ &= \frac{2m}{3a^4} \left(b^2 + 3a^2\right) x^3 dx \end{aligned}$$

The moment of inertia of the disc about z axis is now an integral from $x = 0$ to $x = a$

$$\begin{aligned} I_{zz} &= \frac{2m}{3a^4}\left(b^2 + 3a^2\right)\int_0^a x^3 dx \\ &= \frac{2m}{3a^4}\left(b^2 + 3a^2\right)\left|\frac{x^4}{4}\right|_0^a \\ &= \frac{m}{6a^4}\left(b^2 + 3a^2\right)a^4 \\ &= \frac{1}{6}mb^2 + \frac{1}{2}ma^2 \end{aligned} \tag{11.19.15}$$

(b) Moment of inertia about x axis

Here again the same concept is used, the x axis is perpendicular to rod (strip) EF and passing through its center, so the moment of inertia of mass dm about x axis is

$$\begin{aligned} dI &= \frac{1}{12}dm(2y)^2 \\ &= \frac{m}{6a^2}xdx\ 4\frac{b^2}{a^2}x^2 \\ &= \frac{2mb^2}{3a^4}x^3dx \end{aligned}$$

And the moment of inertia of mass m about x axis is

$$\begin{aligned} I_{xx} &= \frac{2mb^2}{3a^4}\int_0^a x^3dx \\ &= \frac{2mb^2}{3a^4}\left|\frac{x^4}{4}\right|_0^a \\ &= \frac{1}{6}mb^2 \end{aligned}$$

(c) Moment of inertia about y axis

In this case rod (strip) EF is parallel to y axis, so the moment of inertia of mass dm about y axis is

$$\begin{aligned} dI &= dmx^2 \\ &= \frac{2m}{a^2}xdx\ x^2 \\ &= \frac{2m}{a^2}x^3dx \end{aligned}$$

And the moment of inertia of mass m about y axis is

$$\begin{aligned} I_{yy} &= \frac{2m}{a^2}\int_0^a x^3 dx \\ &= \frac{2m}{a^2}\left|\frac{x^4}{4}\right|_0^a \\ &= \frac{1}{2}ma^2 \end{aligned}$$

Note: The moment of inertia about z axis can also be calculated by using theorem of perpendicular axis.

$$\begin{aligned} I_{zz} &= I_{xx} + I_{yy} \\ &= \frac{1}{12}mb^2 + \frac{1}{2}ma^2 \end{aligned}$$

11.19.4 Moment of Inertia of a Uniform Equilateral Triangular Disc (Lamina)

The above triangle can be considered as equilateral triangle of length l by taking

$$l = 2b$$

or

$$b = \frac{l}{2}$$

and

$$\begin{aligned} a &= \sqrt{l^2 - b^2} \\ &= \sqrt{l^2 - \left(\frac{l}{2}\right)^2} \\ &= \frac{\sqrt{3}}{2}l \end{aligned}$$

Then the moment of inertia about axes are

(a) Moment of inertia about z axis

$$\begin{aligned} I_{zz} &= \frac{1}{6}mb^2 + \frac{1}{2}ma^2 \\ &= \frac{1}{6}m\left(\frac{l}{2}\right)^2 + \frac{1}{2}m\left(\frac{\sqrt{3}}{2}l\right)^2 \\ &= \frac{1}{24}ml^2 + \frac{3}{8}ml^2 \\ &= \frac{5}{12}ml^2 \end{aligned} \quad (11.19.16)$$

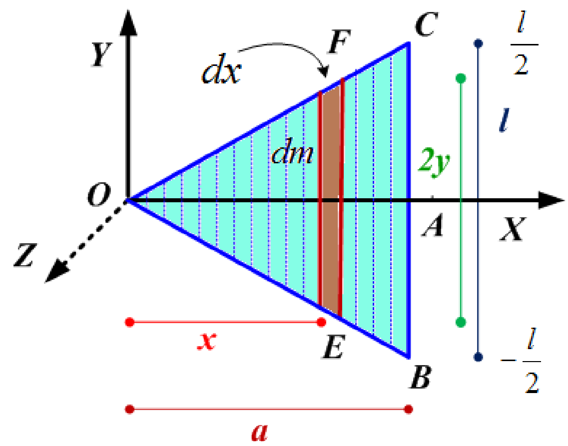

Figure 11.44: Uniform equilateral triangular disc

(b) Moment of inertia about x axis

The moment of inertia of mass m about x axis is

$$\begin{aligned} I_{xx} &= \frac{1}{6}mb^2 \\ &= \frac{1}{6}m\left(\frac{l}{2}\right)^2 \\ &= \frac{1}{24}ml^2 \end{aligned} \tag{11.19.17}$$

(c) Moment of inertia about y axis

The moment of inertia of mass m about y axis is

$$\begin{aligned} I_{yy} &= \frac{1}{2}ma^2 \\ &= \frac{1}{2}m\left(\frac{\sqrt{3}}{2}l\right)^2 \\ &= \frac{3}{8}ml^2 \end{aligned}$$

11.20 Polar coordinates

In this section moment of inertia will be calculated by using polar coordinates.

11.20.1 Moment of Inertia of a Circular Ring (Hoop)

Consider a circular ring of radius r, mass m and center at origin in xy plane. Let z axis be the axis of rotation. Let P be a point on the ring making an angle θ with x axis. Then

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

The arc length of the circular ring is

$$L = 2\pi r$$

Consider a small mass dm with arc length s, having circular measure $d\theta$, then

$$s = rd\theta$$

The mass m has arc length $2\pi r$, and the mass dm has arc length $rd\theta$, so dm can be written as

$$dm = \frac{m}{2\pi}d\theta$$

Now moment of inertia about various axes can be determined as follows.

11.20.2 About the Diameter of the Ring or About an Axis in the Plane of the Ring and Passing Through its Center

Since the ring lies in xy plane having center at origin, so x axis or y axis can be considered as the diameter of the ring/an axis in the plane of the ring and passing through its center. Hence moment of inertia about x axis and y axis is the same. The moment of inertia of mass dm about x axis is

$$\begin{aligned} dI_{xx} &= dmy^2 \\ &= \frac{m}{2\pi}d\theta\,(r\sin\theta)^2 \\ &= \frac{mr^2}{2\pi}\sin^2\theta d\theta \end{aligned}$$

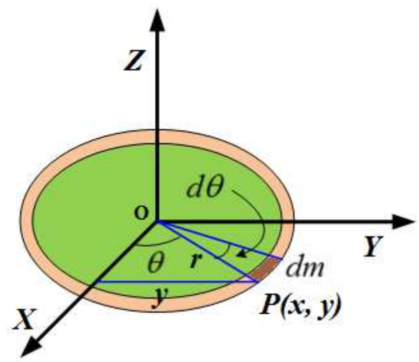

Figure 11.45: Ring

Next the moment of inertia of the ring about x axis is

$$\begin{aligned} I_{xx} &= \frac{mr^2}{2\pi}\int_0^{2\pi}\sin^2\theta d\theta \\ &= \frac{mr^2}{2\pi}\int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta \\ &= \frac{mr^2}{4\pi}\left|\theta-\frac{\sin 2\theta}{2}\right|_0^{2\pi} \\ &= \frac{mr^2}{4\pi}2\pi \\ &= \frac{mr^2}{2} \end{aligned}$$

Similarly the moment of inertia of the ring about y axis is

$$I_{yy} = \frac{mr^2}{2}$$

Hence the moment of inertia about the diameter of the ring is

$$I_d = \frac{mr^2}{2}$$

11.20.3 About an Axis Perpendicular to Plane of the Ring and Passing Through its Center

Since the ring lies in xy plane having center at origin, so z axis can be considered as an axis perpendicular to plane of the ring and passing through its center. Then r be the distance of mass dm from z axis, so its moment of inertia z axis is

$$\begin{aligned} dI_{zz} &= dmr^2 \\ &= \frac{m}{2\pi}d\theta r^2 \end{aligned}$$

Next the moment of inertia of the ring about z axis is

$$\begin{aligned} I_{zz} &= \frac{mr^2}{2\pi}\int_0^{2\pi} d\theta \\ &= \frac{mr^2}{2\pi}\,|\theta|_0^{2\pi} \\ &= \frac{mr^2}{2\pi}2\pi \\ &= mr^2 \end{aligned}$$

Second Approach
Using perpendicular axis theorem, the moment of inertia of mass m about z axis is

$$\begin{aligned} I_{zz} &= I_{xx} + I_{yy} \\ &= \frac{mr^2}{2} + \frac{mr^2}{2} \\ &= mr^2 \end{aligned}$$

11.20.4 About a Line Tangent to Ring

Here two cases arise:

1. The tangent line is parallel to z axis
2. The tangent line is parallel to diameter *i.e.* parallel to x axis or y axis

(1) **About a line tangent to ring and parallel to z axis**
Let AB be the line tangent to ring and parallel to z axis. Then its distance from z axis is r Using theorem of parallel axis, the moment of inertia of mass m about AB axis is

$$\begin{aligned} I_{AB} &= I_{zz} + m(\text{distance between axes})^2 \\ &= mr^2 + mr^2 \\ &= 2mr^2 \end{aligned}$$

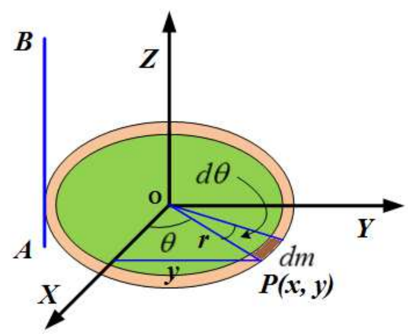

Figure 11.46: Ring

(2) **About a line parallel to diameter of the ring**
Let AB be the line parallel to diameter of the ring. Then its distance from diameter is r Using theorem of parallel axis, the moment of inertia of mass m about AB axis is

$$\begin{aligned} I_{AB} &= I_d + m(\text{distance between axes})^2 \\ &= \frac{1}{2}mr^2 + mr^2 \\ &= \frac{3}{2}mr^2 \end{aligned}$$

11.20.5 Moment of Inertia of a Uniform Circular Disc

Consider a thin uniform disc of mass m and radius R and center at origin in xy plane. Let z axis be the axis of rotation, perpendicular to the plane. Let P be a point on disc making an angle θ with x axis at a distance r from the center. Choose cylindrical coordinates with the coordinates (r, θ) in the plane and the $z - axis$. consider the mass element dm show in the figure below. The area element of mass element dm is the product of arc length $rd\theta$ and the radial width dr. That is

$$dA = rdrd\theta \tag{11.20.1}$$

Since the disc is uniform, the mass per unit area is a constant,

$$\rho = \frac{dm}{dA} = \frac{m_{total}}{A} = \frac{m}{\pi R^2} \tag{11.20.2}$$

From (11.20.1) and (11.20.2)

$$dm = \rho dA = \frac{m}{\pi R^2} rdrd\theta \tag{11.20.3}$$

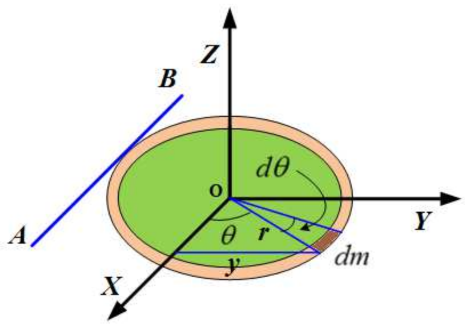

Figure 11.47: Ring

When the disc rotates, the mass element dm traces out a circle of radius r ; that is, the distance from the center is the perpendicular distance from the $z - axis$. Now moment of inertia about various axes can be determined as follows.

11.20.6 About the Diameter of the Disc OR an Axis in the Plane of the Disc and Passing Through its Center

Since the disc lies in xy plane having center at origin, so x axis or y axis can be considered as the diameter of the disc/an axis in the plane of the ring and passing through its center. Hence moment of inertia about x axis and y axis is the same. The moment of inertia of mass dm about x axis is

$$\begin{aligned} dI_{xx} &= dmy^2 \\ &= \frac{m}{\pi R^2} rdrd\theta \left(r\sin\theta\right)^2 \\ &= \frac{mr^3}{\pi R^2}\sin^2\theta drd\theta \end{aligned}$$

The moment of inertia of the disc about x axis integral is now an integral in two dimensions; the angle θ varies from $\theta = 0$ to $\theta = 2\pi$, and the radial coordinate r varies from $r = 0$ to

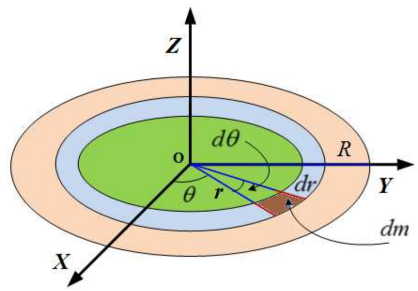

Figure 11.48: Uniform circular disc

$r = R$. Thus the limits of the integral are

$$
\begin{aligned}
I_{xx} &= \frac{m}{\pi R^2}\int_0^R\int_0^{2\pi} \sin^2\theta r^3 d\theta dr \\
&= \frac{m}{\pi R^2}\int_0^R\int_0^{2\pi} \left(\frac{1-\cos 2\theta}{2}\right) r^3 d\theta dr \\
&= \frac{m}{2\pi R^2}\int_0^R \left|\theta - \frac{\sin 2\theta}{2}\right|_0^{2\pi} r^3 dr \\
&= \frac{m}{2\pi R^2}2\pi \left|\frac{r^4}{4}\right|_0^R \\
&= \frac{m}{4R^2}R^4 \\
&= \frac{mR^2}{4} \qquad (11.20.4)
\end{aligned}
$$

Similarly the moment of inertia of the ring about y axis is

$$I_{yy} = \frac{mR^2}{4}$$

Hence the moment of inertia about the diameter of the ring is

$$I_d = \frac{mR^2}{4}$$

11.20.7 About an Axis Perpendicular to Plane of the Disc and Passing Through its Center

Since the disc lies in xy plane having center at origin, so z axis can be considered as an axis perpendicular to plane of the disc and passing through its center. Then r be the distance of mass dm from z axis, so its moment of inertia z axis is

$$\begin{aligned} dI_{zz} &= dmr^2 \\ &= \frac{m}{\pi R^2} rdrd\theta r^2 \\ &= \frac{m}{\pi R^2} r^3 drd\theta \end{aligned}$$

Next the moment of inertia of the ring about z axis is

$$\begin{aligned} I_{zz} &= \frac{m}{\pi R^2} \int_0^R \int_0^{2\pi} r^3 d\theta dr \\ &= \frac{m}{\pi R^2} \int_0^R |\theta|_0^{2\pi} r^3 dr \\ &= \frac{m}{\pi R^2} 2\pi \left| \frac{r^4}{4} \right|_0^R \\ &= \frac{2m}{R^2} \frac{R^4}{4} \\ &= \frac{mR^2}{2} \end{aligned}$$

Second Approach - Perpendicular axis theorem
Using perpendicular axis theorem, the moment of inertia of mass m about z axis is

$$\begin{aligned} I_{zz} &= I_{xx} + I_{yy} \\ &= \frac{mR^2}{4} + \frac{mR^2}{4} \\ &= \frac{mR^2}{2} \end{aligned}$$

11.20.8 About a Line Tangent to Disc - Parallel Axis Theorem

Consider a tangent line to disc passes through a point on the rim of the disc, so two cases arise:

1. The tangent line is parallel to z axis

2. The tangent line is parallel to diameter *i.e.* parallel to x axis or y axis

(1) **About a line tangent to disc and parallel to z axis**
Let AB be the line tangent to ring and parallel to z axis. Then its distance from z axis is R. Using theorem of parallel axis, the moment of inertia of mass m about AB axis is

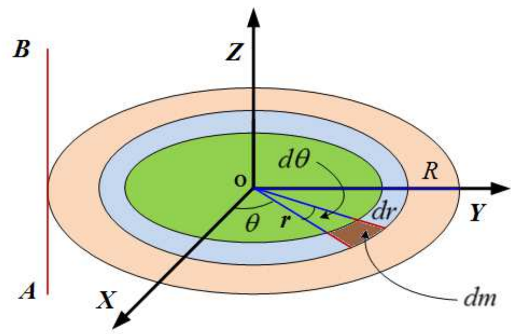

Figure 11.49: Uniform circular disc

$$\begin{aligned} I_{AB} &= I_{zz} + m(\text{distance between axes})^2 \\ &= \frac{mR^2}{2} + mR^2 \\ &= \frac{3}{2}mR^2 \end{aligned}$$

(2) **About a line parallel to diameter of the ring**
Let AB be the line parallel to diameter of the ring. Then its distance from diameter is R. Using theorem of parallel axis, the moment of inertia of mass m about AB axis is

$$\begin{aligned} I_{AB} &= I_d + m(\text{distance between axes})^2 \\ &= \frac{mR^2}{4} + mR^2 \\ &= \frac{5}{4}mR^2 \end{aligned}$$

11.20.9 Moment of Inertia of a Uniform Elliptic Disc

Consider a thin uniform elliptic disc of mass m and center at origin in xy plane. Let its major axis of length $2a$ along x axis and minor axis of length $2b$ along y axis. Let P be a point on disc making an angle θ with x axis at a distance r from the center. Choose

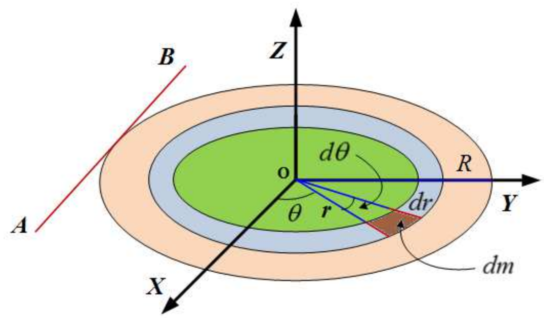

Figure 11.50: Uniform circular disc

cylindrical coordinates with the coordinates (r, θ) in the plane and the $z - axis$. Then

$$\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta \end{aligned}$$

Consider a strip of infinitesimal width dy and mass element dm at a distance y from x axis show in the figure 11.51. The area element of mass element dm is

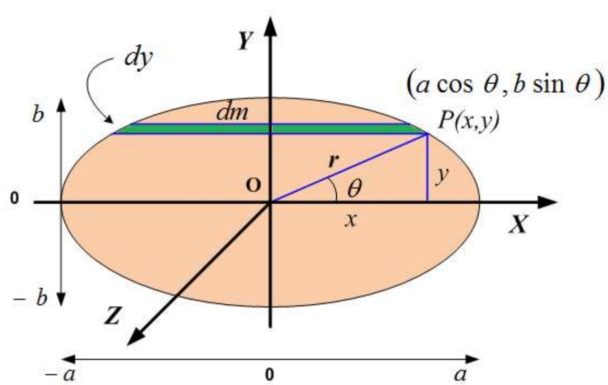

Figure 11.51: Uniform elliptic disc

$$dA = 2xdy \tag{11.20.5}$$

Since the disc is uniform, the mass per unit area is a constant,

$$\rho = \frac{dm}{dA} = \frac{m_{total}}{A} \quad (11.20.6)$$

$$= \frac{dm}{2xdy} = \frac{m}{\pi ab} \quad (11.20.7)$$

From (11.20.6), we can write

$$dm = \rho dA = \frac{2m}{\pi ab} xdy \quad (11.20.8)$$

Now moment of inertia of mass element dm about major axes (x axis) is

$$\begin{aligned} dI_{xx} &= dmy^2 \\ &= \frac{2m}{\pi ab} xdy\ y^2 \\ &= \frac{2m}{\pi b} y^2 \left(1 - \frac{y^2}{b^2}\right) dy \end{aligned}$$

The moment of inertia of the disc about x axis is now an integral from $y = -b$ to $y = b$

$$I_{xx} = \frac{2m}{\pi b} \int_{-b}^{b} y^2 \left(1 - \frac{y^2}{b^2}\right) dy \quad (11.20.9)$$

Next

$$\begin{aligned} y &= b \sin\theta \\ dy &= b \cos\theta d\theta \\ \frac{y}{b} &= \sin\theta \end{aligned}$$

When $y = -b$, then $\theta = -\frac{\pi}{2}$ and when $y = b$, then $\theta = \frac{\pi}{2}$.
Using all above results, Eq. (11.21.79) becomes

$$\begin{aligned}
I_{xx} &= \frac{2mb^2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta d\theta \\
&= \frac{mb^2}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 2\theta d\theta \\
&= \frac{mb^2}{4\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\cos 4\theta)\, d\theta \\
&= \frac{mb^2}{4\pi} \left| \theta - \frac{\sin 4\theta}{4} \right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{mb^2}{4\pi} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) \\
&= \frac{mb^2}{4}
\end{aligned}$$

Similarly the moment of inertia of the disc about minor axis (y axis) is

$$I_{yy} = \frac{ma^2}{4\pi}$$

The moment of inertia about an axis perpendicular to the plane and passing through the center of the disc (z axis) is given by using perpendicular axis theorem

$$\begin{aligned}
I_{zz} &= \frac{ma^2}{4\pi} + \frac{ma^2}{4\pi} \\
&= \frac{m}{4\pi} \left(a^2 + b^2 \right)
\end{aligned}$$

11.21 Three Dimensional

In this section some examples will be presented in the following categories.

1. Cartesian Coordinates.
2. Cylindrical Coordinates.
3. Spherical coordinates.

11.21.1 Cartesian Coordinates

In this section, the moment of inertia of a cube and cupid will be calculated.
Example Calculate the inertia tensor for a homogeneous cube of density ρ, mass m , and side length a. Let one corner be at the origin, and three adjacent edges lie along the coordinate axes (see Figure 11.52).
Solution. We use equation (9.20) to calculate the components of the inertia tensor. Because of the symmetry of the problem, it is easy to see that the three moments of inertia I_{xx}, I_{yy} and I_{zz} are equal and that same holds for all of the products of inertia. So we calculate only I_{xx}. Then volume of mass m is

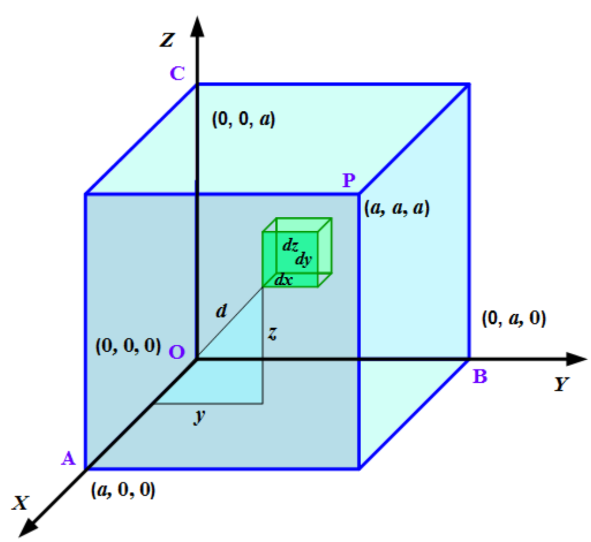

Figure 11.52: A cube with edges along coordinate axis.

$$V = a^3$$

and the density is

$$\rho = \frac{m}{V}$$

or

$$m = \rho\, a^3 \tag{11.21.1}$$

Consider a cube of infinitesimal volume dV and mass element dm at a distance d from x axis show in the figure 11.52. This distance is

$$d = \sqrt{y^2 + z^2}$$

The volume of mass element dm is

$$dV = dxdydz \tag{11.21.2}$$

Since the cube is uniform, the mass per unit area is a constant,

$$\begin{aligned}\rho &= \frac{dm}{dV} = \frac{m_{total}}{V_{total}} \\ &= \frac{dm}{dV} = \frac{m}{V}\end{aligned} \tag{11.21.3}$$

From (11.21.13), we can write

$$dm = \rho dV = \frac{m}{V} dV \tag{11.21.4}$$

Now moment of inertia of mass element dm about x axis is

$$\begin{aligned}dI_{xx} &= dmd^2 \\ &= \rho dV\ d^2 \\ &= \rho \left(y^2 + z^2\right) dV\end{aligned}$$

Using (11.6.4), the moment of inertia of the cube about x axis is now an integral from $x = 0$ to $x = a$, $y = 0$ to $y = a$ and $z = 0$ to $z = a$

$$\begin{aligned}I_{xx} = A &= \iiint\limits_V \rho \left(y^2 + z^2\right) dV \\ &= \rho \int_0^a \int_0^a \int_0^a \left(y^2 + z^2\right)\ dzdydx\end{aligned}$$

First integrating with respect to z

$$= \rho \int_0^a \int_0^a \left[y^2 z + \frac{z^3}{3} \right]_0^a dydx$$

$$= \rho \int_0^a \int_0^a \left(y^2 a + \frac{1}{3} a^3 \right) dydx$$

$$= \rho a \int_0^a \int_0^a \left(y^2 + \frac{1}{3} a^2 \right) dydx$$

Next integrating with respect to y

$$= \rho a \int_0^a \left[\frac{y^3}{3} + \frac{1}{3} a^2 y \right]_0^a dx$$

$$= \frac{1}{3} \rho a \int_0^a \left(a^3 + a^2 a \right) dx$$

$$= \frac{2}{3} \rho a^4 \int_0^a dx$$

Finally integrating with respect to x

$$= \frac{2}{3} \rho a^4 \left[x \right]_0^a$$

$$= \frac{2}{3} \rho a^5$$

$$= \frac{2}{3} \left(\rho\, a^3 \right) a^2$$

Using (11.21.1), (11.21.15) can be written as

$$A = \frac{2}{3} m a^2 \quad (11.21.5)$$

So we have

$$A = \frac{2}{3} m a^2 = B = C \quad (11.21.6)$$

Also using (11.6.7) the products of inertia $w.r.t.$ pair of axes (oy, oz) are as

$$D = \iiint_V \rho yz dV$$

$$= \rho \int_0^a \int_0^a \int_0^a yz\ dzdydx$$

First integrating with respect to z

$$= \rho \int_0^a \int_0^a y \left[\frac{z^2}{2}\right]_0^a dydx$$

$$= \rho \int_0^a \int_0^a y \left(\frac{a^2}{2}\right) dydx$$

$$= \rho \frac{a^2}{2} \int_0^a \int_0^a ydydx$$

Next integrating with respect to y

$$= \rho \frac{a^2}{2} \int_0^a \left[\frac{y^2}{2}\right]_0^a dx$$

$$= \rho \frac{a^2}{2} \int_0^a \left(\frac{a^2}{2}\right) dx$$

$$= \rho \frac{a^4}{4} \int_0^a dx$$

Finally integrating with respect to x

$$= \rho \frac{a^4}{4} \left[x\right]_0^a$$

$$= \rho \frac{a^4}{4} (a)$$

$$= \rho a^3 \left(\frac{a^2}{4}\right)$$

Using (11.21.1), (11.21.7) can be written as

$$D = m \frac{a^2}{4} \tag{11.21.7}$$

Similarly the products of inertia *w.r.t.* pair of axes (oz, ox) and (ox, oy) respectively are as under

$$E = m \frac{a^2}{4} \tag{11.21.8}$$

$$F = m \frac{a^2}{4} \tag{11.21.9}$$

The inertia matrix $[I]$ can be written as

$$[I] \quad = \quad \begin{pmatrix} \frac{2}{3}ma^2 & -\frac{1}{4}ma^2 & -\frac{1}{4}ma^2 \\ \\ -\frac{1}{4}ma^2 & \frac{2}{3}ma^2 & -\frac{1}{4}ma^2 \\ \\ -\frac{1}{4}ma^2 & -\frac{1}{4}ma^2 & \frac{2}{3}ma^2 \end{pmatrix} \tag{11.21.10}$$

If we define $\alpha = ma^2$, the above matrix can be rewritten as

$$[I] \quad = \quad \begin{pmatrix} \frac{2}{3}\alpha & -\frac{1}{4}\alpha & -\frac{1}{4}\alpha \\ \\ -\frac{1}{4}\alpha & \frac{2}{3}\alpha & -\frac{1}{4}\alpha \\ \\ -\frac{1}{4}\alpha & -\frac{1}{4}\alpha & \frac{2}{3}\alpha \end{pmatrix}$$

The **radius of gyration** about axis of rotation can be calculated by using (11.4.3). First about x axis is

$$K_x = \sqrt{\frac{I_{xx}}{m}} \quad = \quad \sqrt{\frac{2}{3}}\ a$$

and about y axis is

$$K_y = \sqrt{\frac{I_{yy}}{m}} \quad = \quad \sqrt{\frac{2}{3}}\ a$$

finally about z axis is

$$K_z = \sqrt{\frac{I_{zz}}{m}} \quad = \quad \sqrt{\frac{2}{3}}\ a$$

Since the inertia matrix is not a diagonal matrix, the coordinate system cannot be the principle axes system. We could find the principle axis system by diagonalizing $[I]$. Since inertia matrix (11.21.21) is symmetric, using (11.11.13) its characteristic equation is

$$\left(\frac{\alpha}{12}\right)^3 \begin{vmatrix} 8-k & - -3 & -3 \\ \\ -3 & 8-k & -3 \\ \\ -3 & -3 & 8-k \end{vmatrix} \quad = \quad 0$$

This equation has three real roots.

$$
\begin{aligned}
(8-k)^3 - 27(8-k) - 54 &= 0 \\
-k^3 + 24^2 - 165k + 242 &= 0 \\
(2-k)\,(k-11)^2 &= 0
\end{aligned}
$$

the roots are $k = 2$ and $k = 11$
or $k_1 = \frac{1}{6}\alpha$, $k_2 = \frac{11}{12}\alpha$ and $k_3 = \frac{11}{12}\alpha$
Using (11.11.13), the inertia matrix for principal axes through O is

$$
\begin{pmatrix} A^* & 0 & 0 \\ 0 & B^* & 0 \\ 0 & 0 & C^* \end{pmatrix} = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{6}\alpha & 0 & 0 \\ 0 & \frac{11}{12}\alpha & 0 \\ 0 & 0 & \frac{11}{12}\alpha \end{pmatrix}
$$

Hence the principal moment of inertia about O are

$$
\begin{aligned}
A^* &= \frac{1}{6}ma^2 \\
B^* &= \frac{11}{12}ma^2 \\
C^* &= \frac{11}{12}ma^2
\end{aligned}
$$

Here $B^* = C^*$ *i.e.* two principal moment of inertia are identical, so there must be one principal axis corresponding to A^* through O.
To find the direction of the principal axis associated with A^*, we use $k = k_1 = \frac{1}{6}\alpha$ in (11.11.6)

$$
\begin{aligned}
\left(\frac{2}{3}\alpha - \frac{1}{6}\alpha\right)\omega_x - \frac{1}{4}\alpha\omega_y - \frac{1}{4}\alpha\omega_z &= 0 \\
-\frac{1}{4}\alpha\omega_x + \left(\frac{2}{3}\alpha - \frac{1}{6}\alpha\right)\omega_y - \frac{1}{4}\alpha\omega_z &= 0 \\
-\frac{1}{4}\alpha\omega_x - \frac{1}{4}\alpha\omega_y + \left(\frac{2}{3}\alpha - \frac{1}{6}\alpha\right)\omega_z &= 0
\end{aligned}
$$

The above system can be written as

$$\begin{aligned} 2\omega_x - \omega_y - \omega_z &= 0 \\ -\omega_x + 2\omega_y - \omega_z &= 0 \\ -\omega_x - \omega_y + \omega_z &= 0 \end{aligned}$$

This system reduces to

$$\begin{aligned} 2\omega_x - \omega_y - \omega_z &= 0 \\ +\omega_y - \omega_z &= 0 \end{aligned}$$

Here ω_z is free variable, let it be 1, then we have

$$\omega_x = \omega_y = \omega_z$$

and the desired ratios are

$$\omega_x : \omega_y : \omega_z = 1 : 1 : 1$$

As a result, when the cube rotates about an axis that has associated with it the moment of inertia $I = \frac{1}{6}\alpha = \frac{1}{6}ma^2$, the projections of ω on the three coordinate axes are all equal. Hence this principal axis corresponds to the diagonal of the cube.
Since $B = C$, two moments are equal, the orientation of the principal axis associated with these moments is arbitrary, need only lie in a plane normal to the diagonal of the cube. Hence we have infinite sets of principal axes, with one fixed principal axis.

Example 11.21.1. *Consider a uniform solid rectangular block of mass m and dimension $2a \times 2b \times 2c$. Find moment of inertia about coordinate axes with O as center of mass.*

Solution Consider $OXYZ$ a regular trihedral system and $PABC$ a solid rectangular block (parallelepiped) with origin O as center of mass. Also

$$PA = 2a,\ PB = 2b \text{ and } PC = 2c$$

Then volume of mass m is

$$V = 8abc$$

and the density is

$$\rho = \frac{m}{V}$$

or

$$m = \rho\, 8abc \tag{11.21.11}$$

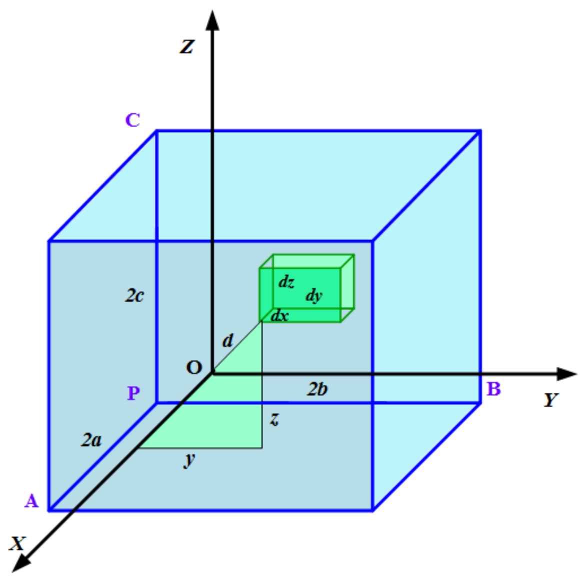

Figure 11.53: uniform solid rectangular block

Consider a cube of infinitesimal volume dV and mass element dm at a distance d from x axis show in the figure 11.53. This distance is

$$d = \sqrt{y^2 + z^2}$$

The volume of mass element dm is

$$dV = dxdydz \tag{11.21.12}$$

Since the cube is uniform, the mass per unit area is a constant,

$$\begin{aligned} \rho &= \frac{dm}{dV} = \frac{m_{total}}{V_{total}} \\ &= \frac{dm}{dV} = \frac{m}{V} \end{aligned} \tag{11.21.13}$$

From (11.21.13), we can write

$$dm = \rho dV = \frac{m}{V} dV \tag{11.21.14}$$

Now moment of inertia of mass element dm about x axis is

$$\begin{aligned} dI_{xx} &= dmd^2 \\ &= \rho dV\ d^2 \\ &= \rho\left(y^2+z^2\right)dV \end{aligned}$$

Using (11.6.4), the moment of inertia of the cube about x axis is now an integral from $x=-a$ to $x=a$, $y=-b$ to $y=b$ and $z=-c$ to $z=c$

$$\begin{aligned} I_{xx}=A &= \iiint\limits_V \rho\left(y^2+z^2\right)dV \\ &= \rho\int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c}\left(y^2+z^2\right)\ dzdydx \end{aligned}$$

First integrating with respect to z

$$\begin{aligned} &= \rho\int_{-a}^{a}\int_{-b}^{b}\left[y^2z+\frac{z^3}{3}\right]_{-c}^{c}\ dydx \\ &= \rho\int_{-a}^{a}\int_{-b}^{b}\left(y^2\left(c+c\right)+\frac{1}{3}\left(c^3+c^3\right)\right)\ dydx \\ &= 2c\rho\int_{-a}^{a}\int_{-b}^{b}\left(y^2+\frac{1}{3}c^2\right)\ dydx \end{aligned}$$

Next integrating with respect to y

$$\begin{aligned} &= 2c\rho\int_{-a}^{a}\left[\frac{y^3}{3}+\frac{1}{3}a^2y\right]_{-b}^{b}\ dx \\ &= \frac{4}{3}\rho c\int_{-a}^{a}\left(b^3+c^2b\right)\ dx \\ &= \frac{4}{3}\rho bc\int_{-a}^{a}\left(b^2+c^2\right)\ dx \end{aligned}$$

Finally integrating with respect to x

$$\begin{aligned} &= \frac{4}{3}\rho bc\left(b^2+c^2\right)\left[x\right]_{-a}^{a} \\ &= \frac{8}{3}\rho abc\left(b^2+c^2\right) \\ &= \frac{1}{3}\left(\rho\ 8abc\right)\left(b^2+c^2\right) \end{aligned}$$

Using (11.21.1), (11.21.15) can be written as

$$A = \frac{1}{3}m\left(b^2+c^2\right) \tag{11.21.15}$$

Similarly

$$B = \frac{1}{3}m\left(a^2+c^2\right) \tag{11.21.16}$$

and

$$C = \frac{1}{3}m\left(a^2+c^2\right) \tag{11.21.17}$$

Also using (11.6.7) the products of inertia *w.r.t.* pair of axes (oy, oz) are as

$$\begin{aligned} D &= \iiint\limits_V \rho yz dV \\ &= \rho \int\limits_{-a}^{a}\int\limits_{-b}^{b}\int\limits_{-c}^{c} yz\ dzdydx \end{aligned}$$

First integrating with respect to z

$$\begin{aligned} &= \rho \int\limits_{-a}^{a}\int\limits_{-b}^{b} y\left[\frac{z^2}{2}\right]_{-c}^{c}\ dydx \\ &= \rho \int\limits_{-a}^{a}\int\limits_{-b}^{b} y\frac{1}{2}\left(c^2-c^2\right)\ dydx \\ &= 0 \end{aligned}$$

Using (11.21.56), (11.21.26) can be written as

$$D = 0 \tag{11.21.18}$$

Similarly the products of inertia *w.r.t.* pair of axes (oz, ox) and (ox, oy) respectively are as under

$$E = 0 \tag{11.21.19}$$

$$F = 0 \tag{11.21.20}$$

The inertia matrix $[I]$ can be written as

$$[I] = \begin{pmatrix} \frac{1}{3}m\left(b^2+c^2\right) & 0 & 0 \\ 0 & \frac{1}{3}m\left(a^2+c^2\right) & 0 \\ 0 & 0 & \frac{1}{3}m\left(a^2+c^2\right) \end{pmatrix} \tag{11.21.21}$$

Example 11.21.2. *Consider a uniform solid rectangular block of mass m and dimension $2a \times 2b \times 2c$. Find M.I. about coordinate axes with Oas center of mass.*

Solution Consider $OXYZ$ a regular trihedral system and $PABC$ a solid rectangular block (parallelepiped) with origin as center of mass. Let G be the center of mass. Then $O = G$. Also

$$PA = 2a,\ PB = 2b \text{ and } PC = 2c$$

Then volume of mass m is

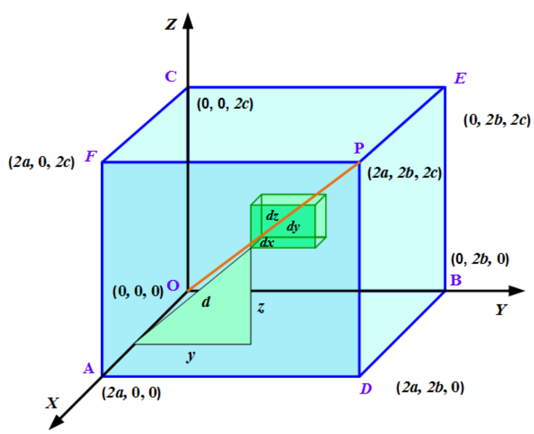

Figure 11.54: uniform solid rectangular block

$$V = 8abc$$

and the density is

$$\rho = \frac{m}{V}$$

or

$$m = \rho\, 8abc \tag{11.21.22}$$

Using (11.6.4) $M.I$ of the block about $x - axis$ is

$$\begin{aligned} I_{xx} = A &= \iiint\limits_V \rho \left(y^2 + z^2\right) dV \\ &= \rho \int\limits_0^{2a} \int\limits_0^{2b} \int\limits_0^{2c} \left(y^2 + z^2\right)\ dzdydx \end{aligned}$$

First integrating with respect to z

$$\begin{aligned}
&= \rho \int_0^{2a} \int_0^{2b} \left[y^2 z + \frac{z^3}{3} \right]_0^{2c} \, dydx \\
&= \rho \int_0^{2a} \int_0^{2b} \left(y^2 2c + \frac{8}{3} c^3 \right) \, dydx \\
&= \rho 2c \int_0^{2a} \int_0^{2b} \left(y^2 + \frac{4}{3} c^2 \right) \, dydx
\end{aligned}$$

Next integrating with respect to y

$$\begin{aligned}
&= \rho 2c \int_0^{2a} \left[\frac{y^3}{3} + \frac{4}{3} c^2 y \right]_0^{2b} \, dx \\
&= \frac{2}{3} \rho c \int_0^{2a} \left(8b^3 + 4c^2 2b \right) \, dx \\
&= \frac{16}{3} \rho bc \int_0^{2a} \left(b^2 + c^2 \right) \, dx
\end{aligned}$$

Finally integrating with respect to x

$$\begin{aligned}
&= \frac{16}{3} \rho bc \left(b^2 + c^2 \right) \left[x \right]_0^{2a} \\
&= \frac{16}{3} \rho bc \left(b^2 + c^2 \right) 2a \\
&= \frac{4}{3} \left(\rho \; 8abc \right) \left(b^2 + c^2 \right)
\end{aligned}$$

Using (11.21.56), (11.21.23) can be written as

$$A = \frac{4}{3} m \left(b^2 + c^2 \right) \tag{11.21.23}$$

Similarly

$$B = \frac{4}{3} m \left(a^2 + c^2 \right) \tag{11.21.24}$$

and

$$C = \frac{4}{3} m \left(a^2 + c^2 \right) \tag{11.21.25}$$

Also using (11.6.7) the products of inertia *w.r.t.* pair of axes (oy, oz) are as

$$\begin{aligned} D &= \iiint\limits_V \rho yz dV \\ &= \rho \int_0^{2a}\int_0^{2b}\int_0^{2c} yz \; dzdydx \end{aligned}$$

First integrating with respect to z

$$\begin{aligned} &= \rho \int_0^{2a}\int_0^{2b} y \left[\frac{z^2}{2}\right]_0^{2c} \; dydx \\ &= \rho \int_0^{2a}\int_0^{2b} y \left(\frac{4}{2}c^2\right) \; dydx \\ &= \rho 2c^2 \int_0^{2a}\int_0^{2b} ydydx \end{aligned}$$

Next integrating with respect to y

$$\begin{aligned} &= \rho 2c^2 \int_0^{2a} \left[\frac{y^2}{2}\right]_0^{2b} \; dx \\ &= \rho 2c^2 \int_0^{2a} \left(\frac{4b^2}{2}\right) \; dx \\ &= 4\rho b^2 c^2 \int_0^{2a} \; dx \end{aligned}$$

Finally integrating with respect to x

$$\begin{aligned} &= 4\rho b^2 c^2 \Big[x\Big]_0^{2a} \\ &= 4\rho b^2 c^2 \,(2a) \\ &= (\rho \; 8abc)\, bc \end{aligned}$$

Using (11.21.56), (11.21.26) can be written as

$$D \;=\; m \; bc \qquad (11.21.26)$$

Similarly the products of inertia *w.r.t.* pair of axes (oz, ox) and (ox, oy) respectively are as under

$$E = m\ ac \tag{11.21.27}$$
$$F = m\ ab \tag{11.21.28}$$

The inertia matrix $[I]$ can be written as

$$[I] = \begin{pmatrix} \frac{4}{3}m\left(b^2+c^2\right) & -\ mab & -\ mac \\ \\ -\ mab & \frac{4}{3}m\left(a^2+c^2\right) & -\ mbc \\ \\ -\ mac & -\ mbc & \frac{4}{3}m\left(a^2+b^2\right) \end{pmatrix}$$

$$= \frac{m}{3}\begin{pmatrix} 4\left(b^2+c^2\right) & -3\ ab & -3\ ac \\ \\ -3\ ab & 4\left(a^2+c^2\right) & -3\ bc \\ \\ -3\ ac & -3\ bc & 4\left(a^2+b^2\right) \end{pmatrix} \tag{11.21.29}$$

If the dimension of the rectangular block is $a \times b \times c$, then the above matrix is.

$$[I] = \begin{pmatrix} \frac{1}{3}m\left(b^2+c^2\right) & -\frac{1}{4}\ mab & -\frac{1}{4}\ mac \\ \\ -\frac{1}{4}\ mab & \frac{1}{3}m\left(a^2+c^2\right) & -\frac{1}{4}\ mbc \\ \\ -\frac{1}{4}\ mac & -\frac{1}{4}\ mbc & \frac{1}{3}m\left(a^2+b^2\right) \end{pmatrix}$$

$$= \frac{m}{12}\begin{pmatrix} 4\left(b^2+c^2\right) & -3\ ab & -3\ ac \\ \\ -3\ ab & 4\left(a^2+c^2\right) & -3\ bc \\ \\ -3\ ac & -3\ bc & 4\left(a^2+b^2\right) \end{pmatrix} \tag{11.21.30}$$

Example 11.21.3. *Consider a uniform solid rectangular block of mass m and dimension $2a \times 2b \times 2c$. Find the equation of the momental ellipsoid for a corner O of the block, referred to the edges through O as co-ordinates axes and hence determine M.I. about OP, where P is the point diagonally opposite to O.*

Solution Consider $OXYZ$ a regular trihedral system and $OABCDEFP$ a solid rectangular block (parallelepiped). Then

$$OA = 2a, \; OB = 2b \text{ and } OC = 2c$$

Then volume of mass m is

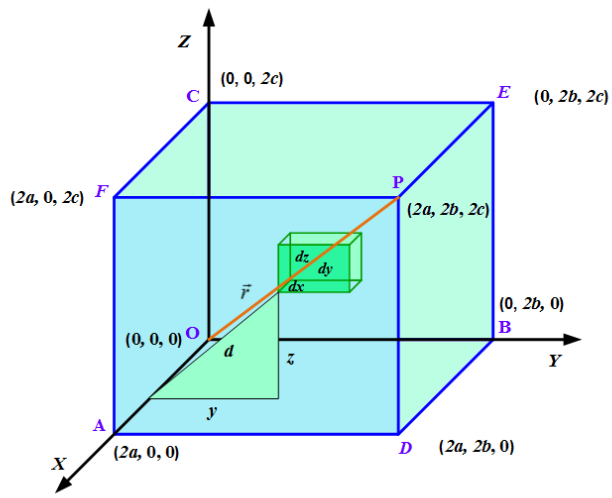

Figure 11.55: Uniform solid rectangular block

$$V = 8abc$$

and the density is

$$\rho = \frac{m}{V}$$

or

$$m = \rho \, 8abc \tag{11.21.31}$$

The $M.I$ of the block about coordinate axis are

$$A = \frac{4}{3}m\left(b^2 + c^2\right)$$

$$B = \frac{4}{3}m\left(a^2 + c^2\right)$$

and

$$C = \frac{4}{3}m\left(a^2+b^2\right)$$

Also the products of inertia *w.r.t.* pair of axes (oy,oz), (oz,ox) and (ox,oy) are as

$$D = m\ bc$$

$$E = m\ ac$$

and

$$F = m\ ab$$

Using these results in (11.16.6) standard equation of momental ellipsoid, we get

$$\begin{aligned} Ir^2 &= \frac{4}{3}m\left[\left(b^2+c^2\right)x^2+\left(c^2+a^2\right)y^2+\left(a^2+b^2\right)z^2\right] \\ &- 2m\left[bcyz+cazx+abxy\right] \end{aligned} \tag{11.21.32}$$

(11.21.32) is the required equation of momental ellipsoid.
Next $M.I.$ about OP axis is:
From the Fig. we can write

$$\vec{OP} = \vec{r} = \langle 2a, 2b, 2c\rangle$$

then

$$r^2 = 4\left(a^2+b^2+c^2\right)$$

Take

$$x = 2a,\ y = 2b \text{ and } z = 2c$$

Using above values in (11.21.32), I_{OP} is

$$\begin{aligned} 4\left(a^2+b^2+c^2\right)I_{OP} &= \frac{4}{3}m\left[4a^2\left(b^2+c^2\right)+4b^2\left(c^2+a^2\right)+4c^2\left(a^2+b^2\right)\right] \\ &- 2m\left[bc(4bc)+ca(4ca)+ab(4ab)\right] \\ 4\left(a^2+b^2+c^2\right)I_{OP} &= \frac{8}{3}m\left[4\left(a^2b^2+a^2c^2+b^2c^2\right)\right]-8m\left(a^2b^2+a^2c^2+b^2c^2\right) \\ I_{OP} &= \frac{2}{3}m\frac{\left(a^2b^2+a^2c^2+b^2c^2\right)}{\left(a^2+b^2+c^2\right)} \end{aligned} \tag{11.21.33}$$

(11.21.33) is the $M.I.$ about OP axis or diagonal.

11.21.2 Cylindrical Coordinates

Moment of Inertia of a Cylinder about its Axis Consider a cylinder of mass m, the origin coincides with center of mass and z axis as the axis of rotation. Let P be a point on cylinder. Take r (distance of P from z axis) in xy plane making an angle θ with x axis. Then cylindrical coordinates of P are

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

Let R be the radius of base and L be the height of the cylinder, then volume of mass m is

$$V = \pi R^2 L$$

Consider an infinitesimal volume dV having mass element dm at a distance d from origin shown in the figure 11.56. The volume of mass element dm is

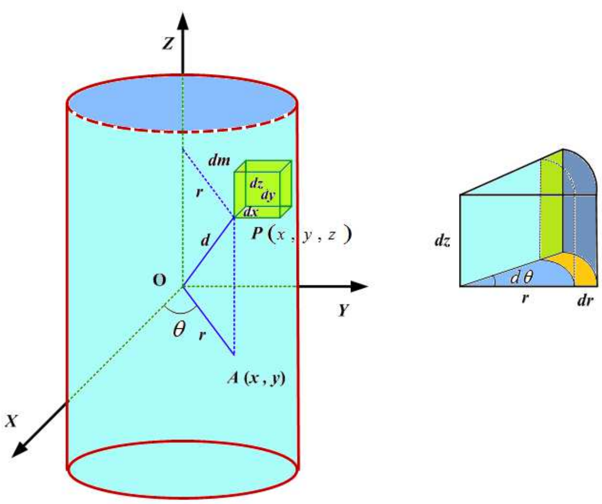

Figure 11.56: Cylinder

$$dV = dxdydz = rdrd\theta dz \tag{11.21.34}$$

Since the cylinder is uniform, the mass per unit volume is a constant,

$$\begin{aligned} \rho &= \frac{dm}{dV} = \frac{m_{total}}{V} \\ &= \frac{dm}{rdrd\theta dz} = \frac{m}{\pi R^2 L} \end{aligned} \tag{11.21.35}$$

From (11.21.35), we can write

$$dm = \frac{m}{\pi R^2 L} r dr d\theta dz \qquad (11.21.36)$$

(a) The moment of inertia of mass element dm about z axis is

$$\begin{aligned} dI_{zz} &= dm r^2 \\ &= \frac{m}{\pi R^2 L} r dr d\theta dz \; r^2 \\ &= \frac{m}{\pi R^2 L} r^3 dr d\theta dz \end{aligned}$$

The moment of inertia of the cylinder about z axis is now an integral from $r = 0$ to $r = R$, $\theta = 0$ to $\theta = 2\pi$ and $z = -\frac{L}{2}$ to $z = \frac{L}{2}$

$$I_{zz} = \frac{m}{\pi R^2 L} \int_{-\frac{L}{2}}^{-\frac{L}{2}} \int_0^{2\pi} \int_0^R r^3 dr d\theta dz$$

First integrating with respect to r

$$= \frac{m}{\pi R^2 L} \left| \frac{r^4}{4} \right|_0^R \int_{-\frac{L}{2}}^{-\frac{L}{2}} \int_0^{2\pi} d\theta dz$$

Next integrating with respect to θ

$$= \frac{mR^2}{4\pi L} \left| \theta \right|_0^{2\pi} \int_{-\frac{L}{2}}^{-\frac{L}{2}} dz$$

Finally integrating with respect to z

$$\begin{aligned} &= \frac{mR^2}{2L} \left| z \right|_{-\frac{L}{2}}^{-\frac{L}{2}} \\ &= \frac{mR^2}{2} \end{aligned}$$

Hence the moment of inertia about z axis is

$$C = \frac{mR^2}{2} \qquad (11.21.37)$$

(b) **Moment of Inertia of a Cylinder about an Axis Normal to the axis of the cylinder and passing through its Center of Mass - I_{xx} or I_{yy}**
Here x axis and y axis both are normal to the axis of the cylinder and passing through its

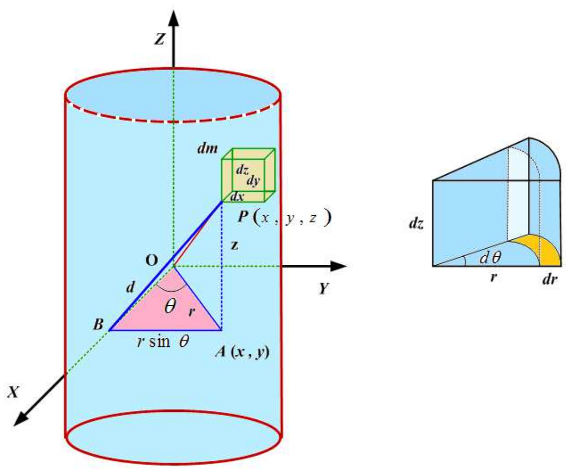

Figure 11.57: Cylinder

center of mass. Hence it is enough to find about any one axis. Let's consider x axis. To calculate moment of inertia about x axis, see Fig. 11.57. Here d is distance of point mass dm from x axis, and can be calculated as the hypotonus of right angle triangle PAB

$$d^2 = z^2 + r^2 \sin^2 \theta$$

The moment of inertia of mass element dm about x axis is

$$\begin{aligned} dI_{xx} &= dmd^2 \\ &= \frac{m}{\pi R^2 L} rdrd\theta dz \left(z^2 + r^2 \sin^2 \theta\right) \\ &= \frac{m}{\pi R^2 L} r \left(z^2 + r^2 \sin^2 \theta\right) drd\theta dz \end{aligned}$$

The moment of inertia of the cylinder about x axis is now an integral from $r = 0$ to $r = R$, $\theta = 0$ to $\theta = 2\pi$ and $z = -\frac{L}{2}$ to $z = \frac{L}{2}$

$$\begin{aligned} I_{xx} &= \frac{m}{\pi R^2 L} \int_{-\frac{L}{2}}^{-\frac{L}{2}} \int_0^{2\pi} \int_0^R r \left(z^2 + r^2 \sin^2 \theta\right) drd\theta dz \\ &= \frac{m}{\pi R^2 L} \int_{-\frac{L}{2}}^{-\frac{L}{2}} \int_0^{2\pi} \int_0^R \left(rz^2 + r^3 \sin^2 \theta\right) drd\theta dz \end{aligned}$$

First integrating with respect to r

$$= \frac{m}{\pi R^2 L} \int_{-\frac{L}{2}}^{-\frac{L}{2}} \int_{0}^{2\pi} \left| \frac{r^2}{2} z^2 + \frac{r^4}{4} \sin^2 \theta \right|_0^R d\theta dz$$

Next integrating with respect to θ

$$= \frac{m}{4\pi L} \int_{-\frac{L}{2}}^{-\frac{L}{2}} \left| 2z^2\theta + \frac{1}{2} R^2 \left(\theta + \frac{1}{2} \sin 2\theta \right) \right|_0^{2\pi} dz$$

$$= \frac{m}{4\pi L} \int_{-\frac{L}{2}}^{-\frac{L}{2}} \left| 4z^2\pi + R^2\pi \right| dz$$

Finally integrating with respect to z

$$= \frac{m}{4L} \left| \frac{4}{3} z^3 + zR^2 \right|_{-\frac{L}{2}}^{-\frac{L}{2}}$$

$$= \frac{m}{4L} \left(\frac{4}{3} \left[\left(\frac{L}{2} \right)^3 - \left(-\frac{L}{2} \right)^3 \right] + \left[\left(\frac{L}{2} \right) - \left(-\frac{L}{2} \right) \right] R^2 \right)$$

$$= m \left(\frac{L^2}{12} + \frac{R^2}{4} \right)$$

Hence the moment of inertia about x axis is

$$A = m \left(\frac{L^2}{12} + \frac{R^2}{4} \right) \tag{11.21.38}$$

Also the moment of inertia about y axis is

$$I_{yy} = B = m \left(\frac{L^2}{12} + \frac{R^2}{4} \right) \tag{11.21.39}$$

To calculate the products of inertia, i.e. the off-diagonal terms in the tensor of inertia, we make use of orthogonal trigonometric relations. First the products of inertia *w.r.t.* pair of axes (oy, oz) are as

$$I_{yz} = I_{zy} = \iiint_V \rho yz dV$$

$$D = \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{0}^{2\pi} \int_{0}^{R} r^2 \sin\theta z \, dr d\theta dz$$

Since

$$\int_0^{2\pi} \sin\theta d\theta \;=\; \int_0^{2\pi} \cos\theta d\theta = 0 \tag{11.21.40}$$

Hence

$$D \;=\; 0$$

Next the products of inertia *w.r.t.* pair of axes (ox, oz) are as

$$\begin{aligned} I_{xz} = I_{zx} &= \iiint_V \rho xz dV \\ E &= \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^{2\pi} \int_0^R r^2 \cos\theta z \; dr d\theta dz \end{aligned}$$

Using (11.21.40)

$$E \;=\; 0 \tag{11.21.41}$$

Finally the products of inertia *w.r.t.* pair of axes (ox, oy) are as

$$\begin{aligned} I_{xy} = I_{yz} &= \iiint_V \rho xy dV \\ F &= \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^{2\pi} \int_0^R r^2 \sin\theta \cos\theta \; dr d\theta dz \end{aligned}$$

Since

$$\int_0^{2\pi} \sin\theta \cos\theta d\theta = 0 \tag{11.21.42}$$

Hence

$$F \;=\; 0$$

Also This is due to the symmetry of the cylinder (under consideration), the x, y and z axes, coincide with its the principal axes, and hence the products of inertia are zero. The inertia matrix $[I]$ can be written as

$$[I] = \begin{pmatrix} m\left(\frac{L^2}{12}+\frac{R^2}{4}\right) & 0 & 0 \\ 0 & m\left(\frac{L^2}{12}+\frac{R^2}{4}\right) & 0 \\ 0 & 0 & \frac{mR^2}{2} \end{pmatrix} \qquad (11.21.43)$$

Now let us assume that the cylinder rotates with angular velocity ω about the z axis. Thus the angular velocity vector for the solid is

$$\vec{\omega} = \langle 0, 0, \omega \rangle$$

Using (11.11.14) the angular momentum of the system is

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$$
$$\vec{L} = C\,\omega\hat{k}$$

Using (11.21.37)

$$\vec{L} = \frac{mR^2}{2}\omega\hat{k} \qquad (11.21.44)$$

represents the rotational motion about a fixed axis.
The kinetic energy of the system is

$$\begin{aligned} K &= \frac{1}{2}\vec{\omega}\cdot\vec{L} \\ &= \frac{1}{4}mR^2\omega^2 \end{aligned}$$

Since $\vec{L}$ is a vector constant, so the torque vanishes,

$$\vec{N} = \frac{d\vec{L}}{dt} = 0$$

Hence the system can continue spinning about its axis of rotation (z axis) without the need to apply any torque to maintain it in this state.

Example 11.21.4. *Consider a uniform solid right circular cone of mass m with hight h and base radius r. Find moment of inertia about its axis of symmetry.*

Solution Consider a regular trihedral system and a solid right circular cone of mass m with height h and vertex at origin O. Let x axis be the axis of symmetry. Then

$$OA = h, \text{ and } AB = r$$

Then volume of mass m is

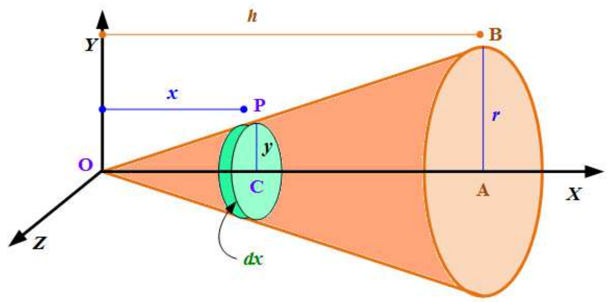

Figure 11.58: uniform solid right circular cone

$$V = \frac{1}{3}\pi r^2 h \tag{11.21.45}$$

and the density is

$$\rho = \frac{m}{V}$$

or

$$m = \frac{1}{3}\rho\pi r^2 h \tag{11.21.46}$$

Consider a disc of mass element dm and infinitesimal width dx having radius y, at a distance x from O, shown in figure 11.58. From similar triangles OAB and OCP, we can write

$$\frac{OA}{AB} = \frac{OC}{CP}$$
$$\frac{h}{r} = \frac{x}{y}$$

and y can be written as

$$y = \frac{r}{h}x \tag{11.21.47}$$

The volume of mass element dm is

$$dV = \pi y^2 dx \tag{11.21.48}$$

Since the cone is uniform, the mass per unit area is a constant,

$$\begin{aligned}\rho &= \frac{dm}{dV} = \frac{m_{total}}{V_{total}} \\ &= \frac{dm}{dV} = \frac{m}{V}\end{aligned} \tag{11.21.49}$$

From (11.21.49), we can write

$$dm = \rho dV = \frac{m}{V} dV$$

From (11.21.45) and (11.21.48), dm can be written as

$$dm = \frac{3my^2}{r^2 h} dx$$

Using (11.21.47), dm is

$$dm = \frac{3m}{h^3} x^2 dx \tag{11.21.50}$$

Using (11.20.5), the moment of inertia of mass element dm about x axis is

$$dI_{xx} = \frac{1}{2} dm y^2$$

Using (11.21.47) and (11.21.50), dI_{xx} is

$$dI_{xx} = \frac{3}{2} \frac{mr^2}{h^5} x^4 dx$$

Using (11.6.4), the moment of inertia of the cone about x axis is now an integral from $x = 0$ to $x = h$.

$$\begin{aligned}I_{xx} = A &= \frac{3}{2} \frac{mr^2}{h^5} \int_0^h x^4 dx \\ &= \frac{3}{2} \frac{mr^2}{h^5} \left[\frac{x^5}{5} \right]_0^h \\ &= \frac{3}{10} mr^2\end{aligned}$$

Hence

$$A = \frac{3}{10} mr^2 \tag{11.21.51}$$

2. About y axis
Using (11.20.5), the moment of inertia of mass element dm about its diameter is

$$I_d = \frac{1}{4} dmy^2 \tag{11.21.52}$$

Since the diameter is passing through the center of mass of small disc, so we can use parallel axis theorem to find moment of inertia about y axis, by taking CP axis (diameter) parallel to y axis.

$$dI_{yy} = \frac{1}{4} dmy^2 + dmx^2$$

Using (11.21.47) and (11.21.50), dI_{xx} is

$$\begin{aligned} dI_{yy} &= \frac{3}{4}\frac{mr^2}{h^5}x^4dx + 3\frac{m}{h^3}x^4dx \\ &= \left[\frac{3}{4}\frac{mr^2}{h^5} + 3\frac{m}{h^3}\right]x^4dx \end{aligned}$$

Using (11.6.4), the moment of inertia of the cone about y axis is now an integral from $x = 0$ to $x = h$.

$$\begin{aligned} I_{yy} = B &= \left(\frac{3}{4}\frac{mr^2}{h^5} + 3\frac{m}{h^3}\right)\int_0^h x^4dx \\ &= \left(\frac{3}{4}\frac{mr^2}{h^5} + 3\frac{m}{h^3}\right)\left[\frac{x^5}{5}\right]_0^h \\ &= \frac{3}{20}m\left(r^2 + 4h^2\right) \end{aligned}$$

Hence

$$B = \frac{3}{20}m\left(r^2 + 4h^2\right) \tag{11.21.53}$$

Similarly about z axis

$$C = I_{zz} = \frac{3}{20}m\left(r^2 + 4h^2\right) \tag{11.21.54}$$

Example 11.21.5. *Consider a uniform solid paraboloid of mass m. Find moment of inertia about its axis of symmetry.*

Solution Consider a regular trihedral system and a paraboloid of mass m with vertex at origin O. Let x axis be the axis of symmetry. Then

$$OA = h, \text{ and } AB = r$$

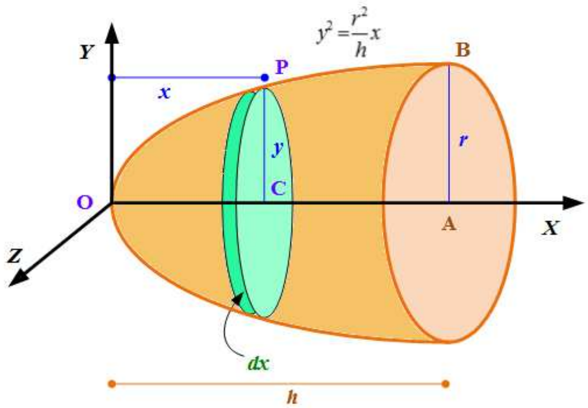

Figure 11.59: solid paraboloid

Then volume of mass m is

$$V = \frac{1}{2}\pi r^2 h \tag{11.21.55}$$

and the density is

$$\rho = \frac{m}{V}$$

or

$$m = \frac{1}{2}\rho\pi r^2 h \tag{11.21.56}$$

For y consider the Fig 11.60, from the definition of a parabola, we have

$$y^2 = 4ax \tag{11.21.57}$$

and

$$|BF| = |BE| \tag{11.21.58}$$

From Fig 11.60, $|BE|$ can be written as

$$|BE| = |AO| + |OD| = h + a \tag{11.21.59}$$

For $|BF|$, consider right angle triangle BAF,

$$\begin{aligned} |BF|^2 &= |BA|^2 + |AF|^2 \\ &= r^2 + (h-a)^2 \end{aligned} \tag{11.21.60}$$

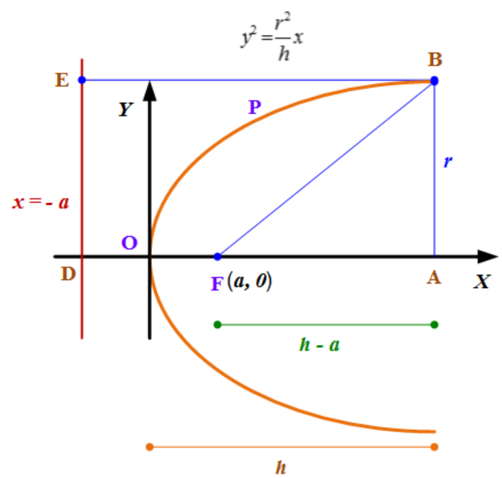

Figure 11.60: Parabola

Using (11.21.59) and (11.21.60), (11.21.58) can be written as

$$\begin{aligned} (h+a)^2 &= r^2 + (h-a)^2 \\ 2ah &= r^2 + -2ah \\ 4a &= \frac{r^2}{h} \end{aligned} \tag{11.21.61}$$

Using (11.21.61), (11.21.57) can be written as

$$y^2 = \frac{r^2}{h}x \tag{11.21.62}$$

Consider a disc of mass element dm and infinitesimal width dx having radius y, at a distance x from O, shown in figure 11.59. The volume of mass element dm is

$$dV = \pi y^2 dx \tag{11.21.63}$$

Since the cone is uniform, the mass per unit area is a constant,

$$\begin{aligned} \rho &= \frac{dm}{dV} = \frac{m_{total}}{V_{total}} \\ &= \frac{dm}{dV} = \frac{m}{V} \end{aligned} \tag{11.21.64}$$

From (11.21.64), we can write

$$dm = \rho dV = \frac{m}{V}dV$$

From (11.21.55) and (11.21.63), dm can be written as

$$dm = 2\frac{my^2}{r^2h}dx$$

Using (11.21.62), dm is

$$dm = 2\frac{m}{h^2}xdx \tag{11.21.65}$$

Using (11.20.5), the moment of inertia of mass element dm about x axis is

$$dI_{xx} = \frac{1}{2}dmy^2$$

Using (11.21.62) and (11.21.65), dI_{xx} is

$$dI_{xx} = \frac{mr^2}{h^3}x^2dx$$

Using (11.6.4), the moment of inertia of the paraboloid about x axis is now an integral from $x = 0$ to $x = h$.

$$\begin{aligned} I_{xx} = A &= \frac{mr^2}{h^3}\int_0^h x^2dx \\ &= \frac{mr^2}{h^3}\left[\frac{x^3}{3}\right]_0^h \\ &= \frac{1}{3}mr^2 \end{aligned}$$

Hence

$$A = \frac{1}{3}mr^2 \tag{11.21.66}$$

2. About y axis

Using (11.20.5), the moment of inertia of mass element dm about its diameter is

$$I_d = \frac{1}{4}dmy^2 \tag{11.21.67}$$

Since the diameter is passing through the center of mass of small disc, so we can use parallel axis theorem to find moment of inertia about y axis, by taking CP axis (diameter) parallel to y axis.

$$dI_{yy} = \frac{1}{4}dmy^2 + dmx^2$$

Using (11.21.47) and (11.21.50), dI_{xx} is

$$\begin{aligned} dI_{yy} &= \frac{3}{4}\frac{mr^2}{h^5}x^4dx + 3\frac{m}{h^3}x^4dx \\ &= \left[\frac{3}{4}\frac{mr^2}{h^5} + 3\frac{m}{h^3}\right]x^4dx \end{aligned}$$

Using (11.6.4), the moment of inertia of the cone about y axis is now an integral from $x = 0$ to $x = h$.

$$\begin{aligned} I_{yy} = A &= \left(\frac{3}{4}\frac{mr^2}{h^5} + 3\frac{m}{h^3}\right)\int_0^h x^4dx \\ &= \left(\frac{3}{4}\frac{mr^2}{h^5} + 3\frac{m}{h^3}\right)\left[\frac{x^5}{5}\right]_0^h \\ &= \frac{3}{20}m\left(r^2 + 4h^2\right) \end{aligned}$$

Hence

$$B = \frac{3}{20}m\left(r^2 + 4h^2\right) \qquad (11.21.68)$$

Similarly about z axis

$$C = I_{zz} = \frac{3}{20}m\left(r^2 + 4h^2\right) \qquad (11.21.69)$$

11.21.3 Spherical Coordinates

Moment of Inertia of a Thin Spherical Shell (Hollow sphere) about an Axis/Diameter
Consider a thin spherical shell of mass m and radius R, the origin coincides with center of mass. Since the spherical bodies are completely symmetrical in all directions about their center, so the moment of inertia about all coordinate axes is same. Select any one axis as the axis of rotation. Let it be z axis. Let P be a point on sphere, such that OP vector makes an angle θ with y axis. Then the distance d of P from z axis is

$$d = R\cos\theta$$

Let the shell be of infinitesimally thin wall of uniform density. In this case the volume density of the mass has to be replaced by the surface density of mass ρ. Then the surface area of mass m is

$$A = 4\pi R^2$$

Consider an infinitesimal mass element dm at a distance R from origin between angles θ and $\theta + d\theta$, shown in the figure 11.61. The mass element dm is a circular ribbon of radius

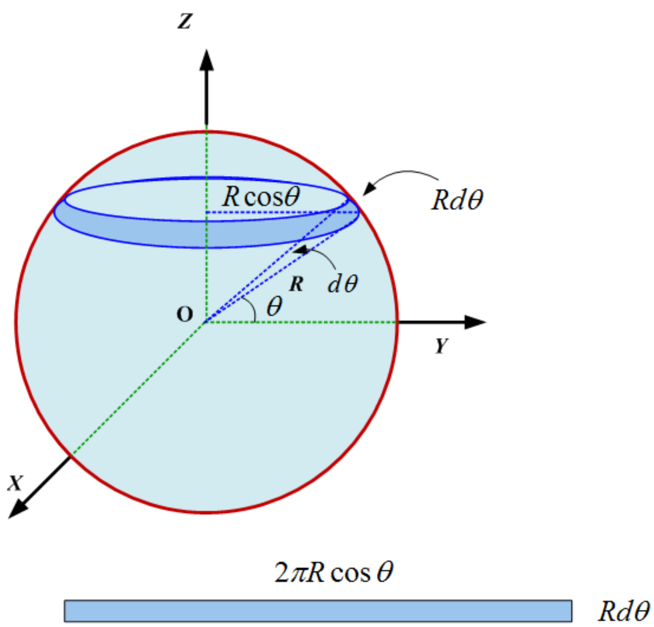

Figure 11.61: Hollow sphere

$R\cos\theta$, having width $Rd\theta$, spread out over the area of the sphere.
Then length of the ribbon = $2\pi(R\cos\theta)$
and width of the ribbon = $Rd\theta$
And area of the ribbon is

$$\begin{aligned} dA &= 2\pi(R\cos\theta)Rd\theta \\ &= 2\pi R^2\cos\theta d\theta \end{aligned} \tag{11.21.70}$$

Since the sphere is uniform, the mass per unit area is a constant,

$$\begin{aligned} \rho &= \frac{dm}{dA} = \frac{m_{total}}{A} \\ &= \frac{dm}{2\pi R^2\cos\theta d\theta} = \frac{m}{4\pi R^2} \end{aligned} \tag{11.21.71}$$

From (11.21.71), we can write

$$dm = \frac{m}{2}\cos\theta d\theta \tag{11.21.72}$$

The moment of inertia of mass element dm about z axis is

$$\begin{aligned}
dI_{zz} &= dmd^2 \\
&= \frac{m}{2}\cos\theta d\theta(R\cos\theta)^2 \\
&= \frac{m}{2}R^2\cos^3\theta d\theta
\end{aligned}$$

The moment of inertia of the mass m about z axis is now an integral from $\theta = 0$ to $\theta = \pi$

$$\begin{aligned}
I_{zz} &= \frac{m}{2}R^2\int_0^{\pi}\cos^3\theta d\theta \\
&= \frac{m}{2}R^2\int_0^{\pi}\left[\cos\theta\left(\cos^2\theta\right)\right]d\theta \\
&= \frac{m}{2}R^2\int_0^{\pi}\left[\cos\theta\left(1-\sin^2\theta\right)\right]d\theta \\
&= \left(\frac{m}{2}R^2\right)2\int_0^{\frac{\pi}{2}}\left(\cos\theta+\sin^2\theta\cos\theta\right) \\
&= mR^2\left|\left(\sin\theta-\frac{1}{3}\sin^3\theta\right)\right|_0^{\frac{\pi}{2}} \\
&= mR^2\left(1-\frac{1}{3}\right) \\
&= mR^2\left(\frac{2}{3}\right) \\
&= \frac{2}{3}mR^2 \qquad (11.21.73)
\end{aligned}$$

The moment of inertia of the mass m about diameter or x axis or y axis is

$$I_{xx} = \frac{2}{3}mR^2$$

11.21.4 Moment of Inertia of a Solid Sphere About z Axis

Consider a solid sphere of mass m and radius R, the origin coincides with center of mass. Since the spherical bodies are completely symmetrical in all directions about their center, so the moment of inertia about all coordinate axes is same. Select any one axis as the axis of rotation. Let it be z axis. Let P be a point on sphere, such that OP vector makes an angle θ with x axis. Then the distance d of P from z axis is

$$d = r\sin\phi$$

Since the sphere is of uniform density, so the volume density of the mass m is ρ. Then the volume of mass m is

$$V = \frac{4}{3}\pi R^3$$

Consider an infinitesimal mass element dm at a distance r from origin between angles θ and ϕ, shown in the figure 11.62. The volume of mass element dm is

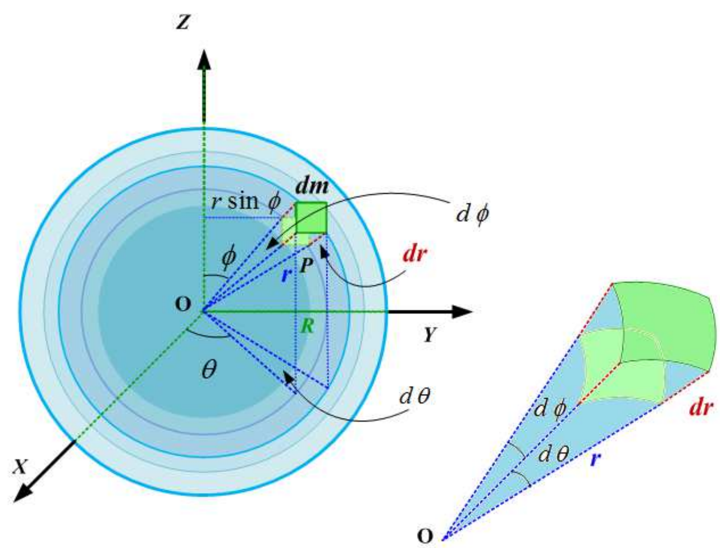

Figure 11.62: Solid Sphere

$$dV = r^2 \sin\phi dr d\theta d\phi$$

Since the sphere is uniform, the mass per unit area is a constant,

$$\begin{aligned} \rho &= \frac{dm}{dV} = \frac{m_{total}}{V} \\ &= \frac{dm}{r^2 \sin\phi dr d\theta d\phi} = \frac{m}{\frac{4}{3}\pi R^3} \end{aligned} \tag{11.21.74}$$

From (11.21.74), we can write

$$dm = \frac{3}{4}\frac{m}{\pi R^3} r^2 \sin\phi dr d\theta d\phi \tag{11.21.75}$$

The moment of inertia of mass element dm about z axis is

$$\begin{aligned} dI_{zz} &= dmd^2 \\ &= \frac{3}{4}\frac{m}{\pi R^3} r^2 \sin\phi dr d\theta d\phi (r\sin\phi)^2 \\ &= \frac{3}{4}\frac{m}{\pi R^3} r^4 \sin^3\phi dr d\theta d\phi \end{aligned}$$

The moment of inertia of the mass m about z axis is now an integral from $r = 0$ to $r = R$, $\theta = 0$ to $\theta = 2\pi$ and $\phi = 0$ to $\phi = \pi$

$$\begin{aligned} I_{zz} &= \frac{3}{4}\frac{m}{\pi R^3} \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^4 \sin^3\phi dr d\theta d\phi \\ &= \frac{3}{4}\frac{m}{\pi R^3} \left(\left| \frac{r^5}{5} \right|_0^R |\theta|_0^{2\pi} \right) \int_0^{\pi} \sin^3\phi d\phi \end{aligned} \tag{11.21.76}$$

The integral can be solved as

$$\begin{aligned} \int_0^{\pi} \sin^3\theta d\theta &= \int_0^{\pi} \left[\sin\theta \left(\sin^2\theta\right)\right] d\theta \\ &= \int_0^{\pi} \left[\sin\theta \left(1 - \cos^2\theta\right)\right] d\theta \\ &= \int_0^{\pi} \left(\sin\theta - \cos^2 \sin\theta d\theta\right) \\ &= \left| \left(-\cos\theta + \frac{1}{3}\cos^3\theta \right) \right|_0^{\pi} \\ &= \left(2 - \frac{2}{3}\right) \\ &= \frac{4}{3} \end{aligned} \tag{11.21.77}$$

$$\begin{aligned} &= \frac{3}{4}\frac{m}{\pi R^3}\frac{R^5}{5}(2\pi)\frac{4}{3} \\ &= \frac{2}{5} m R^2 \end{aligned} \tag{11.21.78}$$

The moment of inertia of the mass m about diameter or x axis or y axis is

$$I_{xx} = \frac{2}{5} m R^2$$

11.21.5 Moment of Inertia of a Hemisphere About Coordinate Axis

Consider a hemisphere of mass m having center of circular base at origin O with radius r. Let y axis be the axis of symmetry. Then

$$OA = r, \; CP = x \text{ and } OC = y$$

Then volume of mass m is

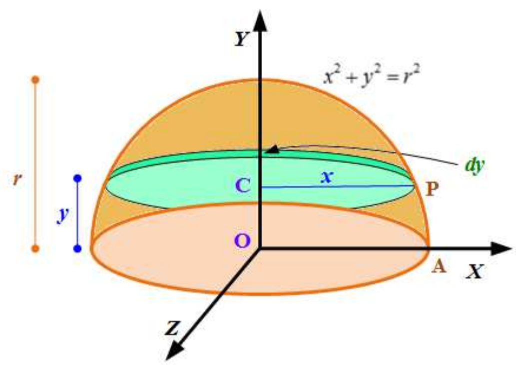

Figure 11.63: Hemisphere

$$V = \frac{2}{3}\pi r^3 \tag{11.21.79}$$

and the density is

$$\rho = \frac{m}{V}$$

or

$$m = \frac{2}{3}\rho\pi r^3 \tag{11.21.80}$$

Consider a disc of mass element dm and infinitesimal width dy having radius x, at a distance y from O, shown in figure 11.63. The equation of the circle (along the boundary of hemisphere) is

$$x^2 + y^2 = r^2$$

and x^2 can be written as

$$x^2 = r^2 - y^2 \tag{11.21.81}$$

The volume of mass element dm is

$$dV = \pi x^2 dy \tag{11.21.82}$$

Using (11.21.83), (11.21.84) can be written as

$$dV = \pi \left(r^2 - y^2\right) dy \tag{11.21.83}$$

Since the cone is uniform, the mass per unit area is a constant,

$$\begin{aligned} \rho &= \frac{dm}{dV} = \frac{m_{total}}{V_{total}} \\ &= \frac{dm}{dV} = \frac{m}{V} \end{aligned} \tag{11.21.84}$$

From (11.21.84), we can write

$$dm = \rho dV = \frac{m}{V} dV$$

Using (11.21.79) and (11.21.83), dm can be written as

$$dm = \frac{3mx^2}{r^2 h} dy$$

Using (11.21.47), dm is

$$dm = \frac{3}{2}\frac{m}{r^3}\left(r^2 - y^2\right) dy \tag{11.21.85}$$

Using (11.20.5), the moment of inertia of mass element dm about y axis is

$$dI_{yy} = \frac{1}{2} dm x^2$$

Using (11.21.85) and (11.21.83), dI_{yy} is

$$dI_{yy} = \frac{3}{4}\frac{m}{r^3}\left(r^2 - y^2\right)^2 dy$$

Using (11.6.4), the moment of inertia of the hemisphere about y axis is now an integral from $y = 0$ to $y = r$.

$$\begin{aligned} I_{yy} = B &= \frac{3}{4}\frac{m}{r^3}\int_0^r \left(r^4 - 2r^2y^2 + y^4\right) dy \\ &= \frac{3}{4}\frac{m}{r^3}\left[r^4 y - \frac{2}{3}r^2y^3 + \frac{1}{5}y^5\right]_0^r \\ &= \frac{3}{4}\frac{m}{r^3} r^5 \left[1 - \frac{2}{3} + \frac{1}{5}\right] \\ &= \frac{2}{5} m r^2 \end{aligned}$$

Hence

$$B = \frac{2}{5}mr^2 \tag{11.21.86}$$

Similarly about x axis

$$A = \frac{2}{5}mr^2 \tag{11.21.87}$$

and z axis

$$C = I_{zz} = \frac{2}{5}mr^2 \tag{11.21.88}$$

11.21.6 Moment of Inertia of a Ellipsoid About x Axis

Consider an ellipsoid of mass m having center at origin O with a along x axis, b along y and c along z axes (see figure 11.64). Then

$$OA = a,\ OB = b \text{ and } OC = c$$

The parametric equations of ellipsoid are

$$\begin{aligned} x &= ar\sin\phi\cos\theta \\ y &= br\sin\phi\sin\theta \\ z &= cr\cos\phi \end{aligned}$$

satisfying

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1$$

Then volume of mass m is

$$V = \frac{4}{3}\pi abc \tag{11.21.89}$$

and the density is

$$\begin{aligned} \rho &= \frac{m}{V} \\ &= \frac{3m}{4\pi abc} \end{aligned}$$

or

$$m = \frac{4}{3}\rho\pi abc \tag{11.21.90}$$

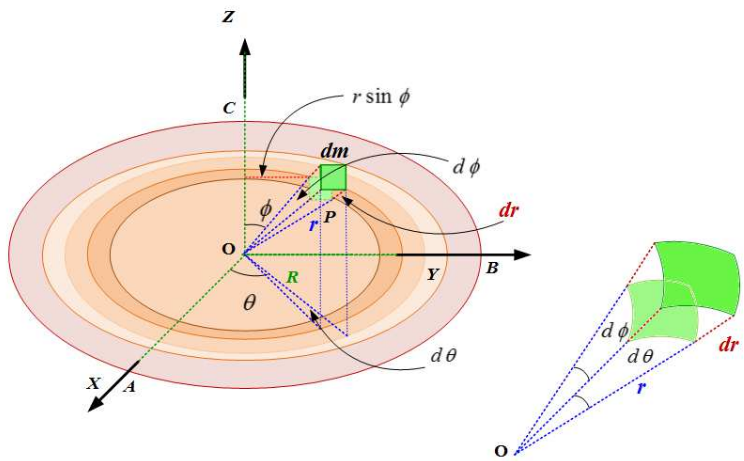

Figure 11.64: Ellipsoid

Consider an infinitesimal mass element dm at a distance r from origin between angles θ and ϕ, shown in the figure 11.64. Its volume is

$$dV = dxdydz = abcr^2 \sin\phi drd\phi d\theta \tag{11.21.91}$$

where $J = abcr^2 \sin\phi$ is the jacobian of the transformation. Using (11.6.4), the moment of inertia of the ellipsoid about x axis is now an integral from $0 \le r \le 1$, $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$ and $-\pi \le \theta \le \pi$. These limits of integration can also be chosen as $0 \le r \le 1$, $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$.

$$\begin{aligned}
I_{xx} &= \frac{3}{4}\frac{m}{\pi abc}\int_0^{2\pi}\int_0^{\pi}\int_0^{1}\left(b^2r^2\sin^2\phi\sin^2\theta + c^2r^2\cos^2\phi\right) abc\, r^2\sin\phi\; drd\phi d\theta \\
&= \frac{3}{4}\frac{m}{\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^4\left(b^2\sin^3\phi\sin^2\theta + c^2\cos^2\phi\sin\phi\right) drd\phi d\theta \\
&= \frac{3}{4}\frac{m}{\pi}\left|\frac{r^5}{5}\right|_0^1 \int_0^{2\pi}\int_0^{\pi}\left(b^2\sin^3\phi\sin^2\theta + c^2\cos^2\phi\sin\phi\right) d\phi d\theta \qquad (11.21.92)
\end{aligned}$$

The integrals

$$\int_0^{\pi}\cos^2\phi\sin\phi d\phi = \frac{2}{3}$$

see (11.21.77)

$$\int_0^{\pi} \sin^3 \phi d\phi = \frac{4}{3}$$

and

$$\int_0^{2\pi} \sin^2 \theta d\theta = \pi$$

Using them in (11.21.92)

$$\begin{aligned} I_{xx} &= \frac{3}{20}\frac{m}{\pi}\left(b^2\ \frac{4}{3}\pi + c^2\ 2\pi\frac{2}{3}\right) \\ &= \frac{1}{5}m\left(b^2+c^2\right) \end{aligned} \tag{11.21.93}$$

The moment of inertia of the mass m about x axis is

$$I_{xx} = \frac{1}{5}m\left(b^2+c^2\right)$$

Similarly about y axis is

$$I_{yy} = \frac{1}{5}m\left(a^2+c^2\right)$$

and about z axis is

$$I_{zz} = \frac{1}{5}m\left(a^2+b^2\right)$$

To calculate the products of inertia, we make use of orthogonal trigonometric relations. First the products of inertia $w.r.t.$ pair of axes (oy, oz) are as

$$\begin{aligned} I_{yz} = I_{zy} &= \iiint_V \rho yz dV \\ D &= \rho \int_0^{2\pi}\int_0^{\pi}\int_0^{1} bcr^2 \sin\phi \sin\theta\ \cos\phi abc\ r^2 \sin\phi\ drd\phi d\theta \\ &= \rho a b^2 c^2 \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^4 \sin\theta\ \sin^2\phi\ \cos\phi\ drd\phi d\theta \end{aligned}$$

Since (11.21.40) is

$$\int_0^{2\pi} \sin\theta d\theta = \int_0^{2\pi} \cos\theta d\theta = 0$$

Hence

$$D = 0$$

Next the products of inertia $w.r.t.$ pair of axes (ox, oz) are as

$$\begin{aligned}
I_{xz} = I_{zx} &= \iiint\limits_V \rho xz dV \\
E &= \rho \int\limits_0^{2\pi} \int\limits_0^{\pi} \int\limits_0^1 acr^2 \sin\phi \cos\theta \ \cos\phi abc\ r^2 \sin\phi\ drd\phi d\theta \\
&= \rho a^2 bc^2 \int\limits_0^{2\pi} \int\limits_0^{\pi} \int\limits_0^1 r^4 \cos\theta \ \sin^2\phi \ \cos\phi\ drd\phi d\theta
\end{aligned}$$

Using (11.21.40)

$$E = 0 \qquad (11.21.94)$$

Finally the products of inertia $w.r.t.$ pair of axes (ox, oy) are as

$$\begin{aligned}
I_{xy} = I_{yz} &= \iiint\limits_V \rho xy dV \\
F &= \rho \int\limits_0^{2\pi} \int\limits_0^{\pi} \int\limits_0^1 abr^2 \sin\phi \cos\theta \ \sin\phi \sin\theta\ abc\ r^2 \sin\phi\ drd\phi d\theta \\
&= \rho a^2 b^2 c \int\limits_0^{2\pi} \int\limits_0^{\pi} \int\limits_0^1 r^4 \cos\theta \sin\theta \ \sin^3\phi \ \cos\phi\ drd\phi d\theta
\end{aligned}$$

Since (11.21.42) is

$$\int\limits_0^{2\pi} \sin\theta \cos\theta d\theta = 0$$

Hence

$$F = 0$$

Also This is due to the symmetry of the ellipsoid (under consideration), the x, y and z axes, coincide with its the principal axes, and hence the products of inertia are zero. The inertia matrix $[I]$ can be written as

$$[I] = \begin{pmatrix} \frac{1}{5}m\left(b^2+c^2\right) & 0 & 0 \\ 0 & \frac{1}{5}m\left(a^2+c^2\right) & 0 \\ 0 & 0 & \frac{1}{5}m\left(a^2+b^2\right) \end{pmatrix} \quad (11.21.95)$$

11.21.7 Moment of Inertia of a Prolate Ellipsoid About x Axis

If a sphere is squash to make a shorter fatter shape (a bit like a burger). In such case it is called an prolate ellipsoid. If we chop it through the middle to get a circle, then the volume is the area of the circle times 2/3rd of the minor axis.
Consider an ellipsoid of mass m having center at origin O with a along x axis and b along y and z axes. Then

$$OA = a,\ OB = b \text{ and } OC = c = b$$

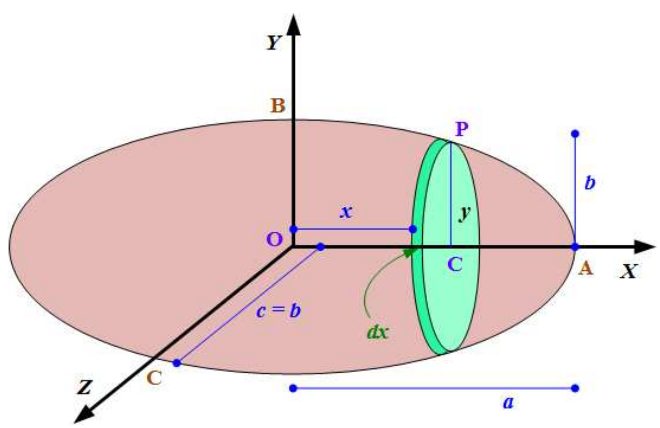

Figure 11.65: Prolate Ellipsoid

Then volume of mass m is

$$V = \frac{4}{3}\pi abc = \frac{4}{3}\pi ab^2 \quad (11.21.96)$$

and the density is

$$\rho = \frac{m}{V}$$

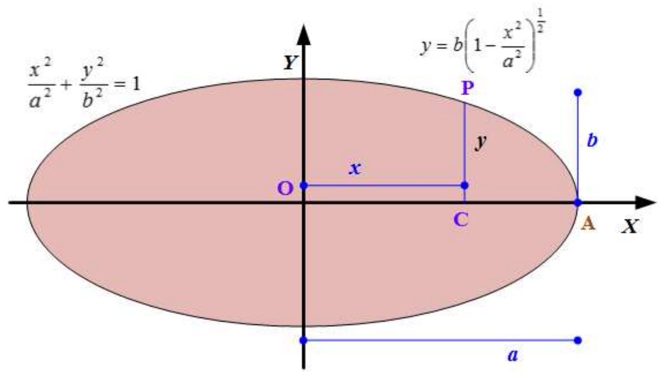

Figure 11.66: Prolate Ellipsoid

or

$$m = \frac{4}{3}\rho\pi ab^2 \tag{11.21.97}$$

Consider a disc of mass element dm of infinitesimal width dx at a distance x from O, shown in figure 11.65. The equation of the ellipse (along the boundary of ellipsoid) is (see figure 11.66)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

and y^2 can be written as

$$y^2 = b^2\left(1-\frac{x^2}{a^2}\right) \tag{11.21.98}$$

The volume of mass element dm is (area of circle $\times$ height)

$$dV = \pi y^2 dx \tag{11.21.99}$$

Using (11.21.98), (11.21.99) can be written as

$$dV = \pi b^2\left(1-\frac{x^2}{a^2}\right)dx \tag{11.21.100}$$

Since the ellipsoid is uniform, the mass per unit volume is a constant,

$$\begin{aligned}\rho &= \frac{dm}{dV} = \frac{m_{total}}{V_{total}} \\ &= \frac{dm}{dV} = \frac{m}{V}\end{aligned} \tag{11.21.101}$$

From (11.21.101), we can write

$$dm = \rho dV = \frac{m}{V}dV$$

Using (11.21.96) and (11.21.100), dm can be written as

$$dm = \frac{3}{4}m\frac{b^2}{a}\left(1-\frac{x^2}{a^2}\right)dx \tag{11.21.102}$$

Using (11.20.5), the moment of inertia of mass element dm about x axis is

$$dI_{xx} = \frac{1}{2}dmy^2$$

Using (11.21.98) and (11.21.102), dI_{xx} is

$$dI_{xx} = \frac{3}{8}\frac{b^2}{a}m\left(1-\frac{x^2}{a^2}\right)^2 dx$$

Using (11.6.4), the moment of inertia of the ellipsoid about x axis is now an integral from $x=-a$ to $x=a$.

$$\begin{aligned} I_{xx} = A &= \frac{3}{8}\frac{mb^2}{a^5}\int_{-a}^{a}\left(a^4-2a^2x^2+x^4\right)dx \\ &= \frac{3}{8}\frac{mb^2}{a^5}2\int_{0}^{a}\left(a^4-2a^2x^2+x^4\right)dx \\ &= \frac{3}{4}\frac{mb^2}{a^5}\left[a^4x-\frac{2}{3}a^2x^3+\frac{1}{5}x^5\right]_0^a \\ &= \frac{3}{4}\frac{mb^2}{a^5}a^5\left[1-\frac{2}{3}+\frac{1}{5}\right] \\ &= \frac{2}{5}mb^2 \end{aligned}$$

Hence

$$A = \frac{2}{5}mb^2 \tag{11.21.103}$$

Example 11.21.6. *Four particle of masses m, $2m$, $3m$ and $4m$ are located at (a,a,a), $(a,-a,-a)$, $(-a,a,-a)$ and $(-a,-a,a)$ respectively. Complete its inertia matrix and hence determine its principal moments of inertia.*

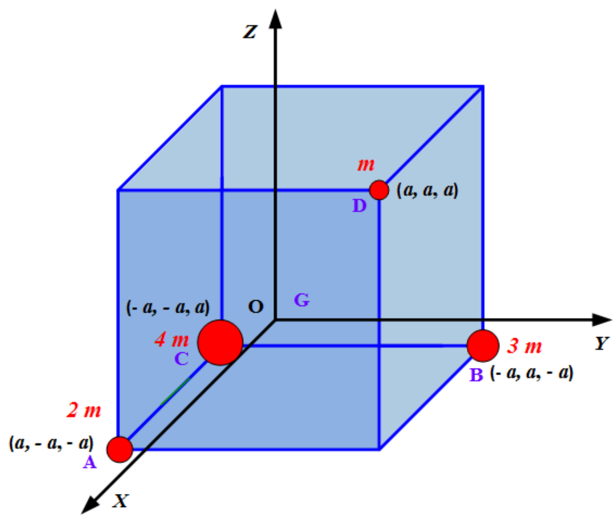

Figure 11.67: Four particles system

Solution: See the Fig. 11.67, the masses $m_1 = m$, $m_2 = 2m$, $m_3 = 3m$ and $m_4 = 4m$ are located at $D = (a, a, a)$, $A = (a, -a, -a)$, $B = (-a, a, -a)$ and $C = (-a, -a, a)$ respectively. Here the origan of regular trihedral system can be considered as the center of the cube formed by these points. For I_{xx}, the distance of m_1 from x axis is

$$d_1 = a^2 + a^2 = 2a^2 \text{ (see Fig. 11.68)}$$

Similarly

$$d_2 = 2a^2 = d_3 = d_4$$

Using (11.1.3), the moment of inertia about x axis is

$$\begin{aligned} I_{xx} = A &= \sum_{i=1}^{4} m_i d_i^2 \\ &= m_1 d_1^2 + m_2 d_2^2 + m_3 d_3^2 + m_4 d_4^2 \\ &= m2a^2 + 2m2a^2 + 3m2a^2 + 4m2a^2 \\ &= (m + 2m + 3m + 4m)\, 2a^2 \\ &= 20ma^2 \end{aligned}$$

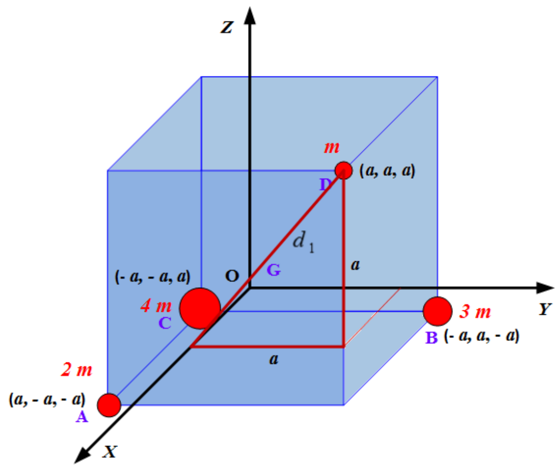

Figure 11.68: Four particles system

I_{xx} can also be calculated by using (11.5.7) as

$$\begin{aligned} I_{xx} = A &= \sum_{i=1}^{4} m_i \left(y_i^2 + z_i^2\right) \\ &= m_1 \left(y_1^2 + z_1^2\right) + m_2 \left(y_2^2 + z_2^2\right) + m_3 \left(y_3^2 + z_3^2\right) + m_4 \left(y_4^2 + z_4^2\right) \\ &= m \left(a^2 + a^2\right) + 2m \left(a^2 + a^2\right) + 3m \left(a^2 + a^2\right) + 4m \left(a^2 + a^2\right) \\ &= 20ma^2 \end{aligned}$$

Similarly

$$I_{yy} = B = 20ma^2$$

and

$$I_{zz} = C = 20ma^2$$

The product of inertia I_{yz} can be calculated by using (11.6.1)

$$\begin{aligned} I_{yz} = D &= \sum_{i=1}^{4} m_i y_i z_i \\ &= m_1 y_1 z_1 + m_2 y_2 z_2 + m_3 y_3 z_3 + m_4 y_4 z_4 \\ &= ma^2 + 2ma^2 - 3ma^2 - 4ma^2 \\ &= -4ma^2 \end{aligned}$$

The product of inertia I_{yz} can be calculated by using (11.6.2)

$$\begin{aligned} I_{xz} = E &= \sum_{i=1}^{4} m_i x_i z_i \\ &= m_1x_1z_1 + m_2x_2z_2 + m_3x_3z_3 + m_4x_4z_4 \\ &= ma^2 - 2ma^2 + 3ma^2 - 4ma^2 \\ &= -2ma^2 \end{aligned}$$

The product of inertia I_{xy} can be calculated by using (11.6.3)

$$\begin{aligned} I_{xy} = F &= \sum_{i=1}^{4} m_i x_i y_i \\ &= m_1x_1y_1 + m_2x_2y_2 + m_3x_3y_3 + m_4x_4y_4 \\ &= ma^2 - 2ma^2 - 3ma^2 + 4ma^2 \\ &= 0 \end{aligned}$$

The inertia matrix $[I]$ can be written as

$$\begin{aligned} [I_{ij}] &= \begin{pmatrix} 20ma^2 & 0 & 2ma^2 \\ 0 & 20ma^2 & 4ma^2 \\ 2ma^2 & 4ma^2 & 20ma^2 \end{pmatrix} \\ &= 2ma^2 \begin{pmatrix} 10 & 0 & 1 \\ 0 & 10 & 2 \\ 1 & 2 & 10 \end{pmatrix} \end{aligned} \tag{11.21.104}$$

Since inertia matrix (11.21.104) is symmetric, using (11.11.13) its characteristic equation is

$$\begin{vmatrix} 10-k & 0 & 1 \\ 0 & 10-k & 2 \\ 1 & 2 & 10-k \end{vmatrix} = 0$$

This equation has three real roots.

$$(10-k)\left((10-k)^2-5\right) = 0$$

the roots are $k = 10$ and $k = 10 \pm \sqrt{5}$
or $k_1 = 20ma^2$, $k_2 = 2ma^2\left(10+\sqrt{5}\right)$ and $k_3 = 2ma^2\left(10-\sqrt{5}\right)$
Using 11.11.13, the inertia matrix for principal axes through O is

$$\begin{pmatrix} A^* & 0 & 0 \\ 0 & B^* & 0 \\ 0 & 0 & C^* \end{pmatrix} = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}$$
$$= \begin{pmatrix} 20ma^2 & 0 & 0 \\ 0 & 2ma^2\left(10+\sqrt{5}\right) & 0 \\ 0 & 0 & 2ma^2\left(10-\sqrt{5}\right) \end{pmatrix}$$

Hence the principal moment of inertia about O are

$$\begin{aligned} A^* &= 20ma^2 \\ B^* &= 2ma^2\left(10+\sqrt{5}\right) \\ C^* &= 2ma^2\left(10-\sqrt{5}\right) \end{aligned}$$

Example 11.21.7. *For the body shown in Fig. 11.69, $I_{AB} = 34.65\ kg-m^2$ and $M = 0.875\ kg$. Find the radius of gyration of the body with respect to the AB axis.*

Solution: Here $I_{AB} = 34.65\ kg-m^2$ and $M = 0.875\ kg$. using (11.4.3), the radius of gyration

$$\begin{aligned} K &= \sqrt{\frac{I}{m}} \\ &= \sqrt{\frac{34.65}{0.875}} = \sqrt{39.6}\ m \\ &= 6.293\ m \end{aligned}$$

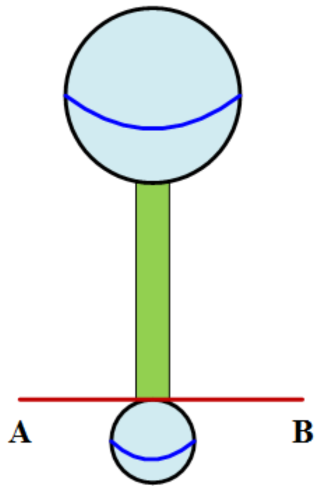

Figure 11.69: Rigid body

Example 11.21.8. *:- A rigid body is free to rotate about its centroid G, the principal moment of inertia about G are* $7, 25, 32$ *units respectively. The body has an angular velocity* ω *about a line through G, whose direction ratios are* $4 : 0 : 3$*. Show that after time t, the angular velocity about the principal axis of inertia about G is*

$$\vec{\omega} = \left\langle \frac{4}{5}\omega\cos\phi,\ \frac{4}{5}\omega\sin\phi,\ \frac{3}{5}\omega\cos\phi \right\rangle$$

Where

$$\tan\left(\frac{\phi}{2}\right) = \tanh\left(\frac{3\omega t}{10}\right) \tag{11.21.105}$$

Solution Consider $OXYZ$ a regular trihedral system, with O coincides with G. Let

$$\vec{\omega} = \langle \omega_x, \omega_y, \omega_z \rangle$$

be the angular velocity about principal axis at any time t. At $t = 0$, the angular velocity is

$$\vec{\omega}(0) = \left\langle \frac{4}{5}\omega, 0, \frac{3}{5}\omega \right\rangle \tag{11.21.106}$$

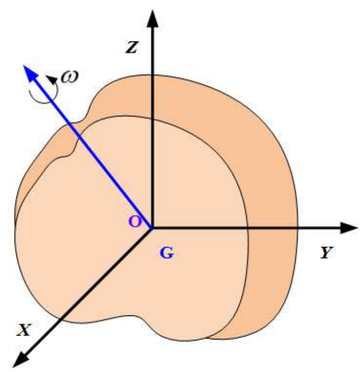

Figure 11.70: Rigid body rotates about its centroid G

Here $A^* = 7$, $B^* = 25$ and $C^* = 32$
Since the body is free to rotate (no torque), then Euler's dynamical equations (11.15.4) are

$$\left.\begin{aligned} 7\dot{\omega}_x - (25-32)\,\omega_y\omega_z &= 0 \\ 25\dot{\omega}_y - (32-7)\,\omega_x\omega_z &= 0 \\ 32\dot{\omega}_z - (7-25)\,\omega_x\omega_y &= 0 \end{aligned}\right\}$$

or

$$\dot{\omega}_x + \omega_y\omega_z = 0 \quad (11.21.107)$$
$$\dot{\omega}_y - \omega_x\omega_z = 0 \quad (11.21.108)$$
$$16\dot{\omega}_z + 9\omega_x\omega_y = 0 \quad (11.21.109)$$

To solve this system, we try to find a relation in one variable *i.e.* in ω_x, ω_y or ω_z and solve it. Let we find a relation for ω_y.
From (11.21.108), we can write

$$\begin{aligned} \dot{\omega}_y &= \omega_x\omega_y \\ \frac{d\omega_y}{dt} &= \omega_x\omega_z \\ \frac{d\omega_y}{\omega_x\omega_z} &= dt \end{aligned} \quad (11.21.110)$$

Now try to find ω_x and ω_z in term of ω_y. For this, multiplying (11.21.107) by ω_x and (11.21.108) by ω_y and then adding, we have

$$\omega_x\dot{\omega}_x + \omega_y\dot{\omega}_y = 0 \quad (11.21.111)$$

Integrating (11.21.111), we have

$$\omega_x^2 + \omega_y^2 = C \tag{11.21.112}$$

Using initial condition (11.21.106), $C = \frac{16}{25}\omega^2$, then (11.21.112) becomes

$$\omega_x^2 + \omega_y^2 = \frac{16}{25}\omega^2$$

or

$$\omega_x^2 = \frac{16}{25}\omega^2 - \omega_y^2$$

Hence ω_x in term of ω_y is

$$\omega_x = \left(\frac{16}{25}\omega^2 - \omega_y^2\right)^{\frac{1}{2}} \tag{11.21.113}$$

Next multiplying (11.21.108) by 9 ω_y and (11.21.109) by ω_z and then adding, we have

$$9\ \omega_y\dot{\omega}_y + 16\omega_z\dot{\omega}_z = 0 \tag{11.21.114}$$

Integrating (11.21.111), we have

$$9\ \omega_y^2 + 16\omega_z^2 = C \tag{11.21.115}$$

Using initial condition (11.21.106), $C = \frac{16}{25}\omega^2$, then (11.21.115) becomes

$$9\ \omega_y^2 + 16\omega_z^2 = \frac{144}{25}\omega^2$$

or

$$\omega_z^2 = \frac{9}{25}\omega^2 - \frac{9}{16}\omega_y^2$$

Hence ω_z in term of ω_y is

$$\omega_z = \frac{3}{4}\left(\frac{16}{25}\omega^2 - \omega_y^2\right)^{\frac{1}{2}} \tag{11.21.116}$$

Using (11.21.113) and (11.21.116) in (11.21.110), we have

$$\frac{d\omega_y}{\left(\frac{4}{5}\omega\right)^2 - \omega_y^2} = \frac{3}{4}dt \tag{11.21.117}$$

(11.21.117) is separable first order differential equation in ω_y and its solution is

$$\begin{aligned}
\frac{5}{4\ \omega} \tanh^{-1}\left(\frac{5}{4\ \omega}\omega_y\right) &= \frac{3}{4}\ t \\
\tanh^{-1}\left(\frac{5}{4\ \omega}\omega_y\right) &= \frac{3}{5}\ \omega t \\
\frac{5}{4\ \omega}\omega_y &= \tanh\left(\frac{3}{5}\ \omega t\right) \\
\omega_y &= \frac{4}{5}\omega\ \tanh\left(\frac{3}{5}\ \omega t\right) \qquad (11.21.118)
\end{aligned}$$

Next consider the trigonometric relation

$$\tanh 2\theta = \frac{2\tanh\theta}{1+\tanh^2\theta}$$

or

$$\tanh\left(\frac{3}{5}\ \omega t\right) = \frac{2\tanh\left(\frac{3}{10}\ \omega t\right)}{1+\tanh^2\left(\frac{3}{10}\ \omega t\right)}$$

Using (11.21.105), right side becomes

$$\begin{aligned}
\tanh\left(\frac{3}{5}\ \omega t\right) &= \frac{2\tan\left(\frac{\phi}{2}\right)}{1+\tan^2\left(\frac{\phi}{2}\right)} \\
&= 2\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right) \\
&= \sin\phi \qquad (11.21.119)
\end{aligned}$$

Using (11.21.119) in (11.21.118), we have

$$\omega_y = \frac{4}{5}\omega\ \sin\phi \qquad (11.21.120)$$

Using (11.21.122) in (11.21.113) and (11.21.116) in (11.21.110), we have

$$\omega_x = \frac{4}{5}\omega\ \cos\phi \qquad (11.21.121)$$

and

$$\omega_z = \frac{3}{5}\omega\ \cos\phi \qquad (11.21.122)$$

Hence the angular velocity about the principal axis of inertia about G is

$$\vec{\omega} = \left\langle \frac{4}{5}\omega\cos\phi,\ \frac{4}{5}\omega\sin\phi,\ \frac{3}{5}\omega\cos\phi \right\rangle$$

Hence the proof.

Exercises

1. In example 11.18.1, find the moment of inertia about an axis passing through one end of the system and perpendicular to the rod.
2. Find moment of inertia of a uniform rod of mass m making an angle $\frac{\pi}{3}$ with $x - axis$ with one end at origin having length a about
   (a) an axis passing through center and perpendicular to the rod.
   (b) Coordinate axes.
   (c) finding products of inertia, hence complete inertia matrix.
3. Find moment of inertia of a uniform rod of mass m making an angle $\frac{\pi}{4}$ with $x - axis$ with center at origin having length a about
   (a) an axis passing through center and perpendicular to the rod.
   (b) Coordinate axes.
   (c) finding products of inertia, hence complete inertia matrix.
4. Calculate the moment of inertia about the diagonal of a rectangular lamina.
5. Calculate the moment of inertia about the y-axis of the square lamina.
6. Calculate the moment of inertia about the diagonal of the square lamina.

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