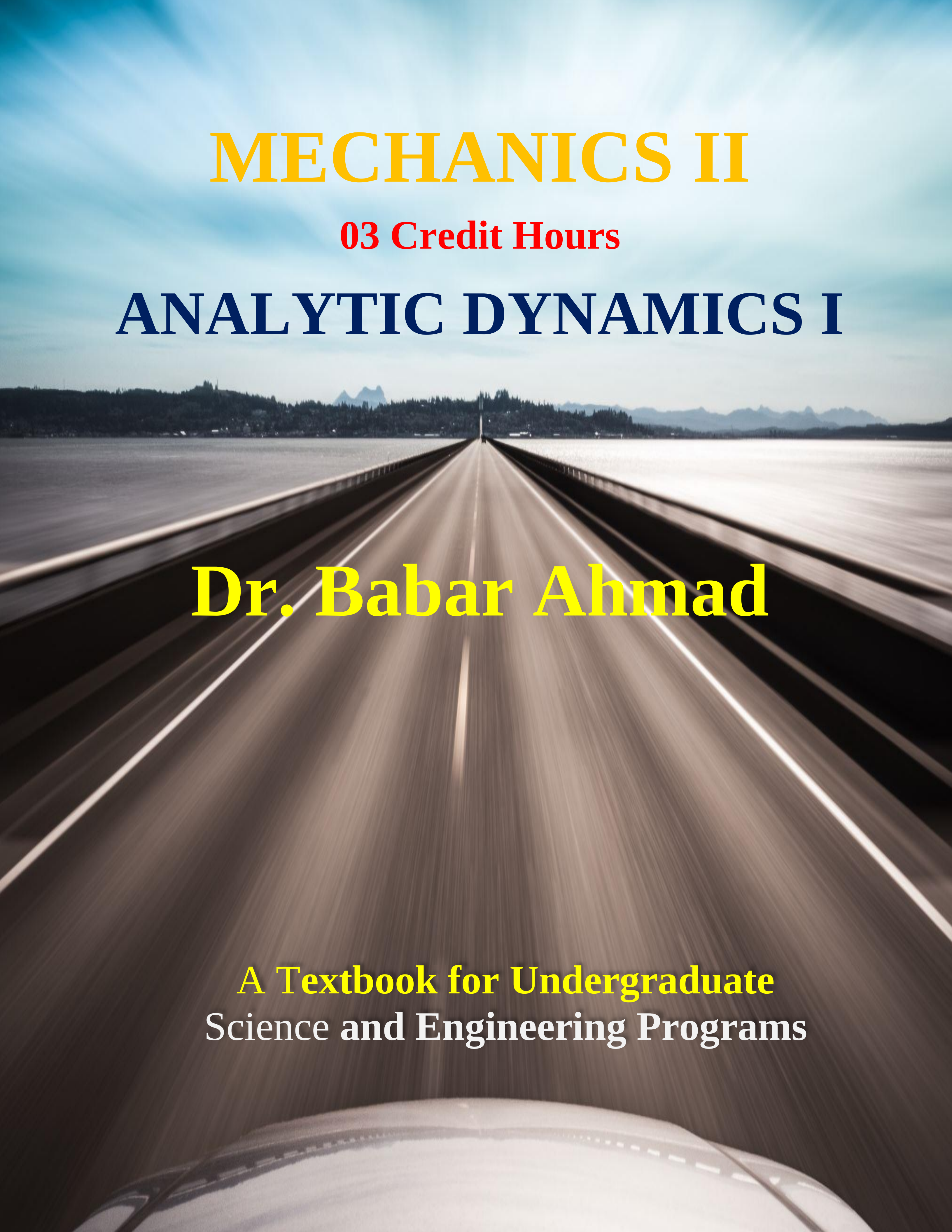
Mechanics II

Research · June 2020

DOI: 10.13140/RG.2.2.21713.76649

1 author:

Babar Ahmad
COMSATS Institute of Information Technology Islamabad, Pakistan



MECHANICS II

03 Credit Hours

ANALYTIC DYNAMICS I

Dr. Babar Ahmad

A Textbook for Undergraduate
Science and Engineering Programs

MECHANICS II

ANALYTIC DYNAMICS I

Dr. Babar Ahmad

A Textbook for Undergraduate
Science and Engineering Programs

babar.sms@gmail.com

Edition: 2020

Price

To My Parents

Mr. & Mrs. Rana Muhammad Hanif (*late*)

(*May ALLAH bless them*)

PREFACE

Mechanics is one of the most important course in maximum disciplines of science and engineering. No matter what your interest in science or engineering, mechanics will be important for you.

Mechanics is a branch of physics which deals with the bodies at rest and in motion. During the early modern period, scientists such as Galileo, Kepler, and Newton laid the foundation for what is now known as classical mechanics. Hence there is an extensive use of mathematics in its foundation.

Mechanics is core course for undergraduate Mathematics, Physics and many engineering disciplines. It appears under different names as Analytical/Classical Mechanics, Theoretical Mechanics, Mechanics I, Mechanics II, Mechanics III, Analytical Dynamics.

This textbook is designed to support teaching activities in Theoretical Mechanics specially Dynamics. It covers the contents of "Mechanics" for many undergraduate science and engineering programs. It presents simply and clearly the main theoretical aspects of mechanics.

It is assumed that the students have completed their courses in Calculus, Linear Algebra and Differential Equations. This book also lay the foundations for further studies in physics, physical sciences, and engineering.

For each concept a number books, documents and lecture notes are consulted. I wish to express my gratitude to the authors of such works.

In Chapter 1, preliminary theory of dynamics of rigid body is given. Newton's laws and some types of motion are also given in this chapter. Chapter 2 is about one dimensional motion. In this chapter, the kinematics of body is discussed by using graphical method, differentiation and integration. In chapter 3, the motion is also one dimensional, but the particle is restricted to move under gravity. This motion is discussed without and with air resistance. In chapter 4, the kinematics of a body in two and three dimensional coordinate systems is presented.

From Chapter 5, we will learn angular kinematics. In this chapter we will learn kinematics of a particle in polar coordinates. In chapter 6, Simple Harmonic Motion is discussed under natural, forced and damping aspects. Two dimensional projectile motion (without and with air resistance) is presented in chapter 7. Chapter 8 starts with Kepler's postulates to discuss the motion under Central Force. Some useful information about planetary motion is also available. Chapter 9 is about Small Oscillation. The stability of horizontally and vertically modulated pendulum under various forces is discussed. Chapter 10 is about rotational dynamics. Chapter 10 contains Rotation of rigid bodies in two space and three space. The concepts of rotational matrix, angle of rotation and axis of rotation are also discussed in this chapter. Euler's theorem and Chasle's theorem are also given in this chapter. This chapter also contains kinematics of a body in cylindrical and spherical coordinate systems.

In a book of this concept, level and size, there may be a possibility that some misprint might have remained uncorrected. If you find such misprints or want to give some suggestions for its improvement, please write me at: babar.sms@gmail.com

Dr. Babar Ahmad

Islamabad, Pakistan
June, 2020

Table of Contents

Chapter 1

Basic Review of Mechanics

In this chapter some basic concepts of Mechanics are presented for the revision of elementary knowledge of the students. These concepts are essential part of mechanics. Some types of Motion are defined, their detail will be in next chapters. Newton's three laws of motion and Newton's gravitational law are also mentioned.

1.1 Definitions of some basic concepts

Mechanics : A branch of physics which describes or predicts the conditions of rest or motion of bodies under the action of forces. It has two branches:

1. statics
2. dynamics

1. **Statics**: Statics deals with the forces, their effects, acting upon the bodies at rest.
2. **Dynamics**: Dynamics deals with the forces, their effects, while acting upon the bodies in motion (accelerated motion).

Parts of Dynamics:

1. **Kinematics** is the study of motion without reference to the forces which cause motion. It is the geometric theory of motion.
2. **Kinetics** is the study of motion with reference to the forces which cause motion.

1.2 Particle and Rigid body

Particle: The smallest thing/part which has no dimension. It is idealization used to represent a phenomena (body).

A man standing on ground is observing the flight of an aeroplane moving away from him. At first instant he can see its shape very clear. At next instant, the shape becomes smaller and next becoming smaller and smaller. At last instant he sees a dot and then nothing. This dot can be regarded as particle (see Fig. 1.1).
Body: An aggregate of particles is known as body.

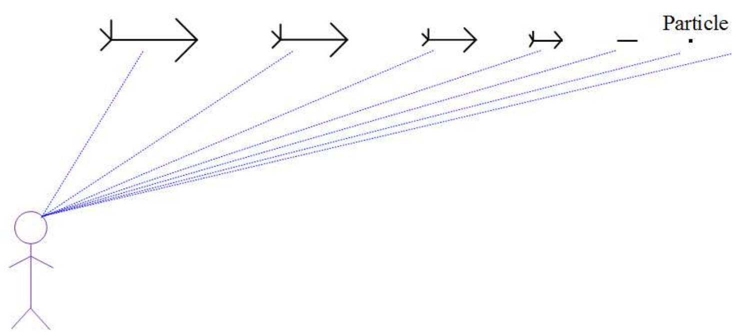


Figure 1.1: Representation of particle

Kinds of bodies: There are various types of bodies *e.g* rigid, elastic, plastic, elasto plastic, etc
The bodies are of two categories:

a) Rigid bodies

b) Deformable bodies

Rigid body A body is considered rigid when the relative movement between its parts is negligible.
In *Mechanics* we will assume the bodies to be perfectly rigid, no deformation. This is never true in the real world, everything deforms a little when a load is applied. These deformations are small and will not significantly affect the conditions of equilibrium or motion, so we will neglect the deformations.

1.3 Kinematics of a Particle

Space: The geometric region occupied by bodies whose positions are described by linear and angular measurements relative to a coordinate system.
Length: The magnitude of displacement vector between two bodies/points is called length. It is usually denoted by l and is measured in m in SI.

Time: It is interval between the existence of events. Time is completely independent quantity, because its relation is with the sun. It is denoted by t and is measured in s in SI.
Mass: The measure of the inertia of a body, which is its resistance to a change of motion. sometimes called "quantity of matter" contained by a particle. It is denoted by m and is measured in kg in SI.
Position: The position of a particle is a point with respect to some reference point (usually origin).
Rest: A body is said to be at rest, if does not change its position with respect to its surrounding.
Motion: A body is said to be in motion, if it changes its position with respect to its surrounding.
Distance: If the rigid body is moved from one position to another, the total length of the path is called distance moved by the body. It is usually denoted by s and is measured in m in SI.
Displacement If the rigid body is moved from one position to another, the change of position is called displacement of the body. It is a vector quantity. It is usually denoted by $\vec{r}$ and is measured in m in SI.
Speed Time rate of change of distance is called speed. It is the only magnitude. It is usually denoted by v and is measured in m/s in SI.
Uniform speed A body has uniform speed if it covers equal distance in equal intervals of time.
Velocity: Rate of change of position vector (displacement) is called velocity. It is usually denoted by $\vec{v}$ and is measured in m/s in SI.
Uniform velocity A body has uniform velocity if it covers equal displacement in equal intervals of time.
Average velocity If a body has initial velocity $\vec{v_1}$ and final velocity $\vec{v_2}$, then the average velocity is

$$\vec{v} = \frac{\vec{v_1} + \vec{v_2}}{2}$$

Acceleration: Acceleration is the rate of change of velocity vector.It is usually denoted by $\vec{a}$ and is measured in m/s^2 in SI.
Uniform acceleration A body has uniform acceleration if it has equal changes in velocity in equal intervals of time.
Force: Force is an agency which changes or tends to change the state of rest or of uniform motion of a body. It is usually denoted by $\vec{F}$ and is measured in N in SI.
Linear Momentum: The linear momentum of a particle of mass m is the product of mass and velocity of the body. It is usually denoted by $\vec{p}$ and is measured in $N.s$ in SI. Mathematically it is given by:

$$\vec{p} = m\vec{v} \tag{1.3.1}$$

In (1.3.1) m is the mass of the particle and $\vec{v}$ is its velocity relative to a reference system.
The concepts of position vector, velocity and acceleration are very useful to describe the kinematics of a particle. These concepts are discussed in detail.

1.3.1 Position Vector

The position vector of a point P with reference to some point is discussed in chapter 1. It can be represented as

$$\vec{r}(t) = r(t)\hat{r}$$

If position function of a particle is given, then differentiation is used to define the notions of instantaneous velocity and acceleration, and if acceleration function is given, then integration is used to define the notions of instantaneous velocity and position. First we define the relations of instantaneous velocity and acceleration for a particle in rectilinear motion in the form of derivatives.

1.3.2 Instantaneous Velocity

If the coordinate of a particle at time t_1 is $r(t_1)$ and $\vec{r}_1$ be its position vector with respect to some reference point, next its coordinate at a later time t_2 is $r(t_2)$ and the position vector be $\vec{r}_2$, then the displacement describes the change in position of the particle and is given as

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$$

then the average velocity is

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t}$$

And instantaneous velocity is defined as

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t}$$

$$\vec{v}(t) = \frac{d}{dt}\vec{r}(t) = \vec{r}' \tag{1.3.2}$$

The velocity vector $\vec{v}$ always points in the direction of motion. The magnitude of the velocity, $v = |\vec{v}|$ is known as the speed.
Units commonly used to express velocity and speed are m/s and ft/sec.

1.3.3 Instantaneous Acceleration

The rate at which the instantaneous velocity of a particle changes with time is called its instantaneous acceleration. Provided the velocity of the particle is known at two points, the average acceleration of the particle during the time interval Δt is defined as

$$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t}$$

Thus, if a particle in rectilinear motion has velocity function $\vec{v}(t)$, then its instantaneous acceleration at time (acceleration function) is defined as

$$\begin{aligned}\vec{a}(t) &= \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} \\ &= \frac{d}{dt}\vec{v}(t) \qquad (1.3.3)\end{aligned}$$

If the velocity changes in either magnitude or direction (or both), the particle must have an acceleration.
Using (4.1.1), the acceleration function in terms of the position function is

$$\vec{a}(t) = \frac{d^2}{dt^2}\vec{r}(t) = \vec{r}'' \qquad (1.3.4)$$

The acceleration vector $\vec{a}$ can point anywhere. Units commonly used to express acceleration are m/s^2 or ft/sec^2

1.3.4 Displacement

If the velocity vector $\vec{v}(t)$ of a moving particle over the time interval $[t_1, t_2]$ is given, then the displacement $\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_1)$ or simply $\vec{r}(t)$ is

$$\vec{r}(t) = \int_{t_1}^{t_2} \vec{v}(t)dt \qquad (1.3.5)$$

and the **distance** is

$$\begin{aligned}r(t) &= \int_{t_1}^{t_2} |v(t)|dt \qquad (1.3.6) \\ &= \int_{t_1}^{t_2} -v(t)dt + \int_{t_2}^{t_3} v(t)dt \qquad (1.3.7)\end{aligned}$$

The examples of these concepts are in next chapters.

1.4 Types of motion

Almost everything in the universe is in motion. Some objects move along a straight line, some move along a curved path, some moves in a circle and some have to and fro motion. These different types of motions are discussed as follow:

1.4.1 Translation

A motion by which a body shifts from one point in space to another in such a way that the straight lines joining the initial and final positions of each of the points of the body are parallel. *i.e.* every point of the body moves an equal distance. If the path of motion is straight line, the motion is said a **rectilinear motion**. The line might be an $x - axis$, a $y - axis$, or a coordinate line inclined at some angle. If the paths are curved lines, the motion is a **curvilinear motion**. The motion of a bullet fired from a gun is the example of such motion.

1.4.2 Rotation

A motion by which an extended body changes orientation, with respect to other bodies in space, without changing position (e.g., the motion of a spinning top). Rotation is of two types.

a) Rotation about a line.

b) Rotation about a point .

a) **Rotation about a line**: Rotation of a rigid body about a line l is such a motion in which the position in space of every point of the body that lies on l in unchanged. The line l is called the axis of rotation.

b) **Rotation about a point**: Rotation of a rigid body about some point P of itself is such a motion in which the position in space of the point P is unchanged.

1.4.3 Oscillation

A motion, which continually repeats in time with a fixed period (e.g., the motion of a pendulum in a grandfather clock).

1.4.4 Circular motion

A motion by which a body executes a circular orbit about another fixed body [e.g., the (approximate) motion of the Earth about the Sun].

1.4.5 Random motion

The disordered or irregular motion of an object is called random motion.
These different types of motion can be combined: for instance, the motion of a swing/spinning bowled ball consists of a combination of translational and rotational motion, whereas wave propagation is a combination of translational and oscillatory motion. Furthermore, the

above mentioned types of motion are not entirely distinct: e.g., circular motion contains elements of both rotational and oscillatory motion.

1.5 Newton's Fundamental Laws

Newton developed the fundamentals of mechanics. The concepts above, space, time, and mass are absolute, independent of each other in Newtonian Mechanics.

- **Newton's First Law of motion** A particle remains at rest or continues to move in a straight line with a constant speed if there is no unbalanced force acting on it (resultant force is zero).

Inertia Newton's first law is also known as Galilian Law Inertia as it was discovered by the Italian astronomer, Galileo Galilei (1564-1642).
Inertia Inertia is the resistance of any physical object to a change in its state of motion or rest, or the tendency of an object to resist any change in its motion. The principle of inertia is one of the fundamental principles of classical physics which are used to describe the motion of matter and how it is affected by applied forces. Inertia comes from the Latin word, iners, meaning idle, or lazy. Isaac Newton defined inertia as his first law in his Philosophiee Naturalis Principia Mathematica, which states:
The vis insita, or innate force of matter, is a power of resisting by which every body, as much as in it lies, endeavours to preserve its present state, whether it be of rest or of moving uniformly forward in a straight line.

- **Newton's Second Law of motion** The time rate of change of linear momentum of a particle is equal to the force producing it, and takes place in the direction in which the force acts.

If a particle of mass m subject to a force $\vec{F}$ moves with velocity $\vec{v}$, at time t its linear momentum is $\vec{p}$. Mathematically the second law is

$$\begin{aligned} \vec{F} &= \frac{dp}{dt} \\ &= \frac{d}{dt}(m\vec{v}) \end{aligned} \tag{1.5.1}$$

If the mass of the particle is constant, the (1.5.1) can be written as

$$\vec{F} = m\vec{a} \tag{1.5.2}$$

It means the acceleration of a particle is proportional to the resultant force acting on it and is in the direction of this force.

- **Newton's Third Law of motion** The forces of action and reaction between interacting bodies are equal in magnitude, opposite in direction, and act along the same line of action (Collinear).

In simple form, the action of one body on another.

1.6 Newton's Law of Gravitation

Every particle of matter in the universe attracts every other particle with a force that is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them.
If m_1 and m_2 are masses of the two particles and r is the distance between the centers of two particles, then the magnitude of the force due to the gravitational interaction between two particles is

$$F = G\frac{m_1 m_2}{r^2} \quad (1.6.1)$$

Here $G = 6.67 \times 10^{11} N - m^2/kg^2$ is the ***universal gravitational constant***.
Weight is the attraction of the earth on a particle located on its surface. If we introduce the constant

$$g = G\frac{M}{r^2} \quad (1.6.2)$$

and let: M = mass of earth
m = mass of a particle
r = radius of earth
g = acceleration of gravity at earth's surface
(1.6.2) can be written as

$$G = g\frac{r^2}{M} \quad (1.6.3)$$

substituting (1.6.3) into (1.5.2)

$$\vec{F} = m\vec{a}$$

Force due to gravity, is the weight of the body

$$\vec{W} = m\vec{g} \quad (1.6.4)$$

comparing (1.6.4) with (1.6.1), the acceleration at the surface of the Earth is

$$\vec{a} = \vec{g}$$

g is dependent upon r. In most cases the value of g in SI system is

$$g = 9.81 m/s^2$$

and in non SI system is

$$g = 32.2 ft/s^2$$

Chapter 2

Rectilinear Motion

2.1 Rectilinear Motion

The motion of a particle along a coordinate line is said to be rectilinear motion. It is also known as motion along a straight line or one dimensional motion. The line might be an $x - axis$, $y - axis$, or a coordinate line inclined at some angle. In general discussions we will designate the coordinate line as $r - axis$. It is the simplest motion in nature.
Mostly the direction of vector quantities is along the axis of motion or an angle is specified, so a plus or minus sign is enough to specify the direction of motion. Hence in rectilinear motion scalar and vector equations are same, and we can use x in place of $\vec{x}$, v in place of $\vec{v}$ and a in place of $\vec{a}$.
We begin observing the motion of the particle at time $t = 0$ with respect to a fixed point on the line, known as reference point. Usually we take origin O as the reference point. As the particle moves along $r - axis$, its coordinate r will be some function of time, say $r = r(t)$ known as the position function of the particle.

2.1.1 Position Vector

The position vector of a point P with reference to some point is discussed in chapter 1. It can be represented as $r(t)$. Usually rectilinear motion is considered along $x - axis$, so the position vector can be represented as $x(t)$.
In rectilinear motion, if position function of a particle is given, then differentiation is used to define the notions of instantaneous velocity and acceleration and if acceleration function is given, then integration is used to define the notions of instantaneous velocity and position. If velocity function is given, then integration is used to define the notions of position and differentiation is used to define the notions of instantaneous acceleration.
First we define the relations of instantaneous velocity and acceleration for a particle in rectilinear motion in the form of derivatives.

2.1.2 Displacement

If the coordinate of a particle at time t_1 is $x(t_1)$ and x_1 be its position vector with respect to some reference point usually is the origin , next its coordinate at a later time t_2 is $x(t_2)$ and the position vector be x_2, then the displacement describes the change in position of the particle and is given as

$$\Delta x = x_2 - x_1$$

Usually we represent $x = \Delta x$. Then the relation

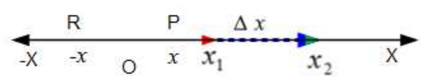


Figure 2.1: Rectilinear motion

$$x = x(t_2) - x(t_1) \tag{2.1.1}$$

gives the displacement of the particle in time interval (t_1, t_2).
Distance: The magnitude of the displacement, $|x|$ is known as the distance. The relation

$$|x| = |x(t_2) - x(t_1)| \tag{2.1.2}$$

gives the distance travelled by the particle in time interval (t_1, t_2) Units commonly used to express displacement and distance are m and ft.

2.1.3 Instantaneous Velocity

The average velocity of a particle in one dimension is

$$v_{avg} = \frac{\Delta x}{\Delta t}$$

And instantaneous velocity is defined as

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

$$v(t) = \frac{d}{dt} x(t) = \dot{x} \tag{2.1.3}$$

The velocity of a moving particle always points in the direction of motion.
Speed: The magnitude of the velocity, $|v|$ is known as the speed. Units commonly used to

express velocity and speed are m/s and ft/sec.
Direction of motion
In rectilinear motion, the sign of the velocity tells the direction of motion. A positive value for $v(t)$ means that $x(t)$ is increasing with time, so the particle is moving in the positive direction, and a negative value for $v(t)$ means that $x(t)$ is decreasing with time, so the particle is moving in the negative direction. If $v(t) = 0$, then the particle has momentarily stopped.

2.1.4 Instantaneous Acceleration

In rectilinear motion, the rate at which the instantaneous velocity of a particle changes with time is called its instantaneous acceleration. If the velocity of the particle is known at two points, the average acceleration of the particle during the time interval Δt is defined as

$$a_{avg} = \frac{\Delta v}{\Delta t}$$

Thus, if a particle in rectilinear motion has velocity function $v(t)$, then its instantaneous acceleration at time (acceleration function) is defined as

$$\begin{aligned} a(t) &= \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \\ a &= \frac{dv}{dt} \end{aligned} \tag{2.1.4}$$

Also we can write

$$\begin{aligned} a &= \frac{dv}{dx}\frac{dx}{dt} \\ a(v) &= v\frac{dv}{dx} \end{aligned} \tag{2.1.5}$$

Equation (2.1.5) represents acceleration as a function of velocity. From equation (2.1.5) we can write

$$adx = vdv \tag{2.1.6}$$

Equation (2.1.6) is known as differential relation.
Using equation (2.1.3) in equation (2.1.4) we can express the acceleration function in terms of the position function as

$$\begin{aligned} a(t) &= \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2}{dt^2}x(t) \\ &= \ddot{x} \end{aligned} \tag{2.1.7}$$

The acceleration vector a can point anywhere. Units commonly used to express acceleration are m/s^2 or ft/s^2.

2.2 Constant and Variable Quantities

To discuss the motion of a particle the quantities displacement, velocity and acceleration may be constant or variable or both. If these quantities are constant, we simply write x, v and a for displacement, velocity and acceleration respectively. If these quantities are variable, they may be functions of time, functions of each other or function of any combination from them. For example if acceleration is a function of time we write $a(t)$, if it is a function of position we write $a(x)$ and if it is a function of velocity we write $a(v)$. It may be a function of any two, or all the three quantities, but will be a complicated case.

2.3 Graphical Methods

It is often very convenient to employ the graphical techniques for solving dynamical problems as they yield results more readily than the ordinary calculations.

2.3.1 Time-Displacement Curve

Consider the displacement function

$$x = x(t)$$

Here t is independent variable and x is dependent variable. For time-displacement curve, we take t along horizontal axis and x along vertical axis. The plane so formed is known as $tx-$plane (see Fig. 2.2). The slope of the tangent to the curve at (x, t)

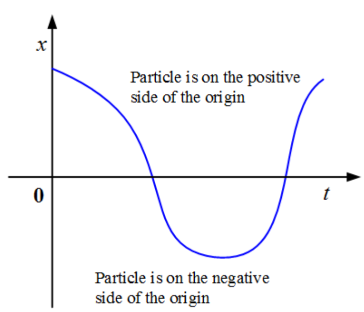


Figure 2.2: Position versus time curve

$$v = \frac{dx}{dt}$$

represents the velocity of the particle at (x, t). If the velocity is constant, the slope of the time-displacement curve remains the same. That is, the curve is a straight line.

2.3.2 Time-Velocity Curve

Consider the velocity function

$$v = v(t)$$

Here t is independent variable and v is dependent variable. For time-velocity curve, we take t along horizontal axis and v along vertical axis. The plane so formed is known as $tv-$plane. The slope of the tangent to the curve at (x, t)

$$a = \frac{dv}{dt}$$

represents the acceleration of the particle at (x, t). If the acceleration is constant, the slope of the time-velocity curve remains the same. That is, the curve is a straight line. Here we have an important result:

Remark 2.3.1. The slope time-velocity curve of a particle moving in a straight line gives its acceleration and area under the curve gives the distance travelled by the particle.

Example 2.3.1. *A particle starts to move from origin O. It moves with constant acceleration a for time interval t_1. At t_1 it acquires a velocity v, then it starts to move with uniform velocity v and continued for time interval t_2. After that its velocity starts to decrease with retardation $2a$ and comes to rest after time interval t_3. If t is the time taken by the particle from rest to rest, find the distance travelled by the particle in terms of v, t and a.*

Solution The time velocity graph of this motion is shown in Fig. 2.3. According to

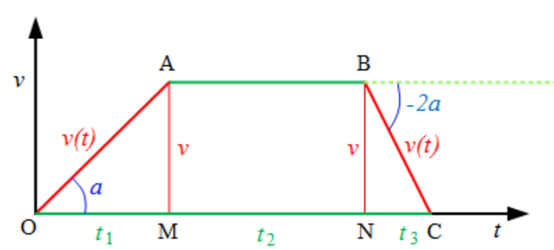


Figure 2.3: Time velocity curve

remark (2.3.1), area under the velocity curve gives the distance travelled by the particle.

The motion of a particle has three phases. In first phase it moves with velocity v and acceleration a for time interval t_1, forming a triangle AOM. Its area will give the distance travelled by the particle along line OA. According to Fig. 2.3

$$|OM| = t_1 \text{ and } \quad |AM| = v$$

Area of the triangle AOM is

$$\begin{aligned} A_1 &= \frac{1}{2}|OM||AM| \\ &= \frac{1}{2}vt_1 \end{aligned}$$

The slope of the line OA is the acceleration.

$$\begin{aligned} a &= \frac{v}{t_1} \\ \text{or} \quad t_1 &= \frac{v}{a} \end{aligned}$$

Then area of the triangle AOM becomes

$$A_1 = \frac{v^2}{2a} \tag{2.3.1}$$

In second phase it moves with constant velocity v for time interval t_2, forming a rectangle $ABMN$. Its area will give the distance travelled by the particle along line AB. According to Fig.

$$|MN| = t_2 \text{ and } \quad |AM| = v$$

Area of the rectangle $ABMN$ is

$$\begin{aligned} A_2 &= |MN||AM| \\ &= vt_2 \end{aligned}$$

In third phase it moves with velocity $-v$ and retardation $2a$ for time interval t_3, forming a triangle BNC. Its area will give the distance travelled by the particle along line BC. According to Fig.

$$|NC| = t_3 \text{ and } \quad |BN| = v$$

Area of the triangle BNC is

$$\begin{aligned} A_3 &= \frac{1}{2}|BN||NC| \\ &= \frac{1}{2}vt_3 \end{aligned}$$

The slope of the line BC is the retardation.

$$\begin{aligned} 2a &= \frac{v}{t_3} \\ \text{or} \quad t_3 &= \frac{v}{2a} \end{aligned}$$

Then area of the triangle BNC becomes

$$A_3 = \frac{v^2}{4a} \tag{2.3.2}$$

The total time of motion is

$$\begin{aligned} t &= t_1 + t_2 + t_3 \\ \text{or} \quad t_2 &= t - t_1 - t_3 \\ &= t - \frac{3v}{2a} \end{aligned}$$

Area of the rectangle $ABMN$ can be rewritten as

$$A_2 = v\left(t - \frac{3v}{2a}\right) \tag{2.3.3}$$

Total area under the velocity curve is the sum of all these area. Add equations (2.3.1) (2.3.2) (2.3.3) to get this sum.

$$\begin{aligned} A &= A_1 + A_2 + A_3 \\ &= \frac{v^2}{2a} + v\left(t - \frac{3v}{2a}\right) + \frac{v^2}{4a} \\ &= v\left(t - \frac{3v}{4a}\right) \end{aligned} \tag{2.3.4}$$

Equation (2.3.4) gives the distance travelled by the particle in terms of v, t and a.

2.3.3 Time-Acceleration Curve

Consider the acceleration function

$$a = a(t)$$

Here t is independent variable and a is dependent variable. For time-acceleration curve, we take t along horizontal axis and a along vertical axis. The plane so formed is known as $ta-$plane. The slope of the tangent to the curve at (x, t)

$$j = \frac{da}{dt}$$

represents the jerk of the particle at (x, t). If the jerk is constant, the slope of the time-acceleration curve remains the same. That is, the curve is a straight line.

Remark 2.3.2. The slope time-acceleration curve of a particle moving in a straight line gives its jerk and area under the curve gives the speed of the particle for defined interval.

2.3.4 Speeding up and Slowing downslowing down

In rectilinear motion, if the speed of a particle is increasing it is speeding up or accelerating and if its speed is decreasing it is slowing down or decelerating. In first case acceleration is positive and in second it is negative. When velocity is constant, acceleration is zero.

The signs of velocity and acceleration

A particle in rectilinear motion is speeding up when its velocity and acceleration have the same sign and slowing down when they have opposite signs.

Example 2.3.2. *Let $x(t) = t^3 - 6t^2$ be the position function of a particle moving along $x-axis$. Find the velocity, speed and acceleration functions, and show the graphs of position, velocity, speed and acceleration versus time. From these graphs, determine when the particle is speeding up and slowing down.*

Solution At any time the position of the particle along $x - axis$ is

$$x(t) = t^3 - 6t^2$$

Using (2.1.3), the velocity at any time is

$$v(t) = \frac{d}{dt}x(t) = 3t^2 - 12t$$

The speed is

$$|v(t)| = |3t^2 - 12t|$$

Using (2.1.4), the acceleration is

$$a(t) = \frac{d}{dt}v(t) = 6t - 12$$

The graphs of position versus time is given in Fig. 2.4
The graphs of speed versus time is given in Fig. 2.5
The graphs of acceleration (magnitude) versus time is given in Fig. 2.6
From figures 2.5 and 2.6, observe that over the time interval $0 < t < 2$ the velocity and acceleration are negative, so the particle is speeding up. Over the time interval $2 < t < 4$ the velocity is negative and the acceleration is positive, so the particle is slowing down. Finally, on the time interval $t > 4$ the velocity and acceleration are positive, so the particle is speeding up.
At $t = 4$, the velocity is zero, so the particle has momentarily stopped.

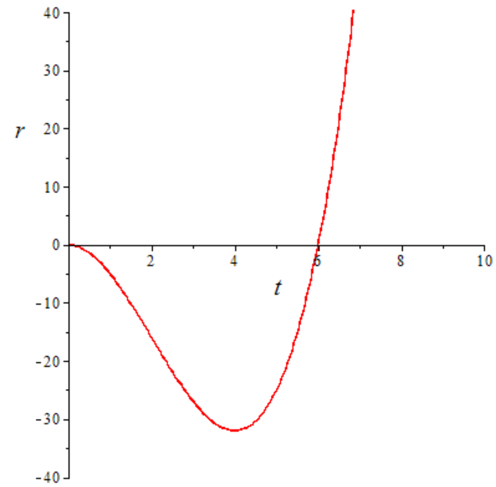


Figure 2.4: Position versus time

Example 2.3.3. *In example 2.3.2.*

(a) Find displacement and distance for time interval $0 \leq t \leq 6$

(b) Find velocity and speed for time interval $0 \leq t \leq 6$

Solution Using (2.1.1), the displacement of the particle for time interval $0 \leq t \leq 6$ is

$$\begin{aligned} x(t) &= x(6) - x(0) \\ &= \left[(6)^3 - 6(6)^2\right] - \left[(0)^3 - 6(0)^2\right] \\ &= 0\ m \end{aligned}$$

Using (2.1.2), the distance of the particle for time interval $0 \leq t \leq 6$ is

$$|x| = |x(6) - x(0)|$$

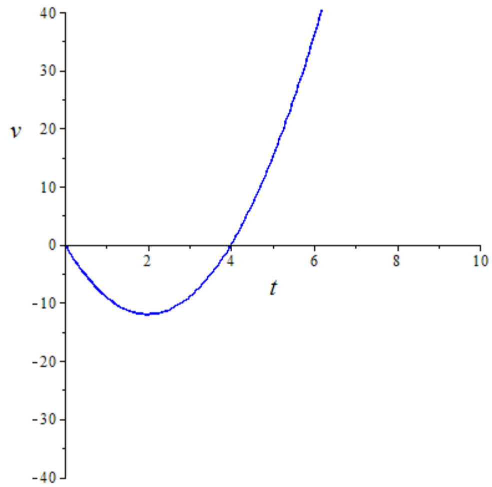


Figure 2.5: Velocity versus time

Using remark 2.3.1, the distance may be calculated by finding the area under the curve

$$\begin{aligned}
|x| &= \left| \int_0^6 \left(3t^2 - 12t\right) dt \right| \\
&= -\left[\int_0^4 \left(3t^2 - 12t\right) dt \right] + \left[\int_4^6 \left(3t^2 - 12t\right) dt \right] \\
&= -\left| t^3 - 6t^2 \right|_0^4 + \left| t^3 - 6t^2 \right|_4^6 \\
&= -(4)^3 + 6(4)^2 + (6)^3 - (4)^3 - 6(6)^2 + 6(4)^2 \\
&= 64\ m
\end{aligned}$$

Using (2.1.3), the velocity at any time is

$$v(t) = \frac{dx}{dt} = 3t^2 - 12t$$

velocity and speed for time interval $0 \le t \le 6$

$$\begin{aligned}
v &= v(6) - v(0) \\
&= \left[3(6)^2 - 12(6)\right] - \left[(0)^3 - 6(0)^2\right] \\
&= 36\ m/s
\end{aligned}$$

The speed is

$$|v(t)| = |3t^2 - 12t|$$

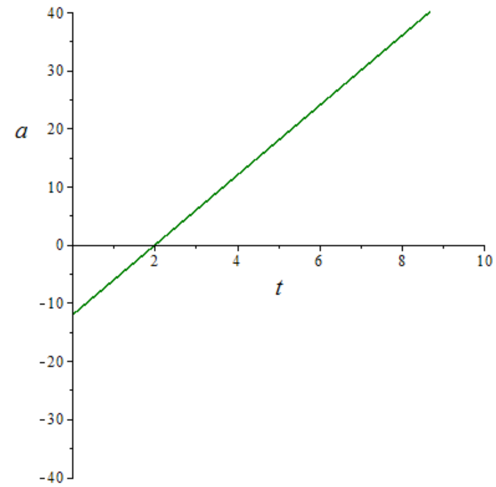


Figure 2.6: acceleration versus time

Using remark 2.3.2, the distance may be calculated by finding the area under the curve

$$\begin{aligned} |v| &= \left| \int_0^6 (6t-12)\,dt \right| \\ &= -\left[\int_0^2 (6t-12)\,dt \right] + \left[\int_2^4 (6t-12)\,dt \right] \\ &= -\left|3t^2-12t\right|_0^2 + \left|3t^2-12t\right|_2^6 \\ &= -3(2)^2+12(2)+3(6)^2-3(2)^2-12(6)+12(2) \\ &= 60\ m/s \end{aligned}$$

2.4 Kinematics by Using Integration

In this section acceleration or velocity functions will be given and we will find velocity and displacement functions by integration. First consider velocity function from acceleration.

2.4.1 Velocity from Acceleration

If we know the acceleration function $a(t)$ of the particle, then by integrating $a(t)$ we can produce a family of velocity functions. If, in addition, we know the velocity v_0 of the particle at any time t_0, then we have sufficient information to find the constant of integration and determine a unique velocity function. If acceleration function is given, the following two cases arise:

a) Acceleration function is constant.

b) Acceleration function is variable.

a) Acceleration function is constant.

consider (2.1.4)

$$\begin{aligned} a(t) = a &= \frac{d}{dt}v(t) \\ dv &= adt \end{aligned}$$

integrating (separation of variables), assuming that initially (at $t = 0$) $v = v_0$

$$\begin{aligned} \int_{v_0}^{v} dv(t) &= a_c \int_0^t dt \\ v\Big|_{v_0}^{v} &= a_c t\Big|_0^t \\ v - v_0 &= a_c(t - 0) \\ v &= v_0 + a_c t \end{aligned} \tag{2.4.1}$$

(2.4.1) gives the velocity of the particle moving with constant acceleration at any time t.

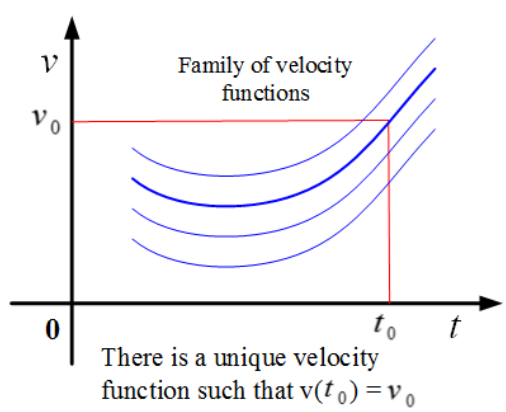


Figure 2.7: Family of velocity functions

b) Acceleration function is variable.

If acceleration is variable, then (2.1.4) can be written as

$$\begin{aligned}\ddot{x} &= a(t)\\ \frac{d}{dt}(\dot{x}) &= a(t)\end{aligned}$$

integrating with respect to t

$$v = \dot{x} = \int a(t)dt + C_1 \tag{2.4.2}$$

Where C_1 is constant of integration and can be determined by applying initial condition. (2.4.2) gives the velocity of the particle moving with variable acceleration at any time t.

2.5 Displacement by Using Integration

We can find displacement function if velocity or acceleration function is given. First consider both velocity and acceleration functions are given.

2.5.1 Position from Velocity and acceleration

If we know the velocity function $v(t)$ of a particle in rectilinear motion, then by integrating $v(t)$ we can produce a family of position functions with that velocity function. If, in addition, we know the position x_0 of the particle at any time t_0, then we have sufficient information to find the constant of integration and determine a unique position function. Consider equations. (2.1.3) and (2.4.1)

$$\begin{aligned}v = \frac{dx}{dt} &= v_0 + at\\ dx &= (v_0 + at)dt\end{aligned}$$

integrating (separation of variables), assuming that initially (at $t = 0$) $x = x_0$

$$\begin{aligned}\int_{x_0}^{x} dx &= \int_0^t (v_0 + at)dt\\ &= v_0 \int_0^t dt + a_c \int_0^t tdt\\ x\Big|_{x_0}^{x} &= v_0 t\Big|_0^t + a\frac{t^2}{2}\Big|_0^t\\ x - x_0 &= v_0 t + \frac{1}{2}at^2\\ x &= x_0 + v_0 t + \frac{1}{2}at^2 \qquad (2.5.1)\end{aligned}$$

If the particle starts from origin O, then $x_0 = 0$, then (2.5.1) becomes

$$x = \frac{1}{2}at^2 + v_0 t \tag{2.5.2}$$

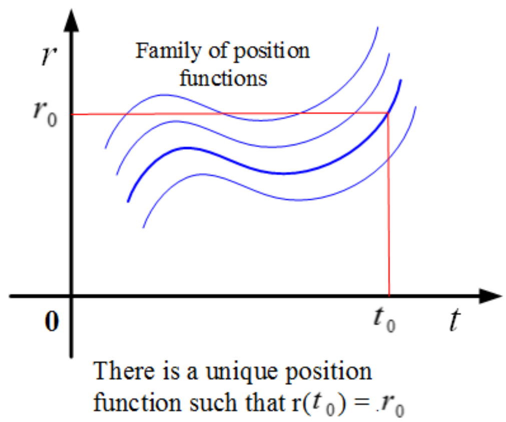


Figure 2.8: Family of position functions

2.5.2 Displacement from Velocity

If velocity function is given, the following two cases arise:

a) Velocity function is constant.

b) Velocity function is variable.

a) Velocity function is constant.

If the velocity function of particle is constant over the time interval $[0, t]$ and an initial condition $x(0) = x_0$, then (2.1.3) can be written as

$$\frac{dx}{dt} = v$$

It is first order separable differential equation, separating variables

$$dx = vdt$$

Integrating

$$\int_{x_0}^{x} dx = v\int_{0}^{t} dt$$
$$x - x_0 = vt$$

or

$$x = x_0 + vt \tag{2.5.3}$$

(2.5.3) gives the position of the particle moving with constant velocity at any time t.

b) Velocity function is variable.

If the velocity function of particle is constant over the time interval $[0, t]$ and an initial condition $x(0) = x_0$, then (2.1.3) can be written as

$$\frac{dx}{dt} = v(t)$$

It is first order separable differential equation, separating variables

$$dx = v(t)dt$$

Integrating

$$\int_{x_0}^{x} dx = \int_{0}^{t} v(t)dt$$

or

$$x = x_0 + \int_{0}^{t} v(t)dt \tag{2.5.4}$$

(2.5.4) gives the position of the particle moving with variable velocity at any time t.

2.5.3 Distance from Velocity

If the velocity $v(t)$ of a moving particle over the time interval $[t_1, t_2]$ is given, then the distance is

$$|x(t)| = \int_{t_1}^{t_2} |v(t)|dt \tag{2.5.5}$$

$$= \int_{t_1}^{t_2} -v(t)dt + \int_{t_2}^{t_3} v(t)dt \tag{2.5.6}$$

2.5.4 Displacement from Acceleration

If acceleration function is given, the following two cases arise:

a) Acceleration function is constant.

b) Acceleration function is variable.

a) Acceleration function is constant.

If the acceleration function of particle is constant over the time interval $[0, t]$ and boundary condition $x(0) = 0$ and $x(t_1) = x_1$, then (4.1.3) can be written as

$$\ddot{x} = a$$

integrating two times with respect to t we have

$$x = \frac{1}{2}at^2 + C_1 t + C_2 \tag{2.5.7}$$

Where C_1 and C_2 are constants of integration. Using $x(0) = 0$, we have $C_2 = 0$, then (2.5.7) becomes

$$x = \frac{1}{2}at^2 + C_1 t \tag{2.5.8}$$

Next use $x(t_1) = x_1$, in (2.5.8)

$$x_1 = \frac{1}{2}at_1^2 + C_1 t_1$$

or

$$C_1 = \frac{x_1}{t_1} - \frac{a}{2}t_1 \tag{2.5.9}$$

Using (2.5.9) in (2.5.8), we have

$$x = \frac{1}{2}at^2 + \left(\frac{x_1}{t_1} - \frac{a}{2}t_1\right)t \tag{2.5.10}$$

(2.5.10) gives the position of the particle moving with constant acceleration at any time t from origin O. A comparison of equations (2.5.2) and (2.5.10) shows that the term $\left(\frac{x_1}{t_1} - \frac{a}{2}t_1\right)$ in (2.5.10) is the velocity of the particle at $t = 0$.

b) Acceleration function is variable.

If the acceleration function of particle is time dependent, then (4.1.3) can be written as

$$\ddot{x} = a(t)$$

integrating with respect to t we have

$$\frac{dx}{dt} = \int a(t)dt + C_1 \tag{2.5.11}$$

Where C_1 is constant of integration. another integrating with respect to t yields

$$x = \int \left(\int a(t)dt \right) dt + C_1 t + C_2 \tag{2.5.12}$$

Where C_2 is constant of integration. These constants of integration can be determined by applying boundary conditions. (2.5.12) gives the position of the particle moving with variable acceleration at any time t.

2.5.5 Velocity as a Function of Position

A differential relation involving the displacement, velocity, and acceleration may be obtained by eliminating the time differential dt from equations (2.1.3) and (2.1.4)

$$vdv = adx \tag{2.5.13}$$

integrating (separation of variables), assuming that initially (at $t = 0$) $v = v_0$ and $x = x_0$

$$\begin{aligned} \int_{v_0}^{v} vdv &= a \int_{x_0}^{x} dx \\ \left.\frac{v^2}{2}\right|_{v_0}^{v} &= at\Big|_{x_0}^{x} \\ \frac{1}{2}(v^2 - v_0^2) &= a(x - x_0) \\ v^2 &= v_0^2 + 2a(x - x_0) \end{aligned} \tag{2.5.14}$$

Equation (2.5.14) gives the velocity of the particle moving with constant acceleration a and initial conditions $v(0) = v_0$ and $x(0) = x_0$, after it has moved a displacement x. If the particle starts from origin, then equation (2.5.14) becomes

$$v^2 = v_0^2 + 2ax \tag{2.5.15}$$

Equation (2.5.15) gives the velocity of the particle starts to move from origin with velocity v_0 and constant acceleration a at any time t after it has moved a displacement x. Equation (2.5.14) can also be written as

$$2a(x - x_0) = v^2 - v_0^2 \tag{2.5.16}$$

Equation (2.5.15) gives the displacement of the particle moving with constant acceleration a and initial conditions $v(0) = v_0$ and $x(0) = x_0$, after it has gained velocity v during time t. If the particle starts from origin, then equation (2.5.15) becomes

$$2ax = v^2 - v_0^2 \tag{2.5.17}$$

Equation (2.5.17) gives the displacement of the particle starts to move from origin with velocity v_0 and constant acceleration a at any time t after it has gained a velocity v. Equation (2.5.17) also gives the acceleration of the particle that would change its velocity from v_0 to v in a given displacement x.

2.5.6 Acceleration as a Function of Velocity

When the acceleration depends only on the velocity v, then consider equation (2.1.5)

$$a(v) = v\frac{dv}{dx}$$

is first order separable differential equation. On separating variables we get

$$dx = \frac{vdv}{a(v)}$$

On integration we have

$$x = \int \frac{vdv}{a(v)} + A \tag{2.5.18}$$

Where A is constant of integration and can be determined if we know the velocity of the particle for some value of x
When the acceleration depends only on the velocity v, then equation (2.1.4) may be written as

$$a(v) = \frac{dv}{dt}$$

is first order separable differential equation. On separating variables we get

$$dt = \frac{dv}{a(v)}$$

On integration we have

$$t = \int \frac{dv}{a(v)} + B \tag{2.5.19}$$

Where B is constant of integration and can be determined if we know the velocity of the particle for some value of t

Example 2.5.1. *A particle starts to move from origin with acceleration $a = kv^3$ along a straight line. If the initial velocity is v_0, find the velocity and the time spend when the particle has moved a displacement x. Also find this displacement.*

Solution Here the acceleration depends only on the velocity v, then equation (2.1.5) can be written as

$$v\frac{dv}{dx} = kv^3 \tag{2.5.20}$$

The particle starts from origin, it means at $x = 0,\ v = v_0$. This initial condition can be written as

$$v(0) = v_0 \tag{2.5.21}$$

Equation (2.5.20) is first order separable differential equation. On separating variables we get

$$\frac{dv}{v^2} = kdx$$

On integration we have

$$-\frac{1}{v} = kx + A$$

Where A is constant of integration and can be determined from the given initial condition.

$$-\frac{1}{v_0} = k(0) + A$$

$$\text{or} \qquad A = -\frac{1}{v_0}$$

Then equation of motion becomes

$$-\frac{1}{v} = kx - \frac{1}{v_0}$$

Hence the velocity of the particle when it has travelled a displacement x, is given by

$$v = \frac{v_0}{1 - kxv_0} \tag{2.5.22}$$

The time spend to travel a displacement x can be calculated by using (2.5.19)

$$\begin{aligned} t &= \int \frac{dv}{kv^3} + B \\ &= -\frac{1}{2kv^2} + B \end{aligned} \tag{2.5.23}$$

Where B is constant of integration and can be determined from the following initial condition. Since the particle starts from origin, it means at $t = 0,\ v = v_0$. Hence the initial condition is

$$v(0) = v_0 \tag{2.5.24}$$

Using equation (2.5.24) in equation(2.5.23), we get

$$\begin{aligned} 0 &= -\frac{1}{2kv_0^2} + B \\ \text{or} \qquad B &= \frac{1}{2kv_0^2} \end{aligned}$$

Hence the time of motion is

$$\begin{aligned} t &= \frac{1}{2kv_0^2} - \frac{1}{2kv^2} \\ &= \frac{1}{2k}\left(\frac{1}{v_0^2} - \frac{1}{v^2}\right) \end{aligned} \tag{2.5.25}$$

Equation (2.5.25) gives the time of motion in terms of velocities. It may be calculated in terms of initial velocity and displacement by using equation (2.5.22) in equation (2.5.25)

$$\begin{aligned} t &= \frac{1}{2k}\left(\frac{1}{v_0^2} - \frac{(1-kxv_0)^2}{v_0^2}\right) \\ &= \frac{1}{2kv_0^2}\left(1 - \left(1 - 2kxv_0 + k^2x^2v_0^2\right)\right) \\ &= \frac{x}{2v_0}\left(2 - kxv_0\right) \end{aligned} \tag{2.5.26}$$

The displacement can be calculated by using equation (2.5.18)

$$\begin{aligned} x &= \int \frac{vdv}{kv^3} + C \\ &= \int \frac{dv}{kv^2} + C \\ &= -\frac{1}{kv} + C \end{aligned} \tag{2.5.27}$$

Using initial condition given by equation (2.5.21), the constant of integration is

$$C = \frac{1}{kv_0}$$

Then equation (2.5.27) becomes

$$x = \frac{1}{k}\left(\frac{1}{v_0} - \frac{1}{v}\right) \tag{2.5.28}$$

Equation (2.5.28) gives the displacement moved by the particle in terms of velocities.

2.5.7 Geometric Interpretation of $a = v\frac{dv}{dx}$

Consider a particle moves in displacement velocity plane. At any time t the particle is at $P(x, v)$. The slope of the curve at any point is given by $\frac{dv}{dx}$. Thus if θ is the angle which the tangent at P of the displacement velocity curve makes with the x axis as shown in Fig. 2.9, then

$$\tan\theta = \frac{dv}{dx} \tag{2.5.29}$$

Let PN be the normal to the curve at P and PM be the perpendicular to the x axis as

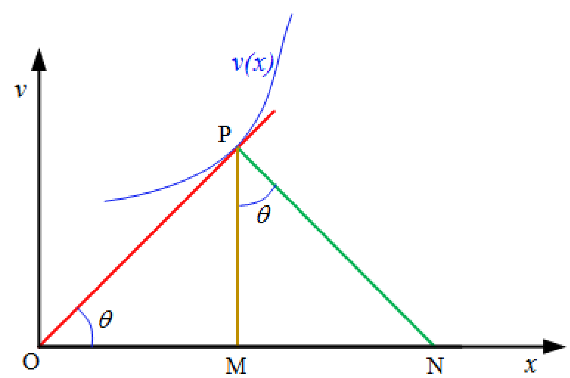


Figure 2.9: velocity versus position curve

shown in Fig. 2.9. Then

$$\angle MPN = \theta$$

and the subnormal MN is given by

$$MN = PM\tan\theta \tag{2.5.30}$$

From Fig. 2.9, we have

$$PM = v \tag{2.5.31}$$

Using eqs. (2.5.29) and (2.5.31) in eq. (2.5.30), we have

$$MN = v\frac{dv}{dx} \tag{2.5.32}$$

Equation (2.5.31) gives the acceleration of the particle at any time t.
Thus the length of the subnormal at a point of the displacement velocity curve, gives the corresponding acceleration of the particle.

2.5.8 Acceleration as a Function of Position

When the acceleration depends only on the velocity x, then consider equation (2.1.5)

$$v\frac{dv}{dx} = a(x)$$

is first order separable differential equation. On separating variables we get

$$vdv = a(x)dx$$

On integration we have

$$\frac{v^2}{2} = \int a(x)dx + A \tag{2.5.33}$$

Where A is constant of integration and can be determined if we know the position of the particle for some value of x. The velocity may be written as

$$v = \pm\sqrt{2\left(\int a(x)dx + A\right)} \tag{2.5.34}$$

$$\text{or} \qquad \frac{dx}{dt} = \pm\sqrt{2\left(\int a(x)dx + A\right)} \tag{2.5.35}$$

Then equation (2.5.35) is first order separable differential equation. On separating variables we get

$$dt = \pm\frac{dx}{\sqrt{2\left(\int a(x)dx + A\right)}}$$

On integration we have

$$t = \pm\int\frac{dx}{\sqrt{2\left(\int a(x)dx + A\right)}} + B \tag{2.5.36}$$

Where B is constant of integration and can be determined if we know the position of the particle for some value of t

Example 2.5.2. *A particle moves along a straight line with acceleration $a = x^3$, where x is the displacement of the particle from a fixed point O. If at $t = 0$, its distance from origin is c and velocity is $\frac{c^2}{\sqrt{2}}$ find the velocity and the time spend when the particle has moved a displacement x.*

Solution Here acceleration depends only on the velocity x, then consider equation (2.1.5)

$$v\frac{dv}{dx} = x^3$$

Following equation (2.5.33), its solution is

$$\begin{aligned} \frac{v^2}{2} &= \int x^3 dx + A \\ &= \frac{x^4}{4} + A \end{aligned} \tag{2.5.37}$$

Where A is constant of integration and can be determined from the given initial condition. That is, at $x = c$, $v = \frac{c^2}{\sqrt{2}}$. Then equation (2.5.37) becomes

$$\begin{aligned} \frac{c^4}{4} &= \frac{c^4}{4} + A \\ \Rightarrow \qquad A &= 0 \end{aligned}$$

Equation (2.5.37) becomes

$$\begin{aligned} \frac{v^2}{2} &= \frac{x^4}{4} \\ \text{or} \qquad v &= \frac{x^2}{\sqrt{2}} \end{aligned} \tag{2.5.38}$$

Equation (2.5.38) gives the velocity of the particle when it has moved a displacement x from fixed point O. Then equation (2.5.38) can be written as

$$\frac{dx}{dt} = \frac{x^2}{\sqrt{2}} \tag{2.5.39}$$

Equation (2.5.39) is first order separable differential equation. On separating variables we get

$$dt = \sqrt{2}\,\frac{dx}{x^2}$$

On integration we have

$$t = -\frac{\sqrt{2}}{x} + B \tag{2.5.40}$$

Where B is constant of integration and can be determined from given initial condition. That is at $t = 0, x = c$. Then equation (2.5.40) becomes

$$\begin{aligned} 0 &= -\frac{\sqrt{2}}{c} + B \\ \text{or} \qquad B &= \frac{\sqrt{2}}{c} \end{aligned}$$

Then equation (2.5.40) becomes

$$t = \sqrt{2}\left(\frac{1}{c} - \frac{1}{x}\right) \tag{2.5.41}$$

Equation (2.5.41) gives the time of the particle when it has moved a displacement x from fixed point O.

Example 2.5.3. *A particle moves along* $x - axis$ *with acceleration* $a(t) = 3\sin 3t$*;* $v(0) = 3$*;* $x(0) = 3$*. Find the velocity and position functions of the particle.*

Solution Here the acceleration is

$$a(t) = \frac{d}{dt}v(t) = 3\sin 3t \tag{2.5.42}$$

The initial conditions are

$$\begin{aligned} v(0) = v(0) &= 3 \\ x(0) &= 3 \end{aligned}$$

The velocity function is obtained by integrating (2.5.42)

$$v(t) = -cos3t + C_1 \tag{2.5.43}$$

(2.5.43) is the family of velocity functions.
Using initial condition $v(0) = v(0) = 3$, the constant of integration is obtained as

$$\begin{aligned} 3 &= -1 + C_1 \\ C_1 &= 4 \end{aligned}$$

Hence the velocity function is

$$v(t) = -cos3t + 4 \tag{2.5.44}$$

(2.5.44) is unique velocity function.
The position function is obtained by integrating (2.5.44)

$$x(t) = -\frac{1}{3}sin3t + 4t + C_2 \tag{2.5.45}$$

(2.5.45) is the family of position functions.
Again using initial condition $x(0) = 3$, the constant of integration is

$$3 = C_2;$$

Hence the position function is

$$x(t) = -\frac{1}{3}sin3t + 4t + 3 \tag{2.5.46}$$

(2.5.46) is unique position function.

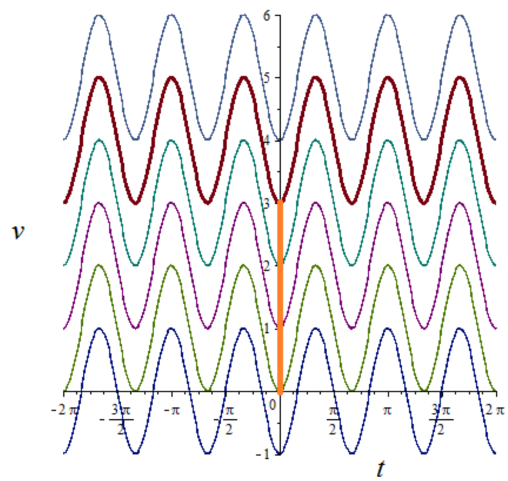


Figure 2.10: Family of velocity functions

Example 2.5.4. *A particle moves along an $r-axis$ with velocity $v(t) = t^2 - 2t$ m/s; (see Fig. 2.12)*

(a) Find the displacement of the particle during the time interval $0 \leq t \leq 3$.

(b) Find the distance of the particle during the time interval $0 \leq t \leq 3$.

Solution (a) Here the velocity is

$$v(t) = t^2 - 2t$$

Using (1.3.5), the displacement of the particle during the time interval $0 \leq t \leq 3$ is

$$\begin{aligned}\int_{t_1}^{t_2} v(t)dt &= \int_0^3 \left(t^2 - 2t\right) dt \\ &= \left[\frac{1}{3}t^3 - t^2\right]_0^3 \\ &= \left(\frac{1}{3}3^3 - 3^2\right) = 9 - 9 \\ &= 0\end{aligned} \tag{2.5.47}$$

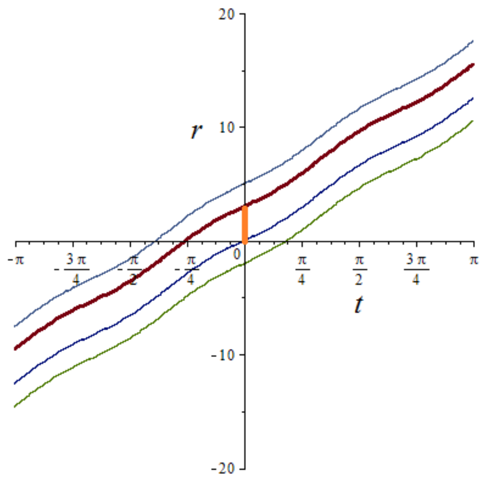


Figure 2.11: Family of position functions

Also from graph 2.17, we see that the displacement or position vector is a zero vector. (b) For distance, see Fig. 2.12, $v(t) \leq 0$ for $0 \leq t \leq 2$ and $v(t) \geq 0$ for $2 \leq t \leq 3$ Thus, it follows from (2.5.5) that the distance traveled is

$$\begin{aligned}
\int_{t_1}^{t_2} |v(t)|dt &= \int_{0}^{2} -\left(t^2 - 2t\right) dt + \int_{2}^{3} \left(t^2 - 2t\right) dt \\
&= -\left[\frac{1}{3}t^3 - t^2\right]_0^2 + \left[\frac{1}{3}t^3 - t^2\right]_2^3 \\
&= -\left(\frac{1}{3}2^3 - 2^2\right) + \left(\frac{1}{3}3^3 - 3^2 - \frac{1}{3}2^3 + 2^2\right) \\
&= \frac{4}{3} + \frac{4}{3} \\
&= \frac{8}{3}\, m \qquad (2.5.48)
\end{aligned}$$

Example 2.5.5. *A particle moves starts to move along a straight line from origin, so at* $t = 0,\ x = 0$ *and* $v = 0$. *At any time* t *its acceleration is* $a(t) = t^2 + \sin t + e^t\ m^2/s$. *Find*

(a) Velocity at any time t

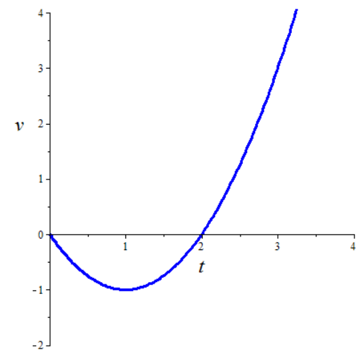


Figure 2.12: velocity at time t

(b) *Speed at any time t*

(c) *Displacement at any time t*

(d) *Distance at any time t*

Solution Here the given data is

$$\ddot{x} = t^2 + \sin t + e^t \tag{2.5.49}$$
$$v(0) = 0 \tag{2.5.50}$$
$$x(0) = 0 \tag{2.5.51}$$

(a) Velocity at any time t is calculated by integrating (2.5.49) with respect to t

$$v(t) = \dot{x} = \frac{1}{3}t^3 - \cos t + e^t + A \tag{2.5.52}$$

Where A is constant of integration and can be calculated by using (2.5.50) in (2.5.52)

$$0 = 0 - 1 + 1 + A$$
$$\Rightarrow \quad A = 0$$

Hence (2.5.52) becomes

$$v(t) = \frac{1}{3}t^3 - \cos t + e^t \tag{2.5.53}$$

(b) The peed at any time t is

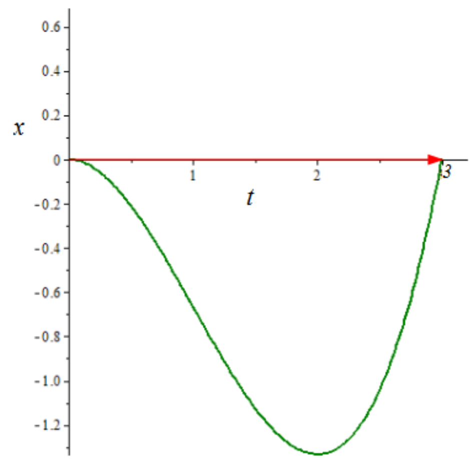


Figure 2.13: Displacement and distance for $0 \leq t \leq 3$.

$$|v(t)| = \left|\frac{1}{3}t^3 - \cos t + e^t\right| \tag{2.5.54}$$

(c) Displacement at any time t is calculated by integrating (2.5.53) with respect to t

$$x = \frac{1}{12}t^4 - \sin t + e^t + B \tag{2.5.55}$$

Where B is constant of integration and can be calculated by using (2.5.51) in (2.5.55)

$$0 = 0 - 0 + 1 + B$$
$$\Rightarrow \quad B = -1$$

Hence (2.5.55) becomes

$$x = \frac{1}{12}t^4 - \sin t + e^t - 1 \tag{2.5.56}$$

(d) Distance at any time t is

$$|x| = \left|\frac{1}{12}t^4 - \sin t + e^t - 1\right| \tag{2.5.57}$$

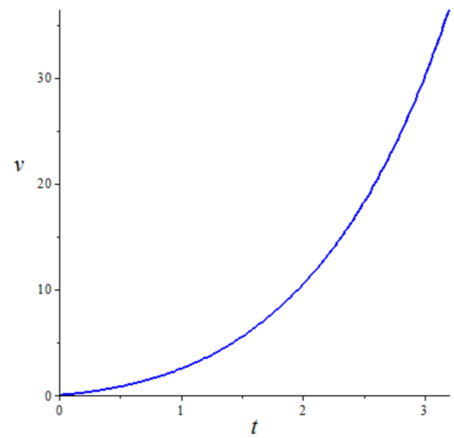


Figure 2.14: velocity and speed any time t.

Example 2.5.6. *A car starts from rest and with constant acceleration achieves a velocity of* $15\,m/s$ *when it travels a distance of* $200\,m$*. Determine the acceleration of the car and the time required.*

Solution 1: Since the car starts from rest, we have
$v_0 = 0\,m/s$ and $v = 15\,m/s$
$x_0 = 0\,m$ and $x = 200\,m$
As acceleration is constant, Using, (2.5.14)

$$\begin{aligned} v^2 &= v_0^2 + 2a_c(x - x_0) \\ (15)^2 &= 0 + 2a_c(200 - 0) \\ a_c &= 0.562 m/s^2 \end{aligned}$$

For time Using (2.4.1)

$$\begin{aligned} v &= v_0 + a_c t \\ 15 &= 0 + 0.562\,t \\ t &= 26.66 s \end{aligned}$$

Example 2.5.7. *A car moves in a straight line such that for a short time its velocity is defined by* $v = (3t^2 + 2t)\,ft/sec$*. Determine its position and acceleration when* $t = 3\,s$*. When* $t = 0$, $x = 0$.

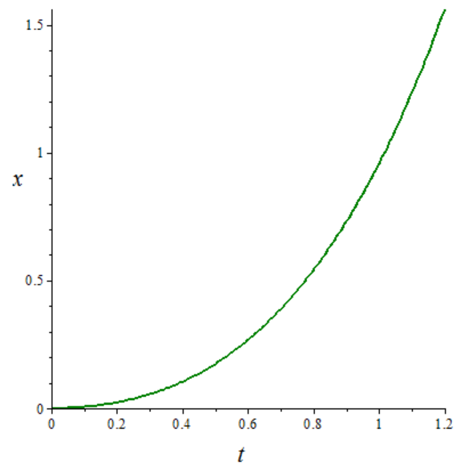


Figure 2.15: Displacement and distance at time t

Solution : For Position: Since $v = v(t)$, and $x = 0$ when $t = 0$, using, (2.1.3)

$$\begin{aligned} v = \frac{dx}{dt} &= 3t^2 + 2t \\ dx &= (3t^2 + 2t)dt \end{aligned}$$

integrating (separation of variables)

$$\begin{aligned} \int_0^x dx &= (3t^2 + 2t) \int_0^t dt \\ x\Big|_0^x &= (t^3 + t^2)\Big|_0^t \\ x &= t^3 + t^2 \end{aligned}$$

When $t = 3\,s$

$$\begin{aligned} x &= 3^3 + 3^2 \\ x &= 36\,ft \end{aligned}$$

For acceleration using, (2.1.4)

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= \frac{d}{dt}(3t^2 + 2t) \\ &= 6t + 2 \end{aligned}$$

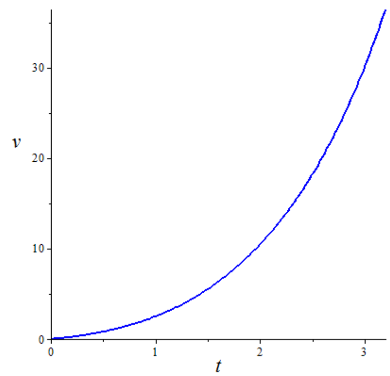


Figure 2.16: velocity at time t

When $t = 3\,s$

$$\begin{aligned} a &= 6(3) + 2 \\ a &= 20\,ft/s^2 \end{aligned}$$

Example 2.5.8. *A particle starts to move from origin O in a straight line with uniform acceleration a. After* 4 *seconds it attains a velocity of* 60 *miles/hour. Determine*

(a) Acceleration of motion.

(b) Displacement travelled in first 3 seconds

(c) Displacement travelled in last 3 seconds.

(d) Displacement travelled in 4th seconds.

Solution: The given data is

$$\begin{aligned} v_0 &= 0 \\ t &= 4\ s \\ v_4 &= 60\ miles/hour \\ &= \frac{60 \times 1760 \times 3}{60 \times 60} \\ &= 88\ ft/s \end{aligned}$$

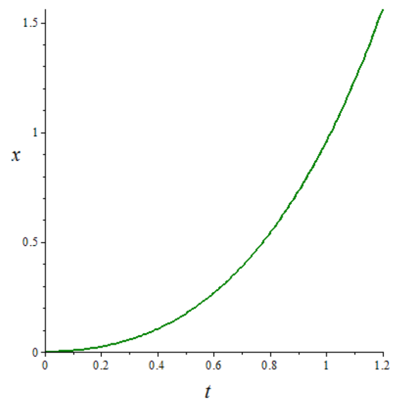


Figure 2.17: Displacement at time t

(a) From equation (2.4.1) we can write

$$\begin{aligned} a &= \frac{1}{t}(v_4 - v_0) \\ &= \frac{1}{4}(88 - 0) \\ &= 22\ ft/s^2 \end{aligned}$$

(b) Let x_1 represents the displacement travelled by the particle in first 3 seconds. It can be determined by using equation (2.5.2)

$$\begin{aligned} x_1 &= v_0 t + \frac{1}{2}at^2 \\ &= 0(3) + \frac{1}{2}22(3)^2 \\ &= 99\ ft \end{aligned}$$

(c) Let x_2 represents the displacement travelled by the particle in last 3 seconds. It can be determined by subtracting displacement travelled in first second from displacement travelled in 4 seconds. Both displacements will be determined by using equation (2.5.2)

$$\begin{aligned} x_2 &= \left(0(4) + \frac{1}{2}22(4)^2\right) - \left(0(1) + \frac{1}{2}22(1)^2\right) \\ &= 176 - 11 \\ &= 165\ ft \end{aligned}$$

(d) Let x_3 represents the displacement travelled by the particle in last $4th$ second. It can be determined by subtracting displacement travelled in first 3 second from displacement travelled in 4 seconds. Both displacements will be determined by using equation (2.5.2)

$$\begin{aligned} x_2 &= \left(0(4) + \frac{1}{2}22(4)^2\right) - \left(0(3) + \frac{1}{2}22(3)^2\right) \\ &= 176 - 99 \\ &= 77\ ft \end{aligned}$$

Corollary 2.5.1. *A particle moves in a straight line with uniform acceleration a. At time $t = 0$ it is at origin O moving with velocity v_0. Hence the displacement travelled by the particle in the nth unit of time is*

$$x = v_0 + \frac{1}{2}a(2n - 1) \tag{2.5.58}$$

Proof: Let x_1 and x_2 be the displacement travelled by the particle in the first n and $n - 1$ units of time respectively. Then it follows from equation (2.5.2) that

$$\begin{aligned} x_1 &= v_0 n + \frac{1}{2}an^2 \\ \text{and} \qquad x_2 &= v_0(n-1) + \frac{1}{2}a(n-1)^2 \end{aligned}$$

Then the distance travelled by the particle in the nth unit of time is

$$\begin{aligned} x = x_1 - x_2 &= \left(v_0 n + \frac{1}{2}an^2\right) - \left(v_0(n-1) + \frac{1}{2}a(n-1)^2\right) \\ &= \left(v_0 n + \frac{1}{2}an^2\right) - \left(v_0 n - v_0 + \frac{1}{2}a(n^2 - 2n + 1)\right) \\ &= v_0 + \frac{1}{2}a(2n - 1) \end{aligned}$$

Example 2.5.9. *In example 2.5.8 the displacement travelled in $4th$ seconds can be calculated by using equation (2.5.58)*

$$\begin{aligned} x_4 &= v_0 + \frac{1}{2}a(2n - 1) \\ &= 0 + \frac{1}{2}22[2(4) - 1] \\ &= 77\ ft \end{aligned}$$

Exercise

1. Find velocity and acceleration of the particle for $t \geq 0$ for the following position curves.
   (a) $r(t) = 2t^2 + 5t^2 - 6t + 4$
   (b) $r(t) = -4t + 3$
   (c) $r(t) = 5t^2 - 20t$
   (d) $r(t) = t^3 - 9t^2 + 24t$
   (e) $r(t) = t^3 - 6t^2 + 9t + 1$
2. Let $x(t) = 2t^3 - 21t^2 + 60t + 3$ be the position function of a particle moving along $x - axis$, Find the velocity, speed and acceleration functions. Also plot the graphs of position, velocity, speed and acceleration versus time. From these graphs, determine when the particle is speeding up and slowing down.
3. Let a particle moves with constant acceleration a along straight line. Obtain the following equations of motion by graphical method.
   (a) $v_f = v_i + at$
   (b) $x = v_i t + \frac{1}{2}at^2$
   (c) $2ax = v_f^2 - v_i^2$
4. Let $r_1 = 15t^2 + 10t + 20$ and $r_2 = 5t^2 + 40t, \quad t \geq 0$, be the position functions of cars A and B that are moving along parallel straight lanes of a highway.
   (a) How far is car A ahead of car B when $t = 0$?
   (b) At what instants of time are the cars next to each other?
   (c) At what instant of time do they have the same velocity?
   (d) Which car is ahead at this instant?
5. Two particles A and B are moving in a straight line in the same direction in such a way that A is accelerated and B is retard. At point O, A is accelerated at the rate of 2 ft/s^2 having velocity 45 $miles/hour$, while B is retard at the rate of 8 ft/s^2 having velocity 90 $miles/hour$.
   (a) At what time both particles have same velocity, find this velocity.
   (b) At what time both particles have same displacement from O, find this displacement.
6. Two particles A and B are moving in a straight line in the same direction in such a way that A is accelerated and B is retarded. At point O, A is accelerated at the rate of 1.2 m/s^2 having velocity 60 $km/hour$, while B is retard at the rate of 7.1 m/s^2 having velocity 120 $km/hour$.

(a) At what time both particles have same velocity, find this velocity.

(b) At what time both particles have same displacement from O, find this displacement.

7. A particle starts to move from rest along straight line from origin. At any time t, its acceleration is given as following. Find velocity, speed, displacement and distance of the particle at ant time t.

   (a) $a(t) = t^n$

   (b) $a(t) = a\cos t + b\sin t$

   (c) $a(t) = 3t^2 - 4t$

   (d) $a(t) = \frac{1}{t}$

   (e) $a(t) = 6e^t + 2t$

8. A particle moves along a straight line with acceleration $a = -n^2x$, where x is the displacement of the particle from a fixed point O. Let the particle starts from rest from origin. Find the velocity and the time spend when the particle has moved a displacement x. Also find this displacement as a function of time.

9. A particle moves along a straight line with acceleration $a = \mu x$, where x is the displacement of the particle from a fixed point O. Let the particle starts from rest at a displacement x_0 from origin. Find the velocity, displacement and the time spend when the particle has moved a displacement x.

10. A particle moves along a straight line with acceleration $a = x$, where x is the displacement of the particle from a fixed point O. If at $t = 0$, its displacement from origin and velocity are c. Find the velocity and the time spend when the particle has moved a displacement x.

11. A particle moves along a straight line with acceleration $a = x^2$, where x is the displacement of the particle from a fixed point O. If at $t = 0$, its distance from origin is c and velocity is $\sqrt{\frac{2c^3}{3}}$. Find the velocity and the time spend when the particle has moved a displacement x.

12. A particle starts to move from origin along a straight line with velocity v_i. If it has acceleration v^3, where v is the velocity of the particle at any time t, find the velocity and the time spend when the particle has moved a displacement x. Also find this displacement as a function of time.

13. A particle starts to move from origin along a straight line with velocity v_i. If it suffers a retardation equal to the square of the velocity of the particle at any time t, find the velocity and the time spend when the particle has moved a displacement x. Also find this displacement as a function of time.

14. A particle starts to move from rest from origin O along a straight line. It moves with uniform acceleration a till it attains a velocity v. The motion is then retarded and the particle comes to rest after travelling a total distance x. Find the retardation and the total time taken by the particle from rest to rest.

15. Two particles A and B are moving in a straight line and are accelerated uniformly such that if acceleration of A is a then acceleration of B is $\frac{1}{2}a$. Both particles starts from origin at the same time. The motion is such that when a particle attains the maximum velocity v, the motion is retarded uniformly in a way that retardation of A is $\frac{1}{2}a$ and retardation of B is a. Then the two particles comes to rest simultaneously at a distance x from the starting point. Find the distance between the points where the two particles attain their maximum velocities.

16. At time $t = 0$, a gunner detects a plane approaching him with a velocity v, the horizontal and vertical displacements of the plane being h and k respectively. His gun can fire a shell vertically upwards with an initial velocity u . Find the time when he should fire the gun and the condition on u so that he may be able to hit the plane if it continues its flight in the same horizontal line.

Chapter 3

Vertical Motion Under Gravity

A motion under the force of gravity is called projectile motion and a body executing such a motion is called projectile. This motion is in the vertically upward or downward direction, hence also called vertical motion. Sometimes air resistance also takes place in this motion. So we will discuss it as free projectile motion and resisted projectile motion. For example, an object dropped from a height, thrown vertically upwards or thrown at an angle (oblique). We can subdivide this motion in two categories namely

a) One dimensional Projectile Motion.

b) Two dimensional Projectile Motion.

In this chapter we will discuss only One dimensional Projectile Motion. Two dimensional Projectile Motion will be discussed in chapter 7.

3.1 One Dimensional Projectile Motion or Vertical Motion

An object dropped from a height or thrown vertically upwards are the examples of such motion. Since this motion takes place along vertical axis so is known as vertical motion.
This motion is one of the most important case of uniform accelerated motions. The objects which are near to earth allowed to free fall. At first, it might seem that different objects accelerate at different rates near the earth depending on their weight. That is what people thought before Galileo did his experiments in the late $1500s$. However, Galileo showed that objects of different weights dropped at the same rate.
The reason that some things drop slower in the air is because air resistance pushes against the moving object. If there was no air resistance then even the open paper would drop at the same rate. (On the moon, where there is no air, a feather and a hammer fell at the same rate). When an object is dropped, the rate of increase of its velocity is 9.80 m/s^2 in SI system and 32 ft/s^2 in BE system. It continues to go faster. If there was no air resistance, this would continue every second. Because there is air resistance, the object eventually reaches a terminal velocity and does not go any faster. But for many applications, we can neglect air resistance. Then, everything near the surface of the earth accelerates toward

the center of the earth with a constat acceleration. Therefore, all of the equations we have derived for constant acceleration apply to an object in free fall, neglecting air resistance). All objects fall with a constant acceleration of about 9.80 m/s^2 which we call g, the acceleration due to gravity.
In this chapter, we will consider vertically free and resisted motions. Assume that both gravity and mass remain constant and, for convenience, choose the downward direction as the positive direction. Since it is one dimensional motion, so the reference point may not be with respect to coordinate system. First of all consider free vertical motion.

3.1.1 Free Vertical Motion

In this motion, the particle is moving vertically under the action of gravitational force only. Then equations for vertically downward motion are the equations of motion for rectilinear motion only replacing a by g. The elementary equations of motion for vertically downward motion are:

$$v_f = v_i + gt \tag{3.1.1}$$

$$x = v_i t + \frac{1}{2}gt^2 \tag{3.1.2}$$

$$2gx = v_f^2 - v_i^2 \tag{3.1.3}$$

Example 3.1.1. *A body is dropped (at rest) from a height of h meters. If the motion is free fall, then at what speed will it hit the ground? Also find the time required for it.*

Solution Since it is one dimensional motion, the reference axis may be $z-axis$ only. As the body starts from rest, so the initial data is

$$\begin{aligned} t_0 &= 0 \\ v_0 &= \frac{dz}{dt}(0) = 0 \\ z_0 &= 0 \end{aligned}$$

At time t the body is at P. At P, the only force acting on the particle is the gravitational force $W = mg$. Hence by Newton's second law of motion

$$\begin{aligned} F &= mg \\ m\frac{d^2z}{dt^2} &= mg \\ \frac{d^2z}{dt^2} &= g \\ \frac{dv}{dt} &= g \end{aligned}$$

is first order differential equation in variable v and can be solved as (separating variables)

$$v(t) = gt + v_0$$

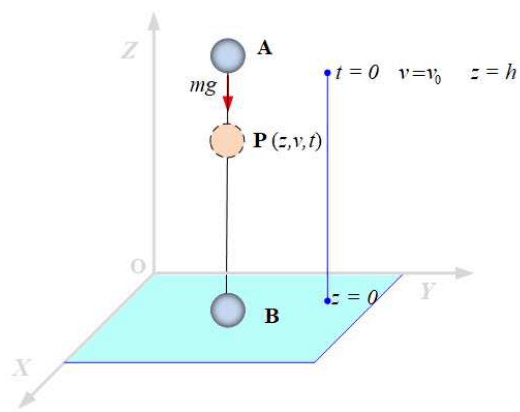


Figure 3.1: Downward motion

Using initial condition $v(0) = 0$, the above relation is

$$v(t) = \frac{dz}{dt} = gt$$

now it is first order differential equation in variable z and can be solved as (separating variables)

$$z(t) = \frac{1}{2}gt^2 + z_0$$

Using initial condition $z(0) = 0$, the above relation is

$$z(t) = \frac{1}{2}gt^2$$

Let it hits the ground after time t_1 then $z(t_1) = h$, then we have

$$h = \frac{1}{2}gt_1^2$$

or time is

$$t_1 = \sqrt{\frac{2h}{g}}$$

at that time its (vertical) velocity is

$$\begin{aligned} v(t_1) &= gt_1 \\ &= \sqrt{2hg}. \end{aligned}$$

Since it is one dimensional motion, the reference axis may be $z - axis$ only.

Example 3.1.2. *A body of mass 5 slugs is dropped from a height of* 100 ft *with zero velocity. Assuming no air resistance, find*

(a) an expression for the velocity of the body at any time t.

(b) an expression for the position of the body at any time t, *and*

(c) the time required to reach the ground.

(d) at what speed will it hit the ground?

Solution: Since it is one dimensional motion, the reference axis may be $x-axis$ only. As the body starts from rest, so the initial data is

$$\begin{aligned} t_0 &= 0 \\ v_0 &= 0 \\ x_0 &= 0 \end{aligned}$$

Choose the coordinate system as in Fig. (3.2). Since there is no air resistance, so the only

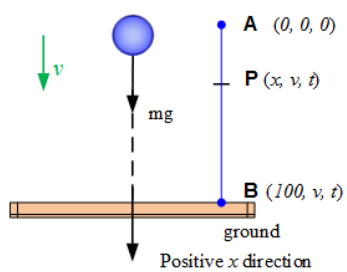


Figure 3.2: Free fall motion

force acting on the body is $W = mg$ and by Newton's second law of motion:

$$\begin{aligned} F &= mg \\ ma &= mg \end{aligned} \tag{3.1.4}$$

Since

$$a = \frac{dv}{dt}$$

then (3.1.4) becomes

$$\frac{dv}{dt} = g$$

This differential equation is linear or, in differential form, separable; its solution is

$$v = gt + C_1 \tag{3.1.5}$$

initially the body has zero velocity *i.e.* at $t = 0, v = 0$ or $v(0) = 0$ using in (3.1.5), we have

$$0 = g(0) + C_1 \tag{3.1.6}$$

or $C_1 = 0$. Thus,

$$v = gt$$

assuming $g = 32 ft/sec$, then

$$v = 32t \tag{3.1.7}$$

Since

$$v = \frac{dx}{dt},$$

then (3.1.7) becomes

$$\frac{dx}{dt} = 32t \tag{3.1.8}$$

This differential equation is also both linear and separable; its solution is

$$x = 16t^2 + C_2 \tag{3.1.9}$$

But at $t = 0, x = 0$ (see Fig. 3.2). Thus,

$$0 = (16)(0)^2 + C_2$$

or $C_2 = 0$. Substituting this value into (3.1.9), we have

$$x = 16t^2 \tag{3.1.10}$$

We require t when $x = 100$. From (3.1.10)

$$t = \sqrt{\frac{100}{16}} = 2.5\ s$$

Using this time in equation (3.1.8) the speed with which it will hit the ground is

$$\begin{aligned} v &= 32(2.5) \\ &= 80\ ft/s \end{aligned}$$

Example 3.1.3. *A body is dropped from a height of h meters with speed v_0. Assuming no air resistance, find* *(a) an expression for the velocity of the body at any time t.*

(b) an expression for the position of the body at any time t, and

(c) the time required to reach the ground.

(d) at what speed will it hit the ground?

Solution Since it is one dimensional motion, the reference axis may be $z - axis$ only. As the body starts from rest, so the initial data is

$$\begin{aligned} t_0 &= 0 \\ v_0 &= \frac{dz}{dt}(0) = v_0 \\ z_0 &= 0 \end{aligned}$$

At time t the body is at P. At P, the only force acting on the particle is the gravitational

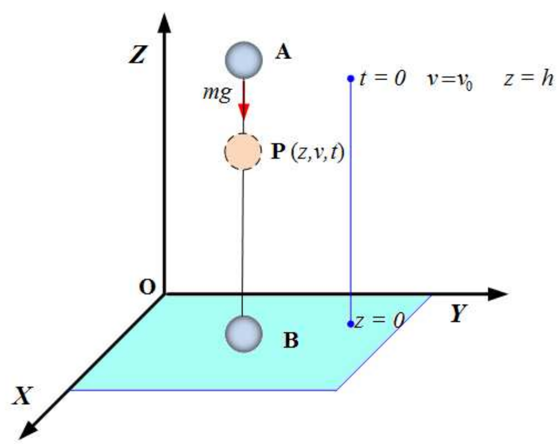


Figure 3.3: Downward motion

force $W = mg$. Hence by Newton's second law of motion

$$\begin{aligned} F &= mg \\ m\frac{d^2z}{dt^2} &= mg \\ \frac{d^2z}{dt^2} &= g \\ \frac{dv}{dt} &= g \end{aligned}$$

is first order differential equation in variable v and can be solved as (separating variables)

$$v(t) \;=\; gt + v_0$$

Using initial condition $v(0) = 0$, the above relation is

$$v(t) \;=\; \frac{dz}{dt} = gt$$

now it is first order differential equation in variable z and can be solved as (separating variables)

$$z(t) \;=\; \frac{1}{2}gt^2 + z_0$$

Using initial condition $z(0) = 0$, the above relation is

$$z(t) \;=\; \frac{1}{2}gt^2$$

Let it hits the ground after time t_1 then $z(t_1) = h$, then we have

$$h \;=\; \frac{1}{2}gt_1^2$$

or time is

$$t_1 \;=\; \sqrt{\frac{2h}{g}}$$

at that time its (vertical) velocity is

$$\begin{aligned} v(t_1) &= gt_1 \\ &= \sqrt{2hg}. \end{aligned}$$

Since it is one dimensional motion, the reference axis may be $z - axis$ only.

Example 3.1.4. *A particle is projected vertically upward with a velocity* $\sqrt{2gh}$ *and at the same time, an other is dropped from a height* h *with zero velocity. Assuming no air resistance, find the height where they meet each other.*

Solution Since it is one dimensional motion, the reference axis may be $x - axis$ only. Choose the coordinate system as in Fig. (3.4). Let both the particles meet at time t at P, first particle has distance x from A and the second particle has distance $h - x$ from B. Let $x(t)$ be the position variable for the first particle and $y(t)$ be the position variable for the second particle Then the first particle has coordinates at $P(x, v_1, t)$ and the second particle has coordinates at $P(h - x, v_2, t) = P(y, v_2, t)$. Since there is no air resistance, so the only force acting on the body is $w = mg$.

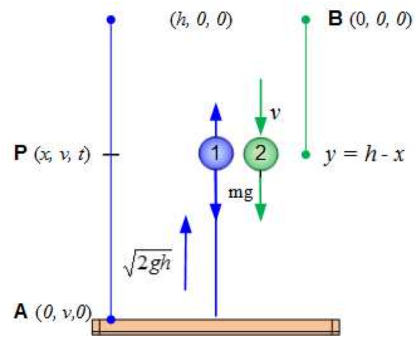


Figure 3.4: Free fall motion

Equation of motion of the first particle:
First particle is moving in the upward direction, so the initial data for it is

$$\begin{aligned} t_0 &= 0 \\ v_1(0) &= \frac{dx}{dt}(0) = \sqrt{2gh} \\ x_0 &= 0 \end{aligned}$$

By Newton's second law of motion, the equation of motion of the first particle is:

$$\begin{aligned} F &= -mg \\ a &= -g \\ \frac{dv_1}{dt} &= -g \end{aligned} \tag{3.1.11}$$

Integrating (3.1.11) with respect to t

$$v_1(t) = -gt + C_1 \tag{3.1.12}$$

Using initial data for first particle, at $t = 0,\ v_1 = \sqrt{2gh}$ *i.e.* ,

$$v_1(0) = \sqrt{2gh}$$

then (3.1.12) becomes

$$\sqrt{2gh} = -g(0) + C_1$$

or

$$\sqrt{2gh} = C_1$$

then (3.1.12) becomes

$$\begin{aligned} v_1(t) &= -gt + \sqrt{2gh} \\ \frac{dx}{dt} &= -gt + \sqrt{2gh} \end{aligned} \tag{3.1.13}$$

(3.1.13) gives the velocity of first ball at any time t. Integrating (3.1.13) with respect to t

$$x(t) = -\frac{1}{2}gt^2 + \sqrt{2gh}\ t + C_2 \tag{3.1.14}$$

Using initial data for first particle, at $t = 0,\ x = 0$ *i.e.* ,

$$x(0) = 0$$

then (3.1.14) becomes

$$C_2 = 0$$

Hence the position of the first particle at t is

$$x(t) = -\frac{1}{2}gt^2 + \sqrt{2gh}\ t \tag{3.1.15}$$

(3.1.15) gives the position of first ball at any time t.

Equation of motion of the second particle:

The second particle is moving in the downward direction, so the initial data for it is

$$\begin{aligned} t_0 &= 0 \\ v_2(0) &= \frac{dy}{dt}(0) = 0 \\ y_0 &= 0 \end{aligned}$$

By Newton's second law of motion, the equation of motion of the second particle is:

$$\begin{aligned} F &= mg \\ a &= g \\ \frac{dv_2}{dt} &= g \end{aligned} \tag{3.1.16}$$

Integrating (3.1.16) with respect to t

$$v_2(t) = gt + C_1 \tag{3.1.17}$$

Using initial data for first particle, at $t = 0,\ v_2 = 0$ *i.e.* ,

$$v_2(0) = 0$$

then (3.1.17) becomes

$$0 = g(0) + C_1$$

or

$$C_1 = 0$$

then (3.1.17) becomes

$$\begin{aligned} v_2(t) &= gt \\ \frac{dy}{dt} &= gt \end{aligned} \tag{3.1.18}$$

(3.1.18) gives the velocity of second ball at any time t. Integrating (3.1.18) with respect to t

$$y(t) = \frac{1}{2}gt^2 + C_2 \tag{3.1.19}$$

Using initial data for first particle, at $t = 0,\ y = 0$ *i.e.* ,

$$y(0) = 0$$

then (3.1.14) becomes

$$C_2 = 0$$

Hence the position of the first particle at t is

$$y(t) = \frac{1}{2}gt^2 \tag{3.1.20}$$

At P, $y = h - x$, then (3.1.20) becomes. Hence the position of the second particle at t is

$$\begin{aligned} h - x(t) &= \frac{1}{2}gt^2 \\ x(t) &= h - \frac{1}{2}gt^2 \end{aligned} \tag{3.1.21}$$

(3.1.21) gives the position of second ball at P. Here (3.1.15) and (3.1.21) are the positions of the two balls at P. From (3.1.15) and (3.1.21), we can write

$$-\frac{1}{2}gt^2 + \sqrt{2gh}\,t = h - \frac{1}{2}gt^2 \tag{3.1.22}$$

Hence both the particles meet at time

$$t = \sqrt{\frac{h}{2g}} \tag{3.1.23}$$

Using (3.1.23) in (3.1.21), the position is

$$\begin{aligned} x(t) &= h - \frac{1}{2}g\left(\frac{h}{2g}\right) \\ &= h - \frac{1}{4}h \\ &= \frac{3}{4}h \end{aligned} \tag{3.1.24}$$

Hence the two particles meet each other at a height $\frac{3}{4}h$

3.1.2 Resisted Vertical Motion

In this motion, the particle is moving under the influence of gravitational force and air resistance force (offered by the atmosphere). The air resistance acts in the opposite direction of the motion, producing a retardation in the motion. It is proportional to the velocity or square of the velocity of the body.

Example 3.1.5. *A body of mass m is thrown vertically into the air with an initial velocity v_0. If the body encounters an air resistance proportional to its velocity, find*

(a) the equation of motion in the coordinate system

(b) an expression for the velocity of the body at any time t, and

(c) the time at which the body reaches its maximum height.

Solution

(a) Equation of motion in the coordinate system

Choose the coordinate system as in Fig. (3.5). Consider a body of mass m is thrown vertically into the air from A. At A, the initial data is

$$\begin{aligned} t_0 &= 0 \\ v(0) &= \frac{dx}{dt}(0) = v_0 \\ x_0 &= 0 \end{aligned}$$

After time t the body is at P having coordinates $P(x, v, t)$. Then by Newton's second law of motion, at time t, the net force acting on a body is

$$F = m\frac{dv}{dt} \tag{3.1.25}$$

where F is the net force on the body and v is the velocity of the body. At P there are two forces acting on the body:

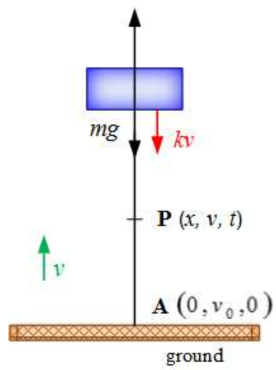


Figure 3.5: Resisted motion

(1) the force due to gravity given by the weight $W = -mg$ of the body, and
(2) the force due to air resistance given by $-kv$, where $k > 0$ is a constant of proportionality. The minus sign is required because this force opposes the velocity; that is, it acts in the downward (see Fig. 3.5). The net force F on the body is, therefore,

$$F = -mg - kv \tag{3.1.26}$$

using (3.1.26) in (3.1.25), we obtain

$$m\frac{dv}{dt} = -mg - kv$$

or

$$\frac{dv}{dt} + \frac{k}{m}v = -g \tag{3.1.27}$$

(3.1.27) is the equation of motion in the coordinate system.

(b) Expression for the velocity of the body at any time t

(3.1.27) is a linear differential equation. Its solution will give the expression for velocity at any time t. The integrating factor is

$$I.F = e^{\frac{k}{m}t}$$

then (3.1.27) can be written as

$$\frac{d}{dt}\left(ve^{\frac{k}{m}t}\right) = -ge^{\frac{k}{m}t} \tag{3.1.28}$$

Integrating (3.1.28) with respect to t

$$\begin{aligned} \left(ve^{\frac{k}{m}t}\right) &= -\frac{mg}{k}e^{\frac{k}{m}t} + C_1 \\ v(t) &= -\frac{mg}{k} + C_1 e^{-\frac{k}{m}t} \end{aligned} \tag{3.1.29}$$

Using initial condition $v(0) = v_0$, then (3.1.29) implies

$$C_1 = v_0 + \frac{mg}{k} \tag{3.1.30}$$

Using (3.1.30), (3.1.29) becomes

$$v(t) = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right) e^{-\frac{k}{m}t} \tag{3.1.31}$$

(3.1.31) is the expression for the velocity of the body at any time t.
When $k > 0$, the limiting velocity v_l is defined by

$$v_l = \frac{mg}{k} \tag{3.1.32}$$

(c) Time required to reach the body at its maximum height.

The body reaches its maximum height when $v = 0$. Hence to calculate time t required to reach maximum height, we put $v = 0$ in (3.1.31)

$$\begin{aligned} 0 &= -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right) e^{-\frac{k}{m}t} \\ \frac{mg}{k} &= \left(v_0 + \frac{mg}{k}\right) e^{-\frac{k}{m}t} \\ \frac{mg}{k}e^{\frac{k}{m}t} &= \left(v_0 + \frac{mg}{k}\right) \\ e^{\frac{k}{m}t} &= \frac{k}{mg}\left(v_0 + \frac{mg}{k}\right) \\ e^{\frac{k}{m}t} &= \left(\frac{kv_0}{mg} + 1\right) \end{aligned}$$

Taking natural log on both sides, we have

$$\frac{k}{m}t = \ln\left(\frac{kv_0}{mg} + 1\right)$$

and time is

$$t = \frac{m}{k}\left[ln\left(\frac{kv_0}{mg} + 1\right)\right] \tag{3.1.33}$$

Hence (3.1.33) gives the time required to reach maximum height.

Example 3.1.6. *A steel ball weighing* 4.9 N *is dropped from a height of* 100 m *with no velocity. As it falls, the ball encounters air resistance numerically equal to* $0.2v$ *(in Newton), where* v *is the velocity of the ball (in* m/s*). Find*

(a) the limiting velocity for the ball and

(b) the time required for the ball to hit the ground.

Solution Choose the coordinate system as in Fig. (3.6). The weight of the ball is $w = 4.9\ N$, its mass is 0.5 kg. Let the ball is dropped from A. At A, the initial data is

$$t_0 = 0$$
$$v_0 = 0$$
$$x_0 = 0$$

After time t the body is at P having coordinates $P(x, v, t)$. Then by Newton's second law

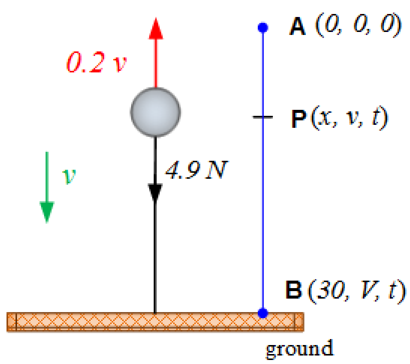


Figure 3.6: Resisted motion

of motion, at time t, the net force acting on a body is

$$F = m\frac{dv}{dt} \tag{3.1.34}$$

where F is the net force on the body and v is the velocity of the body. At P there are two forces acting on the body:
(1) the force due to gravity given by the weight $W = 4.9\ N$ of the body, and
(2) the force due to air resistance is $-0.2\ v$. The net force F on the body is,

$$F = mg - kv \tag{3.1.35}$$

using (3.1.35) in (3.1.34), we obtain

$$\begin{aligned} m\frac{dv}{dt} + 0.2v &= 4.9 \\ 0.5\frac{dv}{dt} + 0.2v &= 4.9 \\ \frac{dv}{dt} + 0.4v &= 9.8 \end{aligned} \tag{3.1.36}$$

(3.1.36) is a linear differential equation, the integrating factor is

$$I.F = e^{0.4t}$$

then (3.1.36) can be written as

$$\frac{d}{dt}\left(ve^{0.4t}\right) = 9.8e^{0.4t} \tag{3.1.37}$$

Integrating (3.1.37) with respect to t

$$\begin{aligned} \left(ve^{0.4t}\right) &= \frac{9.8}{0.4}e^{0.4t} + C_1 \\ v(t) &= 24.5 + C_1 e^{-0.4t} \end{aligned} \tag{3.1.38}$$

Using initial condition $v(0) = v_0 = 0$, then (3.1.38) implies

$$C_1 = 24.5 \tag{3.1.39}$$

Using (3.1.39), (3.1.38) becomes

$$v(t) = 24.5 + 24.5e^{-0.4t} \tag{3.1.40}$$

a) the limiting velocity for the ball is

$$\begin{aligned} \lim_{t\to\infty} v(t) &= \lim_{t\to\infty}\left(24.5 + 24.5e^{-0.4t}\right) \\ &= 24.5\ m/s \end{aligned} \tag{3.1.41}$$

b) the time required for the ball to hit the ground.

From (3.1.40) we can write

$$\frac{dx}{dt} = v(t) = 24.5 + 24.5e^{-0.4t} \tag{3.1.42}$$

Integrating (3.1.42) with respect to t

$$\begin{aligned} x(t) &= 24.5t + 24.5\frac{1}{-0.4}e^{-0.4t} + C_2 \\ &= 24.5t - 61.25e^{-0.4t} + C_2 \end{aligned} \tag{3.1.43}$$

Using initial condition $x(0) = v_0 = 0$, then (3.1.43) implies

$$C_2 = 61.25 \tag{3.1.44}$$

Using (3.1.44), (3.1.43) becomes

$$x(t) = 24.5t - 61.25e^{-0.4t} + 61.25 \tag{3.1.45}$$

The ball hits the ground when $x(t) = 100$, then (3.1.45) has the form

$$\begin{aligned} 100 &= 24.5t - 61.25e^{-0.4t} + 61.25 \\ 0 &= 24.5t - 61.25e^{-0.4t} - 38.75 \end{aligned} \tag{3.1.46}$$

Although (3.1.46) cannot be solved explicitly for t, we can approximate the solution by trial and error, substituting different values of t into (3.1.45) until we locate a solution to the degree of accuracy we need. Such approximation is illustrated in table 3.1
From table 3.1, we see that the ball hits the ground at time $t = 2.5$ s

Table 3.1: Numerical Approximation

Time	Distance
0	0
1	44.69289718
2	82.72860095
2.1	86.25773044
2.2	89.74454666
2.3	93.19070873
2.4	96.59781073
2.5	99.96738423 ≅ 100

Alternatively, we note that for any large value of t, the negative exponential term will be negligible. A good approximation is obtained by setting exponential is essentially zero, then

$$\begin{aligned} 24.5t &= 38.75 \\ t &= 1.58\ s \end{aligned} \tag{3.1.47}$$

Example 3.1.7. *(a) A small stone of mass m is thrown vertically upwards with initial speed V. If the air resistance at speed v is mkv^2, where $k > 0$ constant, show that the stone returns to its starting point with speed $V\left(1 - k\frac{V^2}{g}\right)^{\frac{1}{2}}$*
(b) If $kv^2 << g$ i.e. $\frac{kv^2}{g} << 1$ in the above case, find the velocity with which stone strikes the ground.

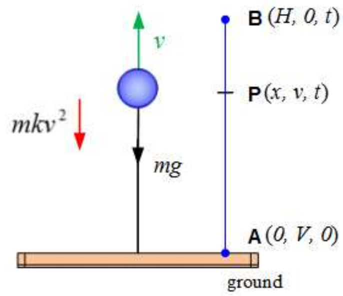


Figure 3.7: Upward motion

Solution Here we consider two way motion, firstly upward and next downward motion.
Upward Motion:
The upward motion of the small stone is illustrated in the Fig 3.7.

Let $OO^{'} = H$ be the maximum height of attained by it. Let at time t, its position be at P with distance $x \leq H$ from the ground. Let its upward velocity at P be $\dot{x}$, and the acceleration in the direction of increasing x is

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \\ &= \frac{d}{dx}\frac{1}{2}v^2 \end{aligned} \tag{3.1.48}$$

For the problem at hand, there are two forces acting on the body at P:
(1) the force due to gravity acting in downward direction given by the weight w of the body, which equals mg, and
(2) the force due to air resistance acting in downward direction given by mkv^2, where $k > 0$ is a constant of proportionality.
The minus sign is required because the forces oppose the velocity; that is, they act in the downward, or negative, direction (see Fig. 3.7). The net force F on the body is, therefore, the equation of motion is:

$$\begin{aligned} F &= -mg - mkv^2 \\ ma &= -m(g + kv^2) \end{aligned}$$

using (3.1.48), we have

$$\begin{aligned} \frac{d}{dx}v^2 + 2kv^2 &= -2g \\ \left(\frac{d}{dx} + 2k\right)v^2 &= -2g \end{aligned} \tag{3.1.49}$$

(3.1.49) is first order linear nonhomogeneous differential equation and can be solved by the method of undetermined coefficient. The homogeneous part of (3.1.49) is:

$$\left(\frac{d}{dx} + 2k\right) v^2 = 0 \tag{3.1.50}$$

The characteristic equation is

$$\begin{aligned} D + 2k &= 0 \\ D &= -2k \end{aligned}$$

and the complementary solution is:

$$v_c^2 = Ae^{-2kx} \tag{3.1.51}$$

For particular solution, let

$$v_p^2 = C \tag{3.1.52}$$

then

$$\frac{d}{dx} v_p^2 = 0 \tag{3.1.53}$$

using above result in (3.1.49), we have

$$\begin{aligned} 2kC &= -2g \\ C &= -\frac{g}{k} \end{aligned} \tag{3.1.54}$$

Hence the particular solution is:

$$v_p^2 = -\frac{g}{k} \tag{3.1.55}$$

Hence the general solution is:

$$v^2(x) = Ae^{-2kx} - \frac{g}{k} \tag{3.1.56}$$

Since the stone is thrown with initial speed V with distance $x = 0$, *i.e.*

$$v(0) = V \tag{3.1.57}$$

Using (3.1.57), (3.1.58) becomes:

$$V^2 = A - \frac{g}{k} \tag{3.1.58}$$

or

$$A = V^2 + \frac{g}{k} \tag{3.1.59}$$

Using (3.1.59), (3.1.58) becomes:

$$v^2(x) = \left(V^2 + \frac{g}{k}\right) e^{-2kx} - \frac{g}{k} \tag{3.1.60}$$

Next when the stone attains maximum height $x = H$, its speed becomes zero, *i.e.*

$$v(H) = 0 \tag{3.1.61}$$

Using (3.1.61), (3.1.62) becomes:

$$\begin{aligned} 0 &= \left(V^2 + \frac{g}{k}\right) e^{-2kH} - \frac{g}{k} \\ \frac{g}{k} &= \left(V^2 + \frac{g}{k}\right) e^{-2kH} \\ e^{2kH} &= \frac{k}{g}\left(V^2 + \frac{g}{k}\right) \\ &= \left(1 + \frac{k}{g}V^2\right) \\ 2kH &= log\left(1 + \frac{k}{g}V^2\right) \\ H &= \frac{1}{2k} log\left(1 + \frac{k}{g}V^2\right) \end{aligned} \tag{3.1.62}$$

is the maximum height attained by the stone.

Downward Motion:

The downward motion of the small stone is illustrated in the Fig 3.8.

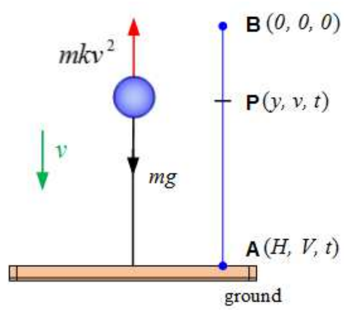


Figure 3.8: Downward motion

After attaining the maximum height H, the stone rests for a while and then returns back. Let at time t, its position be at P with distance $y \leq H$ from the top.

Let its downward velocity at P be $\dot{y}$, and the acceleration in the direction of increasing y is

$$a = \frac{d}{dy}\frac{1}{2}v^2 \tag{3.1.63}$$

For the problem at hand, two forces acting on the body at P:
(1) the force due to gravity acting in downward direction given by the weight w of the body, which equals mg, and
(2) the force due to air resistance acting in upward direction given by mkv^2, where $k > 0$ is a constant of proportionality.
The net force F on the body is $mg - mkv^2$

therefore, the equation of motion is:

$$\begin{aligned} F &= mg - mkv^2 \\ ma &= m(g - kv^2) \end{aligned}$$

using (3.1.63), we have

$$\begin{aligned} \frac{d}{dy}v^2 + 2kv^2 &= 2g \\ \left(\frac{d}{dx} + 2k\right)v^2 &= 2g \end{aligned} \tag{3.1.64}$$

(3.1.64) is first order linear nonhomogeneous differential equation and can be solved by the method of undetermined coefficient.

Hence the general solution in the similar way is:

$$v^2(y) = Be^{-2ky} + \frac{g}{k} \tag{3.1.65}$$

At top $y = 0$ its speed is zero , *i.e.*

$$v(0) = 0 \tag{3.1.66}$$

Using (3.1.66), (3.1.65) becomes:

$$B = -\frac{g}{k} \tag{3.1.67}$$

Using (3.1.67), (3.1.65) becomes:

$$v^2(y) = \frac{g}{k}\left(1 - e^{-2ky}\right) \tag{3.1.68}$$

Next when the stone strikes the ground, then $y = H$, and its speed is:

$$v^2(H) = \frac{g}{k}\left(1 - e^{-2kH}\right) \tag{3.1.69}$$

Using (3.1.62), we have

$$\begin{aligned} v^2(H) &= \frac{g}{k}\left(1 - e^{-log\left(1+\frac{k}{g}V^2\right)}\right) \\ &= \frac{g}{k}\left(1 - \left(1 + \frac{k}{g}V^2\right)^{-1}\right) \end{aligned} \tag{3.1.70}$$

Expand second term upto second order, we have

$$\begin{aligned} v^2 &= \frac{g}{k}\left(1-\left(1-\frac{k}{g}V^2+\left(\frac{k}{g}V^2\right)^2\right)\right) \\ &= \frac{g}{k}\left(\frac{k}{g}V^2\left(1-\frac{k}{g}V^2\right)\right) \end{aligned} \quad (3.1.71)$$

Hence the stone returns to its starting point with speed

$$v = V\left(1-\frac{k}{g}V^2\right)^{\frac{1}{2}} \quad (3.1.72)$$

(b) Expand (3.1.72) upto first order, we have

$$v = V\left(1-\frac{k}{2g}V^2\right) \quad (3.1.73)$$

Exercises

1. A stone of mass 1 kg is dropped from the top of a tower with zero velocity. Assuming no air resistance, it hits the ground after 3 *seconds*. Is the mass is helpful to increase the velocity? Also find
   (a) the height of the tower and
   (b) the velocity with witch the stone hits the ground.

2. A body of mass 5 kg is dropped from a height of 20 m with zero velocity. Assuming no air resistance, find
   (a) an expression for the velocity of the body at any time t.
   (b) an expression for the position of the body at any time t, and
   (c) the time required to reach the ground.

3. A small marble ball is thrown vertically upward with a velocity. Assuming no air resistance, it returns back and hit the ground after 8 *seconds*. Find
   (a) the maximum height reached by the ball and
   (b) the velocity with witch the ball is thrown up.

4. Assuming no air resistance, a particle is projected vertically upward with a velocity v_0 from origin. It passes through a point at a height h from origin at time t_1. After reaching its maximum height it returns back and passes through the same point at time t_2. Show that

$$t_1 + t_2 = \frac{2v_0}{g}$$
$$t_1 t_2 = \frac{2h}{g}$$

5. Two particles are projected simultaneously in the vertically upward direction with velocities $\sqrt{2gh}$ and $\sqrt{2gk}$, $(k > h)$, where h and k are the maximum heights attained by the particles. After a time t, when the two particles are still in flight, another particle is projected upward with a velocity u. Find the condition so that the third particle may meet the first two particles during their upward flights.

6. A particle is projected vertically upward. After a time t, another particle is sent up from the same point with the same velocity and meets the first at height h during the downward flight of the first. Find the velocity of projection.

7. The acceleration of a particle falling freely under the gravitational pull is equal to $\frac{k}{x^2}$, where x is the distance of the particle from the center of the earth and k is some constant. Find the velocity of the particle if it is let fall from an altitude R, on striking the surface of the earth if the radius of earth is r and the air offers no resistance to motion.

8. A ball of weight 10 N is dropped from a height of 50 m with zero velocity. If the body encounters an air resistance proportional to half its velocity, find
   (a) the equation of motion in the coordinate system
   (b) an expression for the velocity of the body at any time t,
   (c) the limiting velocity for the ball and
   (d) the time required to hit the ground.
9. A ball of mass $\frac{1}{2}$ kg is dropped from a height of 25 m with an initial velocity of 10 m/sec. Assume that the air resistance is proportional to the velocity of the body. If the limiting velocity is known to be 100 m/sec, find
   (a) the equation of motion in the coordinate system
   (b) an expression for the velocity of the body at any time t,
   (c) an expression for the position of the body at any time t, and
10. A small stone of mass m is thrown vertically upwards with initial speed V. If the air resistance at speed v is mkv, where $k > 0$ constant, show that the stone returns to its starting point with speed U given by the relation

$$g - kU = (g + kV)\, e^{-\frac{k}{g}(V+U)}$$

Chapter 4

Motion in Two and Three Dimensional Cartesian Coordinate Systems

In this chapter we will discuss the motion of a particle in 2 and 3 dimensional cartesian coordinate system.

4.1 Two and Three Dimensional Cartesian Coordinate System

The concepts of position vector, displacement, velocity and acceleration will be discussed in these systems.

4.1.1 Position Vector

The general expression of position vector of a point P with reference to some point is discussed in chapter 2. In 2 dimensional space, if origin O is the reference point, the position vector $\vec{r}$ of a point $P(x, y)$ is

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$$

And the position vector $\vec{r}$ of a point $P(x, y, z)$ in 3 dimensional space is

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$$

4.1.2 Displacement

(a) Displacement in 2 *space*

Consider a particle moves along a curve C in 2 *space* as shown in Fig. 4.1 . Let at time t_1 it is at point $A = (x_1, y_1)$ whose position vector relative to O is $\vec{r}_1 = x_1\hat{i} + y_1\hat{j}$. At a latter time t_2 it is at point $B = (x_2, y_2)$ whose position vector relative to O is $\vec{r}_2 = x_2\hat{i} + y_2\hat{j}$. Then displacement describes the change in position of the particle and is given as

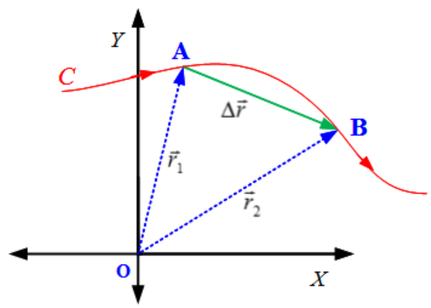


Figure 4.1: Displacement vector in 2 *space*

$$\begin{aligned}\Delta\vec{r} &= \vec{r}_2 - \vec{r}_1\\ &= \left(x_2\hat{i} + y_2\hat{j}\right) - \left(x_1\hat{i} + y_1\hat{j}\right)\\ &= (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j}\\ &= \Delta x\hat{i} + \Delta y\hat{j}\end{aligned}$$

Where $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ are displacements along x *axis* and y *axis* respectively.

(b) Displacement in 3 *space*

Consider a particle moves along a curve C in 3 *space* as shown in Fig. 4.2. Let at time t_1 it is at point $A = (x_1, y_1, z_1)$ whose position vector relative to O is $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$. At a latter time t_2 it is at point $B = (x_2, y_2, z_2)$ whose position vector relative to O is $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$. Then the displacement describes the change in position of the particle and is given as

$$\begin{aligned}\Delta\vec{r} &= \vec{r}_2 - \vec{r}_1\\ &= \left(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}\right) - \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right)\\ &= (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}\\ &= \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}\end{aligned}$$

Where $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$ and $\Delta z = z_2 - z_1$ are displacements along x *axis*, and y *axis* and z *axis* respectively.

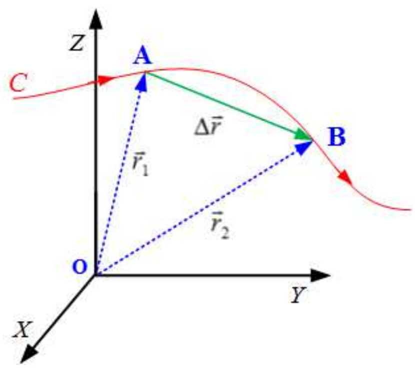


Figure 4.2: Displacement vector in 3 *space*

4.1.3 Average Velocity and Instantaneous Velocity

Continuing above discussion, the particle moves from A to B in time internal

$$\Delta t = t_2 - t_1$$

Then average velocity is

$$\begin{aligned} \text{average velocity} &= \frac{\text{displacement}}{\text{time internal}} \\ \vec{v}_{avg} &= \frac{\Delta \vec{r}}{\Delta t} \end{aligned}$$

And instantaneous velocity is defined as

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$$

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \vec{r}^{\,\prime}(t) \tag{4.1.1}$$

(a) Average Velocity in 2 *space* is

$$\begin{aligned} \vec{v}_{avg} &= \frac{\Delta x \hat{i} + \Delta y \hat{j}}{\Delta t} \\ &= \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} \end{aligned}$$

(b) Similarly Average Velocity in 3 *space* is

$$\vec{v}_{avg} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} + \frac{\Delta z}{\Delta t}\hat{k}$$

(c) Instantaneous Velocity in 2 *space* is

$$\begin{aligned}\vec{v} &= \frac{d\vec{r}}{dt} \\ &= \frac{d}{dt}\left(x\hat{i} + y\hat{j}\right) \\ &= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} \\ &= v_x\hat{i} + v_y\hat{j}\end{aligned}$$

Where $v_x = \frac{dx}{dt}$ is the horizontal scalar component of velocity and $v_y = \frac{dy}{dt}$ is the vertical scalar component of velocity as shown in Fig. 4.3.

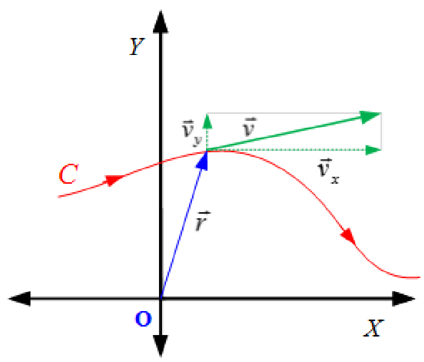


Figure 4.3: Velocity and its components

(d) Similarly Instantaneous Velocity in 3 *space* is

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

4.1.4 Average Acceleration and Instantaneous Acceleration

Continuing above discussion, if the particle at time t_1 is at A having velocity $\vec{v}_1$ and at later time t_2 is at B having velocity $\vec{v}_2$. The change in velocity during the time interval Δt is

$$\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$$

Then average acceleration is

$$\begin{aligned} \text{average acceleration} &= \frac{\text{change in velocity}}{\text{time internal}} \\ \vec{a}_{avg} &= \frac{\Delta \vec{v}}{\Delta t} \end{aligned}$$

And if a particle during motion has velocity function $\vec{v}(t)$, then its instantaneous acceleration at time (acceleration function) is defined as

$$\begin{aligned} \vec{a}(t) &= \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} \\ &= \frac{d}{dt}\vec{v}(t) \end{aligned} \tag{4.1.2}$$

Using (4.1.1), the acceleration function in terms of the position function is

$$\begin{aligned} \vec{a}(t) &= \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2}{dt^2}\vec{x}(t) \\ &= \vec{x}''(t) \end{aligned} \tag{4.1.3}$$

(a) Average Acceleration in 2 *space* is

$$\begin{aligned} \vec{a}_{avg} &= \frac{\Delta v_x \hat{i} + \Delta v_y \hat{j}}{\Delta t} \\ &= \frac{\Delta v_x}{\Delta t}\hat{i} + \frac{\Delta v_y}{\Delta t}\hat{j} \end{aligned}$$

(b) Similarly Average Acceleration in 3 *space* is

$$\vec{a}_{avg} = \frac{\Delta v_x}{\Delta t}\hat{i} + \frac{\Delta v_y}{\Delta t}\hat{j} + \frac{\Delta v_z}{\Delta t}\hat{k}$$

(c) Instantaneous Acceleration in 2 *space* is

$$\begin{aligned} \vec{a} &= \frac{d\vec{v}}{dt} \\ &= \frac{d}{dt}\left(v_x\hat{i} + v_y\hat{j}\right) \\ &= \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} \\ &= a_x\hat{i} + a_y\hat{j} \end{aligned}$$

Where $a_x = \frac{dv_x}{dt}$ is the horizontal scalar component of acceleration and $a_y = \frac{dv_y}{dt}$ is the vertical scalar component of acceleration as shown in Fig. 4.4. the acceleration function in

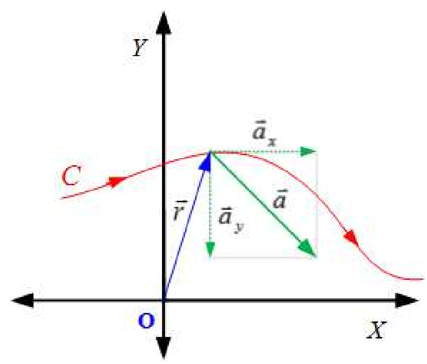


Figure 4.4: Acceleration and its components

terms of the position function is

$$\begin{aligned}\vec{a} &= \frac{d^2\vec{x}}{dt^2} \\ &= \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j}\end{aligned}$$

(d) Similarly Instantaneous Acceleration in 3 *space* is

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

Example 4.1.1. *A car is moving along a path C as shown in Fig. 4.5 Let at time $t_1 = 2s$ the car is at point A having position vector $\vec{r}_1 = (1m)\hat{i} + (4m)\hat{j}$. After a later time $t_2 = 5s$ the car is at point B having position vector $\vec{r}_2 = (6m)\hat{i} + (6m)\hat{j}$. Find its displacement and average velocity.*

Solution The given data is

$$\begin{aligned}t_1 &= 2s \\ t_2 &= 5s \\ \vec{r}_1 &= (1m)\hat{i} + (4m)\hat{j} \\ \vec{r}_2 &= (6m)\hat{i} + (6m)\hat{j}\end{aligned}$$

Time internal to move from A to B is

$$\begin{aligned}\Delta t &= t_2 - t_1 \\ &= 5 - 2 = 3s\end{aligned}$$

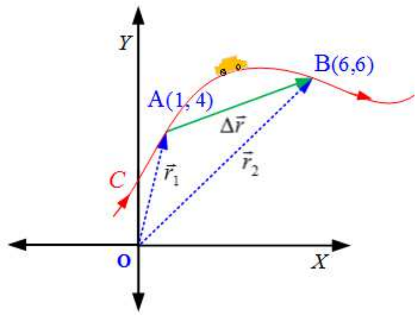


Figure 4.5: Two dimensional motion

The horizontal scalar component of displacement is

$$\begin{aligned}\Delta x &= x_2 - x_1 \\ &= 6 - 1 = 5m\end{aligned}$$

And the vertical scalar component of displacement is

$$\begin{aligned}\Delta y &= y_2 - y_1 \\ &= 6 - 4 = 2m\end{aligned}$$

The displacement is

$$\begin{aligned}\Delta \vec{r} &= \Delta x\hat{i} + \Delta y\hat{j} \\ &= 5\hat{i} + 2\hat{j}\end{aligned}$$

And the average velocity is

$$\begin{aligned}\vec{v}_{avg} &= \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} \\ &= \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j}\end{aligned}$$

Example 4.1.2. *In above example, if car has velocity $\vec{v}_1 = (1m/s)\hat{i} + (2m/s)\hat{j}$.at point A and $\vec{v}_2 = (4m/s)\hat{i} + (8m/s)\hat{j}$ at point B. Find its average acceleration.*

Solution The given data is

$$\begin{aligned}t_1 &= 2s \\ t_2 &= 5s \\ \vec{v}_1 &= \hat{i} + 2\hat{j} \\ \vec{v}_2 &= 4\hat{i} + 8\hat{j}\end{aligned}$$

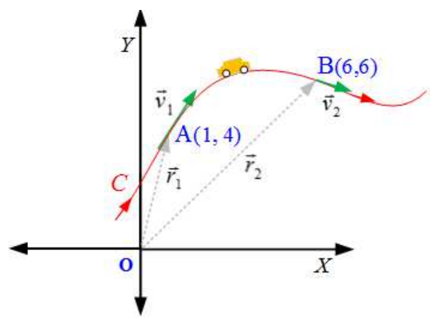


Figure 4.6: Two dimensional motion

Time internal to move from A to B is

$$\begin{aligned} \Delta t &= t_2 - t_1 \\ &= 5 - 2 = 3s \end{aligned}$$

The horizontal scalar component of velocity is

$$\begin{aligned} \Delta v_x &= v_{x_2} - v_{x_1} \\ &= 4 - 1 = 3m/s \end{aligned}$$

And the vertical scalar component of velocity is

$$\begin{aligned} \Delta v_y &= v_{y_2} - v_{y_1} \\ &= 8 - 2 = 6m/s \end{aligned}$$

And the average acceleration is

$$\begin{aligned} \vec{a}_{avg} &= \frac{\Delta v_x}{\Delta t}\hat{\imath} + \frac{\Delta v_y}{\Delta t}\hat{\jmath} \\ &= \frac{3}{3}\hat{\imath} + \frac{6}{3}\hat{\jmath} \\ &= (m/s^2)\hat{\imath} + (2m/s^2)\hat{\jmath} \end{aligned}$$

Example 4.1.3. *A bus is moving along a path C as shown in Fig. 4.7 Let at time $t_1 = 2s$ the bus is at point A having position vector $\vec{r}_1 = (-2m)\hat{\imath} + (4m)\hat{\jmath} + (5m)\hat{k}$. After a later time $t_2 = 5s$ the bus is at point B having position vector $\vec{r}_2 = (3m)\hat{\imath} + (2m)\hat{\jmath} + (5m)\hat{k}$. Find its displacement and average velocity.*

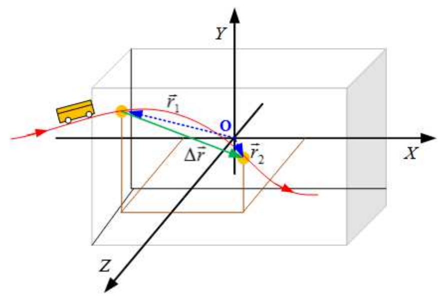


Figure 4.7: Three dimensional motion

Solution The given data is

$$\begin{aligned} t_1 &= 2s \\ t_2 &= 5s \\ \vec{r}_1 &= (-2m)\hat{i} + (4m)\hat{j} + (5m)\hat{k} \\ \vec{r}_2 &= (3m)\hat{i} + (2m)\hat{j} + (5m)\hat{k} \end{aligned}$$

Time internal to move from A to B is

$$\begin{aligned} \Delta t &= t_2 - t_1 \\ &= 5 - 2 = 3s \end{aligned}$$

The x component of displacement is

$$\begin{aligned} \Delta x &= x_2 - x_1 \\ &= 3 - (-2) = 5m \end{aligned}$$

the y component of displacement is

$$\begin{aligned} \Delta y &= y_2 - y_1 \\ &= 2 - 4 = -2m \end{aligned}$$

and the z component of displacement is

$$\begin{aligned} \Delta z &= z_2 - z_1 \\ &= 5 - 5 = 0m \end{aligned}$$

The displacement is

$$\begin{aligned} \Delta\vec{r} &= \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k} \\ &= 5\hat{i} - 2\hat{j} \end{aligned}$$

And the average velocity is

$$\begin{aligned}\vec{v}_{avg} &= \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} + \frac{\Delta z}{\Delta t}\hat{k} \\ &= \frac{5}{3}\hat{i} - \frac{2}{3}\hat{j}\end{aligned}$$

Example 4.1.4. *A particle moves along a path. Let $\vec{r} = \hat{i} + 4t^2\hat{j}$ be its position vector.*

(a) Write an expression for its velocity as functions of time

(b) Write an expression for its acceleration as functions of time

(c) Write expressions for position, velocity and acceleration at time $t = 2s$

Solution

(a) An expressions for its velocity as functions of time is

$$\begin{aligned}\vec{v} &= \frac{d\vec{r}}{dt} \\ &= \frac{d}{dt}\left(\hat{i} + 4t^2\hat{j}\right) \\ &= \frac{d}{dt}(1)\hat{i} + \frac{d}{dt}\left(4t^2\right)\hat{j} \\ &= 0\hat{i} + 8t\hat{j} \\ &= (8t\ m/s)\hat{j}\end{aligned}$$

(b) An expressions for its acceleration as functions of time is

$$\begin{aligned}\vec{a} &= \frac{d\vec{r}}{dt} \\ &= \frac{d}{dt}\left(8t\hat{j}\right) \\ &= (8\ m/s^2)\hat{j}\end{aligned}$$

(c) At time $t = 2s$ position vector is

$$\begin{aligned}\vec{r}(2) &= \hat{i} + 4(2)^2\hat{j} \\ &= (1m)\hat{i} + (16m)\hat{j}\end{aligned}$$

Velocity of a particle at $t = 2s$ is

$$\begin{aligned}\vec{v}(2) &= 8(2)\hat{j} \\ &= (16m/s)\hat{j}\end{aligned}$$

At time $t = 2s$ acceleration is

$$\begin{aligned} \vec{a}(2) &= 8\hat{j} \\ &= (8m/s^2)\hat{j} \end{aligned}$$

Example 4.1.5. *A particle moves along a path. Let the components of its position vector are*

$$\begin{aligned} x &= t \\ y &= 4t^2 - 3 \\ z &= 1 \end{aligned}$$

(a) Write an expression for its velocity as functions of time

(b) Write an expression for its acceleration as functions of time

Solution The position vector can be written as

$$\vec{r} = t\hat{i} + \left(4t^2 - 3\right)\hat{j} + \hat{k}$$

(a) An expressions for its velocity as functions of time is

$$\begin{aligned} \vec{v} &= \frac{d\vec{r}}{dt} \\ &= \frac{d}{dt}\left(t\hat{i} + \left(4t^2 - 3\right)\hat{j} + \hat{k}\right) \\ &= \frac{d}{dt}(t)\hat{i} + \frac{d}{dt}\left(4t^2 - 3\right)\hat{j} + \frac{d}{dt}(1)\hat{k} \\ &= \hat{i} + 8t\hat{j} + 0\hat{k} \\ &= (1m)\hat{i} + (8t\ m/s)\hat{j} \end{aligned}$$

(b) An expressions for its acceleration as functions of time is

$$\begin{aligned} \vec{a} &= \frac{d\vec{r}}{dt} \\ &= \frac{d}{dt}\left(\hat{i} + 8t\hat{j}\right) \\ &= 0\hat{i} + 8\hat{j} \\ &= (8\ m/s^2)\hat{j} \end{aligned}$$

Exercises

1. A particle is moving along a path. Let at time $t_1 = 2s$ the car is at point A having position vector $\vec{r}_1 = (2m)\hat{\imath} + (2m)\hat{\jmath}$. After a later time $t_2 = 4s$ the car is at point B having position vector $\vec{r}_2 = (3m)\hat{\imath} + (5m)\hat{\jmath}$. Find its displacement and average velocity.

2. A car starts to move from point A having position vector $\vec{r}_1 = (3m)\hat{\imath} + (2m)\hat{\jmath} + (4m)\hat{k}$. After a later time $t = 4s$ the car is at point B having position vector $\vec{r}_2 = (3m)\hat{\imath} + (5m)\hat{\jmath}$. Find its displacement and average velocity.

3. A particle is moving along a path. Let at time $t_1 = 2s$ the car is at point A having velocity $\vec{v}_1 = (3m)\hat{\imath} + (2m)\hat{\jmath}$. After a later time $t_2 = 4s$ the car is at point B having velocity $\vec{v}_2 = (3m)\hat{\imath} + (5m)\hat{\jmath}$. Find its average acceleration.

4. A car starts to move from point A having velocity $\vec{r}_1 = (3m)\hat{\imath} + (2m)\hat{\jmath} + (4m)\hat{k}$. After a later time $t = 4s$ the car is at point B having velocity $\vec{v}_2 = (1m)\hat{\imath} + (3m)\hat{\jmath} + (3m)\hat{k}$. Find its average acceleration.

5. A particle moves along a path. Let $\vec{r} = 2t^3\hat{\imath} - 21t^2\hat{\jmath}$ be its position vector.
   (a) Write an expression for its velocity as functions of time
   (b) Write an expression for its acceleration as functions of time
   (c) Write expressions for position, velocity and acceleration at time $t = 1s$

6. A particle moves along a path. Let $\vec{r} = \left(2t^2 - 4\right)\hat{\imath} + 4t^2\hat{\jmath} + e^t\hat{k}$ be its position vector.
   (a) Write an expression for its velocity as functions of time
   (b) Write an expression for its acceleration as functions of time
   (c) Write expressions for position, velocity and acceleration at time $t = 3s$

7. Analyze the motion of the particle for $t \geq 0$ for the following position vectors.
   (a) $\vec{r}(t) = t^3\hat{\imath} + \left(4t^2 + 5\right)\hat{\jmath} + (t+3)\hat{k}$
   (b) $\vec{r}(t) = (-4t + 3)\hat{\imath}$
   (c) $\vec{r}(t) = 5t^2\hat{\imath} - 20t\hat{k}$
   (d) $\vec{r}(t) = t^3\hat{\imath} - 9t^2\hat{\jmath} + 24t\hat{k}$
   (e) $\vec{r}(t) = (9t + 1)\hat{\imath} - 6t^2\hat{\jmath} + t^3\hat{k}$

Chapter 5

Angular Motion

In this chapter we will discuss the motion of a particle with rotational effects. First of all consider some basic concepts of rotational kinematics.

5.1 Angular Kinematics

In this section we will discuss definitions of angular displacement, velocity and acceleration moving along a rotating path.

5.1.1 Angular Displacement

In rotation, the displacement of the body is measured in angular displacement θ (measured in radians). Consider a circle in cartesian plane, with center at origan. Let a particle is initially at A, then moves from A to B along a circular path, the displacement is θ (see Fig 5.1). This displacement is a vector quantity having magnitude and direction (clockwise or anticlockwise).

If the body moves the displacement θ along a circular path, the distance s (*arc length*) is

$$s = r\theta$$

and if the body completes one rotation, the distance s is

$$s = 2\pi r$$

and the angular displacement θ is given as

$$\theta = \frac{s}{r}$$

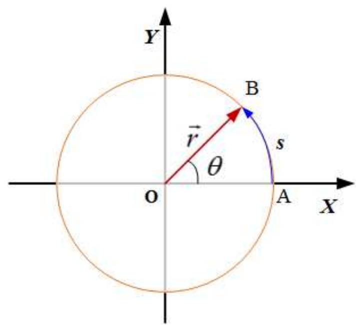


Figure 5.1: Angular displacement

5.1.2 Angular Velocity

Angular velocity is defined as the rate of change of angular coordinate with respect to t (time) and is denoted by $\vec{\omega}$.
If a body is at point A and after a short interval of time Δt it reaches at point B (see Fig. 5.2, then the change in its angular displacement is $\Delta\theta$ and average angular velocity is

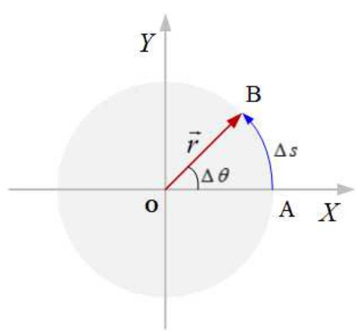


Figure 5.2: Angular velocity

$$\vec{\omega}_{avg} = \frac{\Delta\theta}{\Delta t}\hat{a}$$

where $\hat{a}$ a unit vector in the direction of angular velocity. The instantaneous angular velocity is

$$\begin{aligned}\vec{\omega} &= \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}\hat{a} \\ &= \frac{d\theta}{dt}\hat{a} \\ &= \dot{\theta}\hat{a} \qquad (5.1.1)\end{aligned}$$

And angular speed is

$$\omega = \dot{\theta} \qquad (5.1.2)$$

The directions are measured by right hand rule. The angular displacement (in magnitude) is

$$\theta = \omega t \qquad (5.1.3)$$

5.1.3 Angular Acceleration

Angular acceleration is defined as the rate of change of angular velocity. It is denoted by $\vec{\alpha}$. Average angular acceleration is

$$\vec{\alpha}_{avg} = \frac{\Delta\omega}{\Delta t}\hat{b}$$

where $\hat{b}$ a unit vector in the direction of angular acceleration. The instantaneous angular acceleration is

$$\begin{aligned}\vec{\alpha} &= \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}\hat{b} \\ &= \frac{d\omega}{dt}\hat{b} \\ &= \frac{d^2\theta}{dt^2}\hat{b}\end{aligned}$$

or

$$\vec{\alpha} = \frac{d}{dt}\vec{\omega} = \frac{d^2\vec{\theta}}{dt^2} \qquad (5.1.4)$$

The magnitude of angular acceleration is

$$\alpha = \frac{d\omega}{dt} \qquad (5.1.5)$$

5.2 Motion in Polar Coordinates or Circular Motion

A particle executes circular motion if it travels around a circle or a circular arc. The velocity is always directed tangent to the circle in the direction of the motion and the acceleration is always directed radially inward (see Fig. 5.3).

5.2.1 Uniform Circular Motion

A particle is in uniform circular motion if it travels around a circle or a circular arc at constant (uniform) speed (although speed does not vary, the particle is accelerating because the velocity changes in direction). In this case

$$\frac{dv}{dt} = 0$$

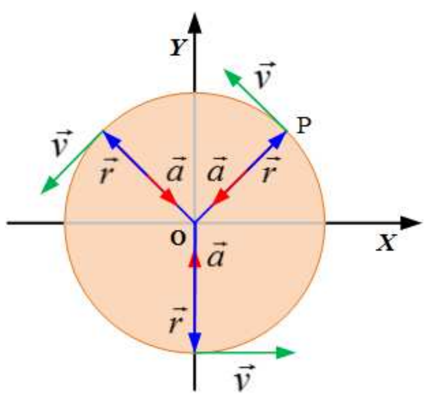


Figure 5.3: Velocity and acceleration for uniform circular motion.

5.2.2 Position Vector

Consider a Cartesian plane coordinate system. Let a particle be at $P(x,\ y)$ whose position vector relative to O is

$$\begin{aligned} \vec{r} &= \vec{OP} \\ &= x\hat{i} + y\hat{j} \end{aligned}$$

Its magnitude is

$$r = |\vec{OP}|$$

Let the position vector $\vec{r}$ makes an angle θ with x $axis$, then completing right angle triangle OAP, we have

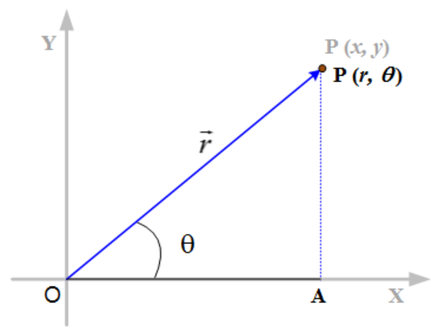


Figure 5.4: Polar Coordinates

$$x = r\cos\theta \quad (5.2.1)$$
$$y = r\sin\theta \quad (5.2.2)$$

If P is a point in polar plane coordinates system then its coordinates are $P(r,\theta)$. The point O is known as pole, x $axis$ as initial line and y $axis$ as terminal line. The number r is called the radial coordinate of P and the number θ the angular coordinate (or polar angle) of P. The number r is given as

$$\begin{aligned} r &= |\vec{OP}| \\ &= \sqrt{x^2 + y^2} \end{aligned} \quad (5.2.3)$$

The number θ is given as

$$\begin{aligned} \theta &= \angle AOP \\ &= \arctan\left(\frac{y}{x}\right) \end{aligned} \quad (5.2.4)$$

Using (5.2.1) and (5.2.2), the position vector of a particle executing circular motion is

$$\begin{aligned} \vec{r} &= r\cos\theta\hat{i} + r\sin\theta\hat{j} \\ &= r\hat{r} \end{aligned} \quad (5.2.5)$$

where

$$\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$$

is a unit vector in the direction of position vector. Its detail will be in next section.

5.2.3 Velocity

Since the velocity is the time rate of change of position vector

$$\begin{aligned}\vec{v} = \frac{d\vec{r}}{dt} &= \frac{d}{dt}\left(r\cos\theta\hat{i} + r\sin\theta\hat{j}\right)\\ &= r\left(-\dot{\theta}\sin\theta\hat{i} + \dot{\theta}\cos\theta\hat{j}\right)\\ &= r\dot{\theta}\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right)\\ &= r\omega\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right) \quad (5.2.6)\\ &= r\omega\hat{\theta} \quad (5.2.7)\end{aligned}$$

$$\hat{\theta} = \left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right)$$

is a unit vector perpendicular to position vector. Its detail will be in next section.
The magnitude of velocity is

$$v = r\omega \quad (5.2.8)$$

(5.2.8) is called the relation between linear and angular speed.
Another form of velocity can be calculated by considering (5.2.1) and (5.2.2)

$$\begin{aligned}\cos\theta &= \frac{x}{r}\\ \sin\theta &= \frac{y}{r}\end{aligned} \quad (5.2.9)$$

Since the velocity $\vec{v}$ of a moving particle is always tangent to the circular path at particle's position, then $\vec{v}$ makes an angle θ with the vertical, that $\vec{r}$ makes with x *axis*. Then its angle with horizontal (x *axis*) is $\left(\theta + \frac{\pi}{2}\right)$. The $\vec{v}$ of the particle at P is

$$\begin{aligned}\vec{v} = \frac{d\vec{r}}{dt} &= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}\\ &= v_x\hat{i} + v_y\hat{j}\end{aligned}$$

Since $\vec{v}$ makes an angle $\left(\theta + \frac{\pi}{2}\right)$ with x *axis* (see Fig. 5.7), so it can be written in its rectangular components as

$$\begin{aligned}\vec{v} &= v\cos\left(\theta + \frac{\pi}{2}\right)\hat{i} + v\sin\left(\theta + \frac{\pi}{2}\right)\hat{j}\\ &= -v\sin\theta\hat{i} + v\cos\theta\hat{j}\\ &= v\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right)\\ &= v\hat{\theta} = r\omega\hat{\theta} \quad (5.2.10)\end{aligned}$$

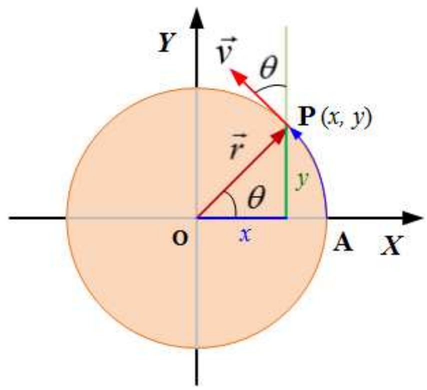


Figure 5.5: Direction of velocity.

Then the velocity components are

$$\begin{aligned} v_x = \frac{dx}{dt} &= -v\sin\theta \\ v_y = \frac{dy}{dt} &= v\cos\theta \end{aligned}$$

Using (5.2.9), (5.2.10) can be written as

$$\begin{aligned} \vec{v} &= -v\frac{y}{r}\hat{i} + v\frac{x}{r}\hat{j} \\ &= -\frac{v}{r}y\hat{i} + \frac{v}{r}x\hat{j} \end{aligned} \tag{5.2.11}$$

5.2.4 Acceleration

The acceleration of a particle at P is the time derivative of its velocity

$$\begin{aligned} \vec{a} = \frac{d\vec{v}}{dt} &= \frac{d}{dt}\left(r\omega\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right)\right) \\ &= r\left[\omega\frac{d}{dt}\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right) + \dot{\omega}\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right)\right] \\ &= r\left[\omega\left(-\dot{\theta}\cos\theta\hat{i} - \dot{\theta}\sin\theta\hat{j}\right) + \dot{\omega}\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right)\right] \\ &= r\left[-\omega^2\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right) + \dot{\omega}\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right)\right] \\ &= -r\omega^2\hat{r} + \frac{dr\omega}{dt}\hat{\theta} \\ &= -r\omega^2\hat{r} + \frac{dv}{dt}\hat{\theta} \end{aligned}$$

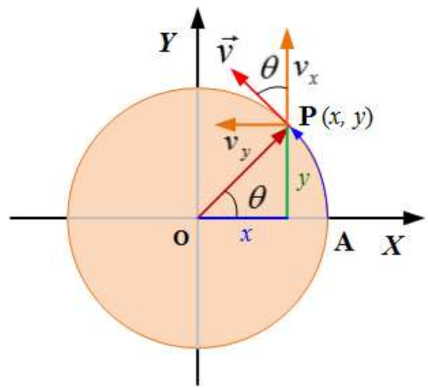


Figure 5.6: Rectangular components of velocity.

Acceleration in circular motion has both the radial (a_r) and the tangential(a_t) components. When the particle is moving with uniform speed then above relation becomes

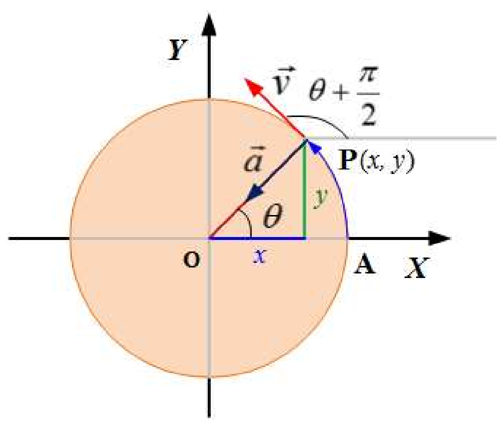


Figure 5.7: Direction of velocity and acceleration.

$$\vec{a} = -r\omega^2\hat{r}$$

Then the magnitude of this acceleration is

$$\begin{aligned} a &= \frac{v^2}{r} = \frac{(r\omega)^2}{r} \\ &= r\omega^2 \end{aligned} \tag{5.2.12}$$

Another form of acceleration can be calculated by considering (5.2.11)

$$\vec{a} = \frac{d\vec{v}}{dt} = -\frac{v}{r}\frac{dy}{dt}\hat{i} + \frac{v}{r}\frac{dx}{dt}\hat{j} \tag{5.2.13}$$

Using (5.2.11), (5.2.13) can be written as

$$\begin{aligned} \vec{a} &= -\frac{v}{r}v\cos\theta\hat{i} + \frac{v}{r}\left(-v\sin\theta\right)\hat{j} \\ &= -\frac{v^2}{r}\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right) \end{aligned} \tag{5.2.14}$$

The magnitude of this acceleration is

$$a = \frac{v^2}{r} \tag{5.2.15}$$

5.2.5 Centripetal Force

The force in action during the motion of bodies whirling at the ends of a string, spinning on the shafts, artificial and natural satellite and nuclear particles in accelerators etc. It is defined as *The force needed to bend the normally straight path of the particle into a circular path*.
By Newton's second law of motion, (in magnitude)

$$F = ma$$

For circular motion, the acceleration is given by (5.2.15), then the centripetal force is

$$F_c = m\frac{v^2}{r} \tag{5.2.16}$$

It acts in the direction of centripetal acceleration.

5.2.6 Relation Between Linear and Angular Acceleration (in magnitude)

The linear acceleration of a particle is:

$$a = \frac{dv}{dt}$$

Using (5.2.8), we can write:

$$a = \frac{rd\omega}{dt}$$

From (5.1.5), we can write:

$$a = r\alpha \tag{5.2.17}$$

(5.2.17) gives the relation between linear and angular acceleration.

Example 5.2.1. *A particle of mass* 2 *kg moves in a circle of radius* 0.5 *m with a linear speed of* 15 *m/s. Find the angular speed and the force required to keep it in circular path.*

Solution: The given data is

$$
\begin{aligned}
m &= 2\ kg \\
r &= 0.5\ m \\
v &= 15\ m/s
\end{aligned}
$$

Using the relation (5.2.8)

$$
\begin{aligned}
\omega &= \frac{v}{r} \\
&= \frac{15}{0.5} \\
&= 30\ rad/s
\end{aligned}
$$

The force required to keep it in circular path is the centripetal force and is given by (5.2.16)

$$
\begin{aligned}
F_c &= m\frac{v^2}{r} \\
&= 2\frac{(15)^2}{0.5} \\
&= 900\ N
\end{aligned}
$$

Example 5.2.2. *A particle moves in a circle of radius* 1 *m. If its speed uniformly increases from* 5 *m/s to* 10 *m/s in time interval* 2 *s, find the angular acceleration.*

Solution : The given data is

$$
\begin{aligned}
r &= 1\ m \\
v_i &= 5\ m/s \\
v_f &= 10\ m/s \\
dt &= 2\ s
\end{aligned}
$$

The particle is executing uniform circular motion The tangential acceleration is average acceleration and given by

$$
\begin{aligned}
a &= \frac{dv}{dt} \\
&= \frac{v_f - v_i}{dt} = \frac{10-5}{2} \\
&= 2.5\ m/s^2
\end{aligned}
$$

Next using (5.2.17), angular acceleration is

$$\begin{aligned} \alpha &= \frac{a}{r} \\ &= \frac{2.5}{1} \\ &= 2.5\ rad/s^2 \end{aligned}$$

5.3 Motion in Radial and Transverse Plane

In some problems a particle P may not moving along a circular path but may be located in a better way using its polar coordinates. In such cases it becomes convenient to resolve the velocity and acceleration into components parallel and perpendicular to the tip of the vector $\vec{OP}$. These components are called the radial and transverse components. In order to do this, attach 2 unit vectors to the tip of P. The vector $\hat{r}$ is directed along OP and the vector $\hat{\theta}$ is obtained by rotating $\hat{r}$ about $\frac{\pi}{2}$ radians counterclockwise. The vector $\hat{r}$ defines the radial direction, that is the direction in which P would move if r were increased and θ kept constant. And the vector $\hat{\theta}$ defines the radial direction, that is the direction P would move if θ were increased and r kept constant. Its graphical representation is given in Fig. 5.8 Notice that in this coordinate system, unlike the xy coordinate system, the unit vectors

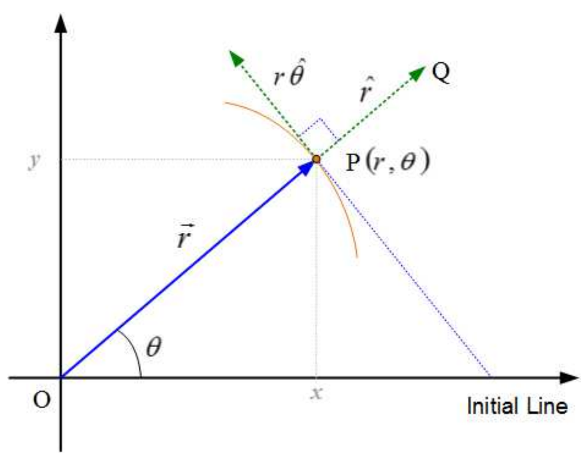


Figure 5.8: Position vector in radial and transverse plane.

constantly change direction, thus they have time derivatives. These two unit vectors are given by

$$\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j} \tag{5.3.1}$$

$$\begin{aligned}\hat{\theta} &= \left(\cos\left(\theta + \frac{\pi}{2}\right), \sin\left(\theta + \frac{\pi}{2}\right)\right) \\ &= -\sin\theta\hat{i} + \cos\theta\hat{j}\end{aligned} \tag{5.3.2}$$

5.3.1 Position Vector

If a particle move along a curve $r = r(\theta)$, its position at any time t relative to O is $P(r, \theta)$ and its position vector is

$$\vec{r} = r\hat{r} \tag{5.3.3}$$

The above relation is already given by (5.2.5). The position vector in this system is same as in polar plane system. Also it has only radial component as shown in Fig. 5.9.

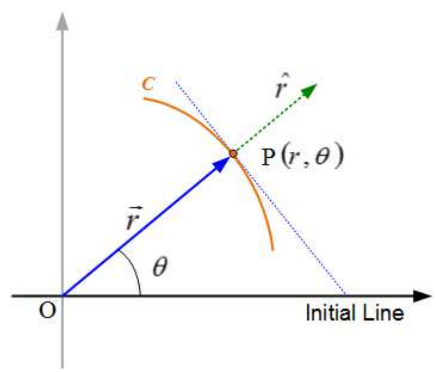


Figure 5.9: Position vector in radial and transverse plane.

5.3.2 Radial and Transverse Components of Velocity

The velocity of P at any time t is

$$\begin{aligned}\vec{v} &= \frac{d\vec{r}}{dt} \\ &= \frac{d(r\hat{r})}{dt} \\ &= \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}\end{aligned} \tag{5.3.4}$$

Let $\vec{OQ} = \hat{r}$ is fixed but its direction varies with θ (radial direction). Consider

$$\begin{aligned}\frac{d\hat{r}}{dt} &= \frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} \\ &= (-\sin\theta, \cos\theta)\dot{\theta} \\ &= \dot{\theta}(-\sin\theta, \cos\theta)\end{aligned}$$

Using (5.3.2)

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta} \qquad (5.3.5)$$

Using (5.3.5), (5.3.4) becomes

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \qquad (5.3.6)$$

is the velocity of the particle at any time t. The radial component of velocity is

$$v_r = \dot{r}$$

and the transverse component of velocity is

$$v_t = r\dot{\theta}$$

Radial and transverse components of velocity are illustrated in Fig. 5.10

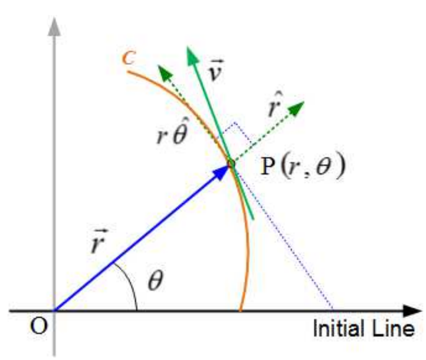


Figure 5.10: Radial and transverse components of velocity

5.3.3 Radial and Transverse Components of Acceleration

The acceleration of the particle of P at any time t is

$$\begin{aligned}
\vec{a} &= \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) \\
&= \dot{r}\frac{d\hat{r}}{dt} + \ddot{r}\hat{r} + r\dot{\theta}\frac{d\hat{\theta}}{dt} + r\ddot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} \\
&= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + r\dot{\theta}(-\dot{\theta}\hat{r}) + r\ddot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta} \qquad (5.3.7)
\end{aligned}$$

The radial component of acceleration is

$$a_r = \ddot{r} - r\dot{\theta}^2$$

and the transverse component of acceleration is

$$a_t = 2\dot{r}\dot{\theta} + r\ddot{\theta}$$

Radial and transverse components of acceleration are illustrated in Fig. 5.11

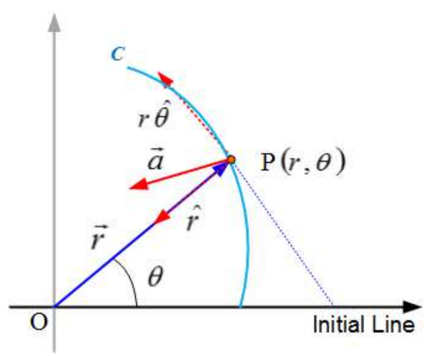


Figure 5.11: Radial and transverse components of acceleration

Example 5.3.1. *A particle is constrained to move along the equiangular spiral $r = ae^{b\theta}$, so that the radius vector moves with constant angular velocity ω. Determine the velocity and acceleration components.*

Solution : From (5.1.3), angular distance at any time t is

$$\begin{aligned}
\theta(t) &= \omega t \\
\dot{\theta} &= \omega \\
\ddot{\theta} &= 0
\end{aligned}$$

The path of the particle is

$$r = ae^{b\theta}$$

Using (5.1.3), the path of the particle is

$$\begin{aligned} r(t) &= ae^{b\omega t} \\ \dot{r} &= \omega abe^{b\omega t} \\ \ddot{r} &= \omega^2 ab^2 e^{b\omega t} \end{aligned}$$

Using (5.3.6), the velocity of the particle at any time t is

$$\begin{aligned} \vec{v} &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \\ &= \omega abe^{b\omega t}\hat{r} + a\omega e^{b\omega t}\hat{\theta} \\ &= \left\langle ab\omega e^{b\omega t}, \omega ae^{b\omega t} \right\rangle \end{aligned}$$

The radial component of velocity is

$$v_r = ab\omega e^{b\omega t}$$

and the transverse component of velocity is

$$v_t = a\omega e^{b\omega t}$$

Using (5.3.7), the acceleration of the particle at any time t is

$$\begin{aligned} \vec{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta} \\ &= \left(\omega^2 ab^2 e^{b\omega t} - ae^{b\omega t}\omega^2\right)\hat{r} + \left(2\omega abe^{b\omega t}\omega + ae^{b\omega t}(0)\right)\hat{\theta} \\ &= \left(a(b^2-1)\omega^2 e^{b\omega t}\right)\hat{r} + \left(2ab\omega^2 e^{b\omega t}\right)\hat{\theta} \\ &= \left\langle a(b^2-1)\omega^2 e^{b\omega t}, 2ab\omega^2 e^{b\omega t} \right\rangle \end{aligned}$$

The radial component of acceleration is

$$a_r = a(b^2-1)\omega^2 e^{b\omega t}$$

and the transverse component of acceleration is

$$a_t = 2ab\omega^2 e^{b\omega t}$$

Example 5.3.2. *Suppose we have an object that travels in a circle of constant radius* 3 *meters with a constant angular velocity of* 2 *radians per second. What are the expressions for the position, velocity and acceleration as functions of time?*

Solution :
The given data is

$$\begin{aligned} r &= 3 \\ \omega &= 2 \end{aligned}$$

From $r = 3$, we can find

$$\begin{aligned} \dot{r} &= 0 \\ \ddot{r} &= 0 \end{aligned}$$

And from $\omega = 2$, we can find

$$\begin{aligned} \dot{\theta} &= \omega = 2\ rad/s \\ \theta &= \omega t = 2t\ rad \\ \ddot{\theta} &= 0 \end{aligned}$$

The position as a function of time is trivial as

$$r(t) = 3$$

Using the relations between cartesian and polar coordinates, we have

$$\begin{aligned} x(t) &= r\cos(\theta) = 3\cos(2t) \\ y(t) &= r\sin(\theta) = 3\sin(2t) \end{aligned}$$

In Cartesian coordinates, the expression for position would be

$$\vec{r}(t) = 3\cos(2t)\hat{\imath} + 3\sin(2t)\hat{\jmath}$$

The velocity would be given by

$$\begin{aligned} v &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \qquad (5.3.8) \\ &= 0\hat{r} + 3(2)\hat{\theta} \\ &= 0\hat{r} + 6\hat{\theta} \end{aligned}$$

The velocity would have a radial speed of $0m/s$ and a tangential speed of $6m/s$, both of which were constant; *i.e.* the object would be traveling in a circle with a constant speed of $6m/s$. In Cartesian coordinates, the expression for velocity would be

$$v(t) = = -6cos(2t)\hat{\imath} + 6sin(2t)\hat{\jmath}. \qquad (5.3.9)$$

The acceleration would be given by

$$a = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

so that

$$a = (0 - 3(2)^2)\hat{r} + (2(0)2 + 3(0))\hat{\theta}$$

which is simply

$$a = -12\hat{r} + 0\hat{\theta}$$

The acceleration has a constant magnitude of $12m/s^2$ and is directed towards the origin. (Remember that positive $\hat{r}$ points away from the origin.) This is the centripetal acceleration as

$$\begin{aligned} a_c &= \frac{v^2}{r} \\ &= \frac{6^2}{3} = 12\ m/s^2 \end{aligned}$$

There is no tangential component of the acceleration.
In Cartesian coordinates, the expression for acceleration would be:

$$a(t) = -12cos(2t)\hat{\imath} - 12sin(2t)\hat{\jmath}$$

5.4 Motion in Tangent and Normal Plane

We will discuss position, Velocity and acceleration of a Particle in this plane.
Let us consider the motion of a particle, which describes the curve C from A to B in xy *plane*. The distance moved from A to B is the arc AB and its measure is the arc length denoted by s and is defined as

$$\widehat{AB} = s = \int_{t_A}^{t_B} \left\| \frac{dr}{dt} \right\| dt \tag{5.4.1}$$

Let P be the position of the particle at any time t. Let Q be the neighboring position along the curve at time $t + \Delta t$. Then the small distance from P to Q is

$$\widehat{PQ} = \Delta s$$

Next the position vectors of P and Q are

$$\begin{aligned} \vec{OP} &= \vec{r} \\ \vec{OQ} &= \vec{r} + \Delta\vec{r} \end{aligned}$$

Let $\vec{v}$ and $\vec{v} + \Delta\vec{v}$ be the velocities along the tangents to the curve at P and Q respectively. Then from the Fig.

$$\vec{PQ} = \Delta\vec{r} \tag{5.4.2}$$

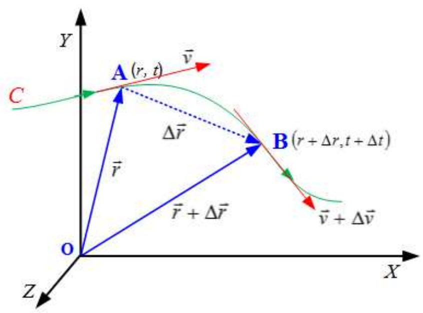


Figure 5.12: Path of the particle

Which is the displacement during the time Δt. Then the average velocity is

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} \tag{5.4.3}$$

and proceeding to the limit $\Delta t \to 0$, we obtain the actual velocity of the particle at any time t

$$\begin{aligned}\vec{v} &= \lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t} \\ &= \frac{d\vec{r}}{dt}\end{aligned}$$

Here Δr and Δt are very very small quantities and are differentiable functions of time t. Again consider Eq.(5.4.3)

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta\vec{r}}{\Delta s}\frac{\Delta s}{\Delta t}$$

Now proceeding limit as $Q \to P$, we have

$$\begin{aligned}\vec{v} &= \lim_{Q\to P}\frac{\Delta\vec{r}}{\Delta t} = \lim_{Q\to P}\frac{\Delta\vec{r}}{\Delta s}\frac{\Delta s}{\Delta t} \\ &= \frac{dr}{dt} = \lim_{\Delta s\to 0}\frac{\Delta\vec{r}}{\Delta s}\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} \\ &= \frac{d\vec{r}}{ds}\frac{ds}{dt} \\ &= \dot{s}\frac{d\vec{r}}{ds}\end{aligned} \tag{5.4.4}$$

Here $\frac{d\vec{r}}{ds}$ is a vector in the direction of $\vec{PQ}$. Next we show that it is a unit vector as

$$\begin{aligned}\left|\frac{\Delta r}{\Delta s}\right| &= \frac{|\Delta r|}{\Delta s} \\ &= \frac{chord\ PQ}{arc\ PQ} = \frac{|PQ|}{\widehat{PQ}}\end{aligned}$$

In limiting case $\Delta s \to 0$, then *chord* $PQ \approx$ *arc* PQ and we have $\Delta s \approx \Delta r$. Hence

$$\lim_{\Delta s \to 0} \frac{|\Delta r|}{\Delta s} = \frac{\Delta r}{\Delta r} = 1$$

or we can write as

$$\begin{aligned} \frac{d\vec{r}}{ds} &= \lim_{\Delta s \to 0} \frac{\Delta \vec{r}}{\Delta s} \\ &= \lim_{\Delta s \to 0} |\frac{\Delta \vec{r}}{\Delta s}|\hat{t} = 1 \cdot \hat{t} \\ &= \hat{t} \ (tangent\ unit\ vector) \end{aligned} \tag{5.4.5}$$

Using (5.4.5), (5.4.4)can be written as

$$\vec{v} = \dot{s}\hat{t} \tag{5.4.6}$$

The velocity has only tangential component. The magnitude of velocity is

$$|\vec{v}| = v = \dot{s} \tag{5.4.7}$$

From (5.4.28), the path or arc length moved by the particle in time interval T is

$$s = \int_0^T \sqrt{v} dt \tag{5.4.8}$$

Next the acceleration is given as

$$\begin{aligned} \vec{a} &= \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{s}\hat{t}\right) \\ &= \ddot{s}\hat{t} + \dot{s}\frac{d}{dt}\hat{t} \end{aligned}$$

consider

$$\begin{aligned} \frac{d\hat{t}}{dt} &= \frac{d\hat{t}}{ds}\frac{ds}{dt} \\ &= \dot{s}\frac{d\hat{t}}{ds} \end{aligned}$$

Here $\frac{d\hat{t}}{ds}$ is a vector whose magnitude is

$$|\frac{d\hat{t}}{ds}| = \kappa$$

then

$$\frac{d\hat{t}}{ds} = \kappa\hat{n}$$

Hence the acceleration is given as

$$\vec{a} = \ddot{s}\hat{t} + \dot{s}^2\kappa\hat{n} \tag{5.4.9}$$

When the particle is moving along a circular path with uniform speed, then the tangential component of acceleration is zero, and we have only its normal component.

$$\begin{aligned}\vec{a} &= \dot{s}^2\kappa\hat{n} \\ &= v^2\kappa\hat{n}\end{aligned} \tag{5.4.10}$$

Let $\kappa = \frac{1}{r}$, then the magnitude of this acceleration is

$$a = \frac{v^2}{r} \tag{5.4.11}$$

If a particle of mass m is stretched by a massless string and allowed to move along a circular path with uniform speed, as shown in Fig. 5.13 then according to Newton's second law of

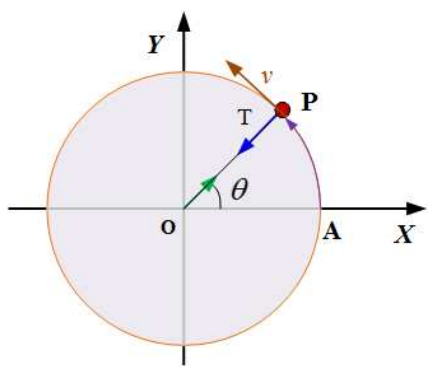


Figure 5.13: Circular Motion

motion, the magnitude of the centripetal force F_c is given by

$$\begin{aligned}F_c &= ma_c \\ &= m\frac{v^2}{r}\end{aligned} \tag{5.4.12}$$

This force is provided by the tension T in the string. Hence its magnitude is

$$T = m\frac{v^2}{r}$$

Suppose that the string is such that it breaks up whenever the tension in it exceeds a certain critical value T_m. It follows that there is a maximum velocity with which the weight can be whirled around, and that speed is

$$v_m = \sqrt{\frac{rT_m}{m}} \tag{5.4.13}$$

If v exceeds v_m then the string will break, and the weight will cease to be subject to a centripetal force, so it will fly off with velocity v_m along the straight-line which is tangential to its executed circular path.
There is another force, known as centrifugal force, equal in magnitude but acts in the opposite direction of centripetal force.

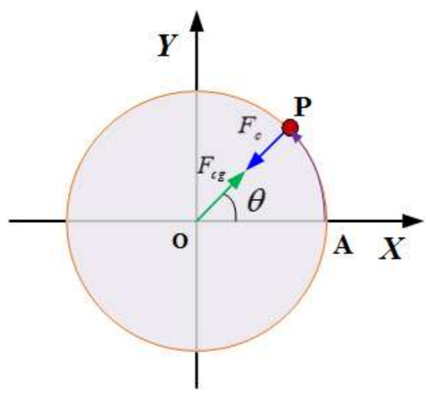


Figure 5.14: Centripetal force and centrifugal force

Example 5.4.1. *A particle of mass* $0.50\ kg$ *is attached with a massless string of length one meter. Let it is moving in a circle with speed* $10\ m/s$*. Find the tension in the string.*

Solution The given data is

$$\begin{aligned} m &= 0.50\ kg \\ r &= 1\ m \\ v &= 5\ m/s \end{aligned}$$

$$\begin{aligned} F_c &= m\frac{v^2}{r} \\ T &= 0.5\frac{(10)^2}{1} \\ &= 50\ N \end{aligned}$$

Example 5.4.2. *A particle is constrained to move along a circular helix represented by*

$$\vec{r} = a\cos\omega t\hat{i} + a\sin\omega t\hat{j} + bt\hat{k} \tag{5.4.14}$$

Show that the distance moved by the particle in one full turn of the helix is $\frac{2\pi}{\omega}\sqrt{a^2\omega^2+b^2}$

Solution : For $z = 0$, $\vec{r}$ represents a circle, a projection of $\vec{r}$ in xy plane.

$$\vec{r} = a\cos\omega t\hat{i} + a\sin\omega t\hat{j} \tag{5.4.15}$$

The particle starts at $t = 0$, and rotates in a circle of radius a about z axis with angular velocity ω. So the time required to complete one full turn of the helical path is $T = \frac{2\pi}{\omega}$
At any time $t(o < t < T)$, the position vector is

$$\begin{aligned}\vec{r} &= \vec{r}(t)\\ &= a\cos\omega t\hat{i} + a\sin\omega t\hat{j} + bt\hat{k}\\ &= \langle a\cos\omega t, a\sin\omega t, bt\rangle \end{aligned}\tag{5.4.16}$$

the velocity is

$$\begin{aligned}\vec{v} &= \frac{d\vec{r}}{dt}\\ &= \langle -a\omega\sin\omega t, a\omega\cos\omega t, b\rangle \end{aligned}\tag{5.4.17}$$

The speed (magnitude of velocity) is given by

$$\begin{aligned}v &= \left\|\frac{dr}{dt}\right\|\\ &= \sqrt{(-a\omega\sin\omega t)^2 + (a\omega\cos\omega t)^2 + (b)^2}\\ &= \sqrt{a^2\omega^2+b^2}\end{aligned}$$

using (5.4.8), we can find the distance moved by the particle as

$$\begin{aligned}s &= \int_0^T vdt = \int_0^{\frac{2\pi}{\omega}} \sqrt{a^2\omega^2+b^2}dt\\ &= \frac{2\pi}{\omega}\sqrt{a^2\omega^2+b^2}\end{aligned}\tag{5.4.18}$$

Hence the result.

Example 5.4.3. *A particle is constrained to move along a circular helix so that its coordinates at any time t are $(a\cos\theta, a\sin\theta, a\theta\tan\alpha)$, where $a > 0$, $0 < \alpha < \frac{\pi}{2}$ are constants. The speed increases linearly with time t from zero at $t = 0$ to V at $t = T$. Find the acceleration at any time $t < T$, the motion taking place in the sense of θ increasing and starting from the point $(a, 0, 0)$.*

Solution : At any time t $(o < t < T)$, the position vector is

$$\begin{aligned} \vec{r} &= \vec{r}(\theta) \\ &= a\langle \cos\theta, \sin\theta, \theta\tan\alpha \rangle \end{aligned} \tag{5.4.19}$$

the velocity is

$$\begin{aligned} \vec{v} &= \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{d\theta}\frac{d\theta}{dt} \\ &= a\dot{\theta}\langle -\sin\theta, \cos\theta, \tan\alpha \rangle \end{aligned} \tag{5.4.20}$$

The speed (magnitude of velocity) is given by

$$\begin{aligned} v &= \left\| \frac{d\vec{r}}{dt} \right\| \\ &= a\dot{\theta}\sqrt{(-\sin\theta)^2 + (\cos\theta)^2 + (\tan\alpha)^2} \\ &= a\dot{\theta}\sqrt{1 + (\tan\alpha)^2} \\ &= a\dot{\theta}\sec\alpha \end{aligned} \tag{5.4.21}$$

Using Eq. (5.4.21), Eq. (5.4.20) becomes

$$\vec{v} = a\dot{\theta}\sec\alpha\langle -\cos\alpha\sin\theta, \cos\alpha\cos\theta, \sin\alpha \rangle \tag{5.4.22}$$

But

$$\vec{v} = v\hat{t} \tag{5.4.23}$$

From Eqs. (5.4.22) and (5.4.23) we have

$$\hat{t} = \langle -\cos\alpha\sin\theta, \cos\alpha\cos\theta, \sin\alpha \rangle \tag{5.4.24}$$

Next the speed at any time t is

$$v = at \tag{5.4.25}$$

The speed increases linearly with time t from zero at $t = 0$ to V at $t = T$, then acceleration can be found as

$$a = \frac{V}{T} \tag{5.4.26}$$

Using Eq. (5.4.26), Eq. (5.4.25) becomes

$$\dot{s} = v = \frac{V}{T}t \tag{5.4.27}$$

Using Eq. (5.4.27). Eq. (5.4.21) becomes

$$\vec{v} = \left(\frac{V}{T}t\right)\hat{t} \tag{5.4.28}$$

Which is the velocity of the particle at any time t, with $\hat{t}$ is given by Eq.(5.4.24)
Next the acceleration of the particle is given by

$$\vec{a} = \ddot{s}\hat{t} + \dot{s}^2\kappa\hat{n} \tag{5.4.29}$$

Consider Eq. (5.4.27), $\ddot{s}$ is given as

$$\ddot{s} = \frac{V}{T} \tag{5.4.30}$$

Next we need only κ and $\hat{n}$, given by the relation

$$\frac{d\hat{t}}{ds} = \kappa\hat{n} \tag{5.4.31}$$

where s is the arc length, $\hat{n}$ is the unit vector, and κ is

$$\kappa = |\frac{d\hat{t}}{ds}|$$

Since $\hat{t} = \hat{t}(\theta)$, then Eq. (5.4.31) can be written as

$$\frac{d\hat{t}}{ds} = \frac{d\hat{t}}{d\theta}\frac{d\theta}{ds} \tag{5.4.32}$$

first right part of Eq. (5.4.32) is

$$\frac{d\hat{t}}{d\theta} = \langle -\cos\alpha\cos\theta, -\cos\alpha\sin\theta, 0\rangle \tag{5.4.33}$$

For $\frac{d\theta}{ds}$, consider Eq. (5.4.21) and Eq. (5.4.27), we have

$$\begin{aligned} a\dot{\theta}\sec\alpha &= \frac{V}{T}t \\ \dot{\theta} &= \frac{V}{aT}(\cos\alpha)t \end{aligned} \tag{5.4.34}$$

Integrating Eq. (5.4.34) with respect to t

$$\theta = \frac{V}{aT}(\cos\alpha)\frac{t^2}{2} + C$$

at $t = 0$, $\theta = 0 \Rightarrow\ C = 0$

$$\theta(t) = \frac{V}{aT}(\cos\alpha)\frac{t^2}{2} \tag{5.4.35}$$

Here we find that θ is a function of t, so we can write

$$\begin{aligned} \frac{d\theta}{ds} &= \frac{d\theta}{dt}\frac{dt}{ds} = \frac{d\theta}{dt}\frac{1}{\frac{ds}{dt}} \\ &= \frac{\dot{\theta}}{\dot{s}} \end{aligned} \tag{5.4.36}$$

From Eqs. (5.4.27) and (5.4.34), Eq. (5.4.36) can be written as

$$\begin{aligned}\frac{d\theta}{ds} &= \frac{V}{aT}(\cos\alpha)t \times \frac{T}{Vt} \\ &= \frac{\cos\alpha}{a}\end{aligned} \tag{5.4.37}$$

Using Eqs. (5.4.33) and (5.4.37), Eq. (5.4.32) becomes

$$\frac{d\hat{t}}{ds} = \frac{\cos^2\alpha}{a}\langle -\cos\theta, -\sin\theta, 0\rangle \tag{5.4.38}$$

so κ is

$$\kappa = \left|\frac{d\hat{t}}{ds}\right| = \frac{\cos^2\alpha}{a} \tag{5.4.39}$$

and $\hat{n}$ is

$$\hat{n} = \langle -\cos\theta, -\sin\theta, 0\rangle \tag{5.4.40}$$

Finally the acceleration is given by

$$a = \frac{V}{T}\hat{t} + \left(\frac{Vt}{T}\right)^2\frac{\cos^2\alpha}{a}\hat{n} \tag{5.4.41}$$

Where $\hat{t}$ and $\hat{n}$ are given by Eqs. (5.4.24) and (5.4.40) respectively.

5.4.1 Equation of Motion of Simple Pendulum

Consider OXY a cartesian coordinate system. Let a particle of m is attached with a massless string of length l, with other end fixed at O, forming a simple pendulum, as shown in Fig. 5.15. At any time t, the particle be at $P(x, y) = P(r, \theta)$. Then by Newton's second law of motion its equation of motion is

$$\begin{aligned}F &= -mg\sin\theta \\ ma &= -mg\sin\theta \\ a &= -g\sin\theta\end{aligned} \tag{5.4.42}$$

Here g is the acceleration due to gravity near the surface of the earth. The negative sign on right hand side of (5.4.42) implies that θ and a always acts in opposite directions. Since $r = l$ is fixed, then using trigonometric relation

$$s = r\theta = l\theta$$

differentiating with respect to t

$$\begin{aligned}\frac{ds}{dt} &= l\frac{d\theta}{dt} \\ v &= l\dot{\theta}\end{aligned}$$

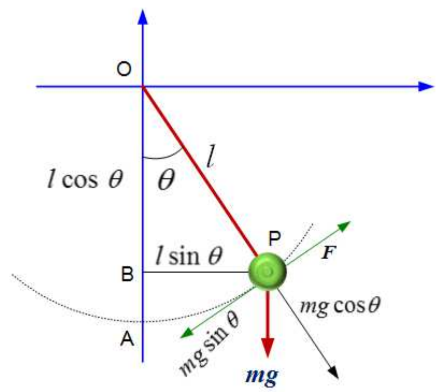


Figure 5.15: Simple Pendulum

$$a = \frac{du}{dt} = l\ddot{\theta} \tag{5.4.43}$$

Using (5.4.43) in (5.4.42)

$$\begin{aligned} l\ddot{\theta} &= -g\sin\theta \\ \ddot{\theta} - \frac{g}{l}\sin\theta &= 0 \\ \ddot{\theta} + \omega^2 \sin\theta &= 0 \end{aligned} \tag{5.4.44}$$

with $\omega = \sqrt{\frac{g}{l}}$ is the frequency of oscillation.

(5.4.44) is equation of motion of a simple pendulum.

Exercises

1. A particle of mass 10 kg moves in a circle of radius 15 m with an angular speed of 2 rad/s. Find
   (a) Its linear speed
   (b) Its linear acceleration
   (c) Its angular acceleration
   (d) the centripetal force

2. A radar fixed on the ground, tracks the circular motion of a rocket under the gravitational force only. At an instant, the rocket is at a distance of $r = 75\ km$ away from the radar with an inclination of $\theta = \frac{\pi}{3}$ with the ground. At that instant the rocket has linear speed of $\dot{r} = 1700\ m/s$ and angular speed as $\omega = 0.8\ degree/s$. Find
   (a) Its transverse component of velocity
   (b) Its velocity and speed
   (c) Its radial and transverse components of acceleration.
   (d) Magnitude of acceleration.
   (d) $\ddot{r}$ and $\ddot{\theta}$

3. A particle is constrained to move along a circle helix so that its coordinates at any time t are $(a\cos\theta, a\sin\theta)$, where $a > 0$, is constants. The speed increases linearly with time t from zero at $t = 0$ to V at $t = T$. Find the acceleration at any time $t < T$, the motion taking place in the sense of θ increasing and starting from the point $(a, 0)$.

4. A particle is constrained to move along a circular helix so that its coordinates at any time t are $(a\cos\theta, a\sin\theta, b\theta)$, where $a > 0$, $b \neq 0$ are constants. The speed increases linearly with time t from zero at $t = 0$ to V at $t = T$. Find the acceleration at any time $t < T$, the motion taking place in the sense of θ increasing and starting from the point $(a, 0, 0)$.

Chapter 6

Simple Harmonic Motion

A periodic motion in which the displacement is symmetrical about a point is called harmonic motion. It may be discussed as simple harmonic motion, damped harmonic motion, forced harmonic motion and forced and damping harmonic motion. First consider simple harmonic motion.

6.1 Simple Harmonic Motion

A very special kind of motion occurs when the force acting on a body is proportional to the displacement of the body from some equilibrium position. If this force is always directed toward the equilibrium position, and repetitive back and forth motion occurs about this position, such motion is called periodic, harmonic, oscillation, or vibration motion (the four terms are completely equivalent). To explain this oscillatory motion, we present spring mass system.

6.1.1 Spring Mass System with Horizontal Oscillatory Motion.

Consider a physical system that consists of a block of mass m attached to the end of a spring, whose other end is fixed in the wall, and the block is free to move on a horizontal, frictionless surface. See Fig. 6.1. When the spring is neither stretched nor compressed, the block is at the position $x = 0$ called the equilibrium position of the system. Let this position is O. When the block is displaced a small distance x from equilibrium, the spring exerts a force on the block that is proportional to the displacement and given by Hookes law

$$F_r = -kx \tag{6.1.1}$$

where k is a constant of proportionality called the spring constant. The spring is essentially characterized by the number k. This force is called restoring force as it is always directed toward the equilibrium position, opposite the displacement. That is, when the block is

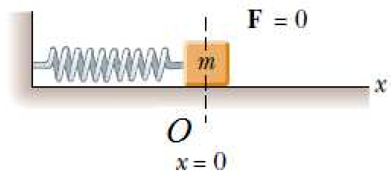


Figure 6.1: The system is at rest

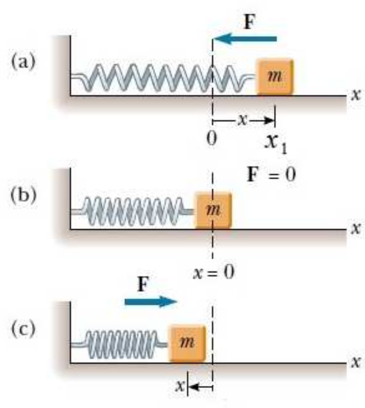


Figure 6.2: Simple harmonic motion

displaced to the right of O in Figure 6.2, then the displacement is positive and the restoring force is directed to the left. When the block is displaced to the left of O, then the displacement is negative and the restoring force is directed to the right. This is the only force acting on the block. Then by Newtons second law of motion, its equation of motion is

$$\begin{aligned} F &= F_r \\ ma &= -kx \\ a &= -\frac{k}{m}x \end{aligned} \tag{6.1.2}$$

Take

$$\omega = \sqrt{\frac{k}{m}} \tag{6.1.3}$$

is the angular frequency, then (6.1.2) can be written as

$$a(t) = -\omega^2 x(t) \tag{6.1.4}$$

(6.1.4) shows that the motion is with variable acceleration and this acceleration is proportional to the displacement of the block, and is directed in the opposite direction of the displacement (directed towards the equilibrium position). Systems that behave in this way are said to exhibit simple harmonic motion. Thus simple harmonic motion can be defined as

An object is said to execute simple harmonic motion whenever its acceleration is proportional to its displacement from some equilibrium position and is oppositely directed.

This equilibrium position is called center of simple harmonic motion. If $x = x_2$ is the center of simple harmonic motion, then equation of simple harmonic motion is

$$a(t) = -\omega^2 (x(t) - x_2) \tag{6.1.5}$$

6.1.2 Spring Mass System with Vertical Oscillatory Motion.

Suppose that a flexible spring is suspended vertically from a rigid support (see Fig. 6.3 (a)) and then a mass m is attached to its free end. The amount of stretch, or elongation, of the spring depends on the mass. By Hookes law the restoring force is

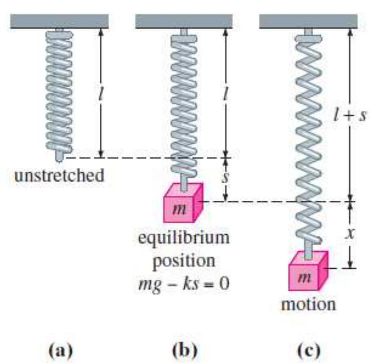


Figure 6.3: Simple harmonic motion

$$F_r = -ks$$

And the mass attains an equilibrium position at which its weight $W = mg$ is balanced by the restoring force ks (see Fig. 6.3 (b)).

$$\begin{aligned} mg &= ks \\ mg - ks &= 0 \end{aligned} \tag{6.1.6}$$

If the mass is pulled down a distance x from its equilibrium position, the new restoring force of the spring is

$$F_r = -k(x+s)$$

Assuming that there are no retarding forces acting on the system, then by Newtons second law of motion, its equation of motion is

$$\begin{aligned} F &= F_r + W \\ ma &= -kx - ks + mg \end{aligned} \tag{6.1.7}$$

Using (6.1.6), (6.1.7) becomes

$$a = -\frac{k}{m}x$$

The same expression for acceleration as given in (6.1.2) and consequently we have the same equation of motion. Hence we have same equation of motion for both horizontal and vertical oscillatory motions.

6.2 Expression of velocity and Position in Simple Harmonic Motion

: Let the block of mass m is pulled a distance $x = x_1$ from its equilibrium position (from O at $x = 0$) and is released from rest at time $t = 0$. Then its initial data (at P) is

$$\begin{aligned} t = t_0 &= 0 \\ v = v_0 &= 0 \\ x = x_0 &= x_1 \end{aligned}$$

From (6.1.4), we can say that the motion is taking place in such a way that when the particle is moving towards O (equilibrium position), the acceleration is acting along it so that as the time progresses, the velocity becomes higher and higher and when the particle is moving away from O, the acceleration is acting against it so that as the time progresses, the velocity becomes lesser and lesser. Let at time $t = t_1$, the particle is at O with velocity $v = v_1$. Then the final data (at O) is

$$\begin{aligned} t &= t_1 \\ v &= v_1 \\ x &= x_1 \end{aligned}$$

Since acceleration can be expressed as a function of velocity

$$a = v\frac{dv}{dx} \tag{6.2.1}$$

Using (6.2.1), (6.1.4) can be written as

$$v\frac{dv}{dx} = -\omega^2 x \tag{6.2.2}$$

(6.2.2) is separable first order differential equation and can be solved as

$$\frac{v^2}{2} = -\omega^2\frac{x^2}{2} + A \tag{6.2.3}$$

The constant of integration A can be evaluated by using initial data

$$\begin{aligned} 0 &= -\omega^2\frac{x_1^2}{2} + A \\ A &= \omega^2\frac{x_1^2}{2} \end{aligned} \tag{6.2.4}$$

Using (6.2.4), (6.2.3) can be written as

$$v^2 = \omega^2\left(x_1^2 - x^2\right)$$

Hence the velocity is

$$v = \omega\sqrt{\left(x_1^2 - x^2\right)} \tag{6.2.5}$$

Equation (6.2.5) gives the velocity of the particle at any time t. Since $v = \frac{dx}{dt}$, then (6.2.5) can be written as

$$\frac{dx}{dt} = \omega\sqrt{\left(x_1^2 - x^2\right)} \tag{6.2.6}$$

(6.2.6) is a separable first order differential equation, and can be solved as

$$\frac{dx}{\sqrt{\left(x_1^2 - x^2\right)}} = \omega dt$$

Integrating

$$\sin^{-1}\frac{x}{x_1} = \omega t + B \tag{6.2.7}$$

The constant of integration B can be evaluated by using initial data

$$\begin{aligned} \sin^{-1}\frac{x_1}{x_1} &= \omega(0) + B \\ B &= \sin^{-1}(1) \\ &= \frac{\pi}{2} \end{aligned} \tag{6.2.8}$$

Using (6.2.8), (6.2.7) can be written as

$$\begin{aligned} \sin^{-1}\frac{x}{x_1} &= \omega t + \frac{\pi}{2} \\ \frac{x}{x_1} &= \sin\left(\omega t + \frac{\pi}{2}\right) \\ x &= x_1 \cos(\omega t) \qquad (6.2.9) \end{aligned}$$

Since

$$\cos(\omega t) \leq 1$$

it follows that the numerical value of x cannot be greater than x_1. Thus the particle is bound to stay within a distance x_1 from the fixed point O. The length x_1 is called the amplitude of the motion. The point O is referred to as the centre of motion. Following equation (6.2.5), the velocity of the particle at amplitude of the motion is zero and at the centre of motion

$$()v = v_1 = \omega x_1 \qquad (6.2.10)$$

Equation (6.2.10) gives the maximum velocity of the particle executing simple harmonic motion. If we consider the motion starts from O, then the initial data (at O) is

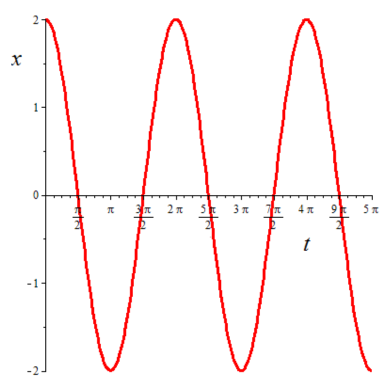


Figure 6.4: Plot of displacement vs. time for Simple harmonic motion.

$$\begin{aligned} t_0 &= 0 \\ v_0 &= v \\ x_0 &= 0 \end{aligned}$$

Then the constant of integration B can be evaluated as

$$\begin{aligned} \sin^{-1}\frac{0}{x_1} &= \omega(0) + B \\ B &= \sin^{-1}(0) \\ &= 0 \end{aligned} \tag{6.2.11}$$

Using (6.2.11), (6.2.7) can be written as

$$\begin{aligned} \sin^{-1}\frac{x}{x_1} &= \omega t \\ \frac{x}{x_1} &= \sin(\omega t) \\ x &= x_1 \sin(\omega t) \end{aligned} \tag{6.2.12}$$

(6.2.9) gives the displacement of the object from equilibrium position O, at any time t,

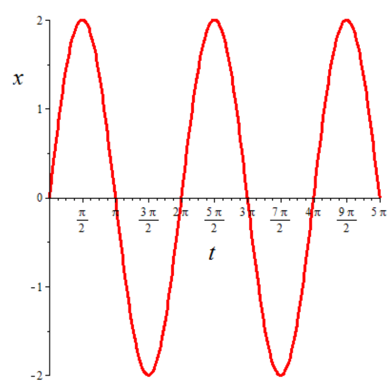


Figure 6.5: Plot of displacement vs. time for Simple harmonic motion.

when we consider its motion starts from P and (6.2.12) gives the displacement of the object from equilibrium position P, at any time t, when we consider its motion starts from O. From (6.2.9) and (6.2.12), the displacement has different expression due to different boundary conditions. Similarly other boundary conditions give still different forms for displacement.
For a general solution of (6.1.4), the acceleration can be written as

$$a = \frac{d^2x}{dt^2} \tag{6.2.13}$$

Using (6.2.13), (6.1.4) can be written as

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\omega^2 x \\ \frac{d^2x}{dt^2} + \omega^2 x &= 0 \end{aligned} \qquad (6.2.14)$$

(6.2.14) is second order homogeneous differential equation. Its characteristic equation is

$$m^2 + \omega^2 = 0 \qquad (6.2.15)$$

(6.2.15) has imaginary roots as

$$m = \pm\, i\omega$$

and the solution is

$$x(t) = C_1 \cos\omega t + C_2 \sin\omega t \qquad (6.2.16)$$

If we suppose the constants as $C_1 = A\cos\phi$ and $C_2 = -A\sin\phi$, then (6.2.16) can be written as

$$\begin{aligned} x(t) &= A(\cos\phi\cos\omega t - \sin\phi\sin\omega t) \\ &= A\cos(\omega t + \phi) \end{aligned} \qquad (6.2.17)$$

Similarly

$$x(t) = A\sin(\omega t + \phi) \qquad (6.2.18)$$

Here ϕ is known as phase constant.

$$\phi = \tan^{-1}\left(-\frac{C_1}{C_2}\right) \qquad (6.2.19)$$

(6.2.17) and (6.2.18) are the general expressions of displacement for simple harmonic motion.

6.2.1 Simple Harmonic Motion is Periodic

: A motion which repeats itself in time is called periodic motion. (6.2.9) can also be written as

$$\begin{aligned} x &= x_1\ \cos\left(\omega t + 2\pi\right) \\ &= x_1\ \cos\left(\omega\left(t + \frac{2\pi}{\omega}\right)\right) \end{aligned} \qquad (6.2.20)$$

The velocity of the object is the time derivative of (6.2.9)

$$\begin{aligned} v = \frac{dx}{dt} &= x_1\ \omega \sin(\omega t) \\ &= x_1\ \omega\ \sin(\omega t + 2\pi) \\ &= x_1\ \omega\ \sin\left(\omega\left(t + \frac{2\pi}{\omega}\right)\right) \end{aligned} \tag{6.2.21}$$

And the acceleration can also be written as

$$\begin{aligned} a = \frac{dv}{dt} = \frac{d^2x}{dt^2} &= -x_1\ \omega^2\ \cos(\omega t) \\ &= -x_1\ \omega^2\ \cos(\omega t + 2\pi) \\ &= -x_1\ \omega^2\ \cos\left(\omega\left(t + \frac{2\pi}{\omega}\right)\right) \end{aligned} \tag{6.2.22}$$

From (6.2.20), (6.2.21) and (6.2.22), we observe that the displacement, velocity and acceleration of the object executing simple harmonic motion are the same after an addition of $\frac{2\pi}{\omega}$ in time t. Therefore simple harmonic motion is periodic of period $\frac{2\pi}{\omega}$. This time period is denoted by T.

$$T = \frac{2\pi}{\omega} \tag{6.2.23}$$

Equation (6.2.23) indicates that time period depends only on frequency of oscillation and is independent of amplitude.

6.2.2 Maximum Speed and Acceleration

From (6.2.14), we can find maximum speed as

$$v_{max} = \text{maximum of } \left[x_1\ \omega\ \sin\left(\omega\left(t + \frac{2\pi}{\omega}\right)\right)\right]$$

This maximum occurs when $\sin(\omega t + 2\pi)$ is maximum, and

$$\max(\sin(\omega t + 2\pi)) = 1$$

Hence maximum speed is

$$v_{max} = x_1\ \omega \tag{6.2.24}$$

The maximum speed occurs at equilibrium position of the oscillation (at $x = 0$).
From (6.2.15), we can find the magnitude of maximum acceleration as

$$a_{max} = \text{maximum of } \left[-x_1\ \omega^2\ \cos\left(\omega\left(t + \frac{2\pi}{\omega}\right)\right)\right]$$

This maximum occurs when $\cos(\omega t + 2\pi)$ is maximum, and

$$\max(\cos(\omega t + 2\pi)) = 1$$

Hence magnitude of maximum acceleration is

$$a_{max} = x_1\,\omega^2 \tag{6.2.25}$$

The magnitude of acceleration is greatest at the ends of the oscillation (at $x = \pm x_1$). The

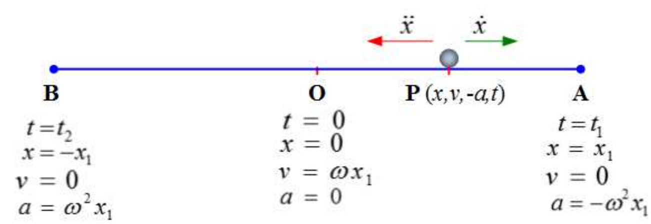


Figure 6.6: Displacement velocity and acceleration of Simple harmonic motion at mean and end points.

displacement, velocity and acceleration of an object at different positions are illustrated in Fig. 6.6

6.3 Simple Harmonic Motion with Centre other than Origin

If the centre of simple harmonic motion is other than origin, say $x = b$, its equation of motion is

$$\ddot{x} + \omega^2(x - b) = 0$$

Before going next, we present some definitions, considering the above spring mass example.
Oscillation or Vibration: The distance covered by the particle in time period T is called oscillation
In spring mass example, the particle is at rest at P having distance $x = x_1$ from O. Next it is released from P to move towards O, at time $t = 0$. As the time progresses, it moves towards O and its velocity increases to its maximum value $v = x_1\,\omega$ at O. The particle then moves to the left of O and its velocity decreases and becomes zero at $x = -x_1$. Next the particle retrace its motion backwards and comes to rest at $x = x_1$, *i.e.* it reaches at P and completes one round known as one oscillation or vibration. The motion is then repeated.

Frequency: The number of oscillation per unit time is called frequency. It is denoted by f. Mathematically can be written as

$$f = \frac{1}{T} \tag{6.3.1}$$

The frequency of spring mass system is

$$f = \frac{\omega}{2\pi} \tag{6.3.2}$$
$$\text{or} \quad = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Equation (6.3.2) indicates that the angular frequency ω and the frequency f are closely related with a factor 2π. We normally express ω in rad/s and f in $cycle/s$ or Hz (Hertz). However, the real dimensions of both are $1/s$ in SI system.

Amplitude: The maximum distance covered by the particle on either side of equilibrium position (*i.e.* from O) is called amplitude of simple harmonic motion. In above motion, $\sin\omega t \leq 1$ so $x = x_1$ is the amplitude. It is also equal to one-half of the total range of motion $\frac{1}{2}(x_{max} - x_{min})$. In spring mass example, $x_{max} = x_1$ and $x_{min} = -x_1$, and the amplitude is

$$\begin{aligned} A &= \frac{1}{2}(x_{max} - x_{min}) \\ &= \frac{1}{2}(x_1 + x_1) \\ &= x_1 \end{aligned} \tag{6.3.3}$$

Phase of the Motion: The time varying quantity $(\omega t + \phi)$ is called the phase of the motion.

Phase Constant or Phase Angle: The constant ϕ is called the phase constant or phase angle. Its value depends on the displacement and velocity of the particle at $t = 0$. The general expression for displacement is

$$x(t) = A\cos(\omega t + \phi)$$

at $t = 0$, the displacement is

$$x(0) = A\cos\phi \tag{6.3.4}$$

and the velocity is

$$v(t) = \; = -\omega A\sin(\omega t + \phi)$$

at $t = 0$, the displacement is

$$v(0) = -\omega A\sin\phi \tag{6.3.5}$$

From (6.3.4) and (6.3.5), we can write

$$\frac{v(0)}{x(0)} = -\omega \tan\phi$$

or

$$\phi = \tan^{-1}\left(-\frac{v(0)}{\omega x(0)}\right) \tag{6.3.6}$$

(6.3.6) gives the phase angle.

6.4 Kinetic and Potential Energies of Spring Mass System

The kinetic and potential energies of this system are as under.

6.4.1 Kinetic Energy

The energy due to motion of a particle is kinetic energy. It is denoted by T. In this unit T stands for time period, so K is used to denote the kinetic energy. The kinetic energy of a system is

$$K = \frac{1}{2}mv^2$$

The velocity of the system is given by (6.2.5), using $\omega^2 = \frac{k}{m}$, the velocity of the system is

$$v = \sqrt{\frac{k}{m}\left(x_1^2 - x^2\right)} \tag{6.4.1}$$

Using (6.4.1), the kinetic energy of the system is

$$K = \frac{1}{2}k\left(x_1^2 - x^2\right) \tag{6.4.2}$$

The graph of the kinetic energy of spring mass system is illustrated in Fig. 6.7. The kinetic energy is maximum at its mean position (at $x = 0$) and is zero (minimum) at its endpoints ($x = \pm x_1$)

6.4.2 Potential Energy

In spring mass system, the force (restoring force) is given by (6.4.3)

$$F = -kx$$

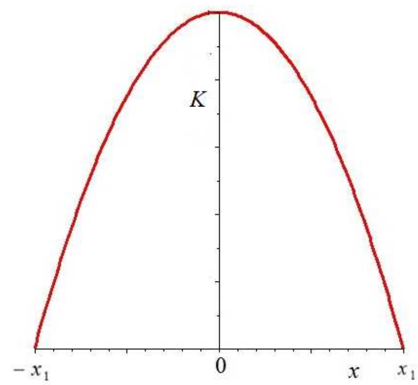


Figure 6.7: Kinetic energy of spring mass system

Since the force is conservative, there exist a potential function U such that

$$F = -\frac{dU}{dx}$$

Or we can write

$$U = \int_a^b F = \int_a^b kx$$

Let the limits of integration are from $0 \to x$, then the potential energy of the system is

$$U = \int_0^x kx$$
$$= \frac{1}{2}kx^2 \qquad (6.4.3)$$

The graph of the potential energy of spring mass system is illustrated in Fig. 6.8. The potential energy is maximum at its endpoints ($x = \pm x_1$) and is zero (minimum) at its mean position (at $x = 0$)

6.4.3 Total Energy of Spring Mass System is Constant

Let F be the force acting on the system, then by Newton's second law of motion, we can write

$$F = ma$$

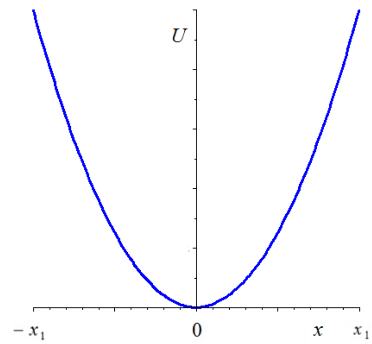


Figure 6.8: Potential energy of spring mass system

Using the force given by (6.4.3) and $a = \ddot{x}$, we have

$$-kx = m\ddot{x} \tag{6.4.4}$$

Multiplying $\dot{x}$ on both sides of (6.4.4), we have

$$-kx\dot{x} = m\dot{x}\ddot{x} \tag{6.4.5}$$

(6.4.5) can also be written as

$$\begin{aligned} -\frac{1}{2}k\frac{dx^2}{dt} &= \frac{1}{2}m\frac{d\dot{x}^2}{dt} \\ -\frac{d}{dt}\left(\frac{1}{2}kx^2\right) &= \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2\right) \\ \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) &= 0 \\ \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 &= C \ (constant) \\ E = K + U &= C \end{aligned} \tag{6.4.6}$$

Hence the total energy of spring mass system is constant as shown in Fig. 6.9

Example 6.4.1. *A block of mass* 680 *g is fastened to a spring whose spring constant is* 65 *N/m. The block is pulled a distance* 11 *cm from its equilibrium position at* $x = 0$ *on a frictionless surface and released from rest at* $t = 0$.

a) *What is the angular frequency?*

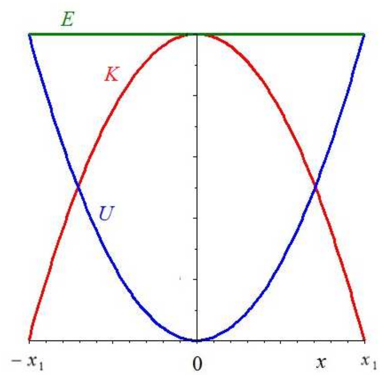


Figure 6.9: Total energy of spring mass system

b) *What is the time of the resulting motion (time period)?*

c) *What is the frequency of oscillation?*

d) *What is the amplitude of oscillation?*

e) *What is the maximum speed* v_{max} *of the oscillating block, and where is the block when it has this speed?*

f) *What is the magnitude of maximum acceleration* a_{max} *of the motion, and where is the block when this acceleration occur?*

g) *What is the phase constant* ϕ *of the motion?*

h) *What is the displacement function* $x(t)$ *for the moving block?*

i) *What is the kinetic energy function for the moving block?*

j) *What is the potential energy function for the moving block?*

k) *Find the extreme values of kinetic and potential energies.*

l) What is the total energy function for the moving block?

Solution The given data is

$$\begin{aligned} m &= 680\ g = 0.68\ kg \\ k &= 65\ N/m \end{aligned}$$

And the initial data is (at $t = 0$)

$$\begin{aligned} x(0) &= 11\ cm = 0.11\ m \\ v(0) &= \dot{x}(0) = 0 m/s \end{aligned}$$

a) Using (6.1.3), the angular frequency is:

$$\begin{aligned} \omega &= \sqrt{\frac{k}{m}} \\ &= \sqrt{\frac{65}{0.68}} \\ &= 9.87\ rad/s \end{aligned}$$

b) Using (6.2.23), the time period is:

$$\begin{aligned} T &= \frac{2\pi}{\omega} \\ &= \frac{2\pi}{9.87} \\ &= 0.636594 \cong 0.64\ s \end{aligned}$$

c) Using (6.3.1), the frequency is:

$$\begin{aligned} f &= \frac{1}{T} \\ &= \frac{1}{0.636594} = 1.57086 \cong 1.6\ Hz \end{aligned}$$

d) Using (6.3.3), the amplitude is:

$$\begin{aligned} A &= x_1 \\ &= 0.11\ m \end{aligned}$$

e) Using (6.2.24), the maximum speed v_{max} is:

$$\begin{aligned} v_{max} &= x_1\ \omega \\ &= (0.11)(9.87) = 1.0857 \\ &\cong 1.1\ m/s \end{aligned}$$

The block has maximum speed at $x = 0$

f) Using (6.2.25), the greatest magnitude of acceleration a_{max} is:

$$\begin{aligned} a_{max} &= x_1\ \omega^2 \\ &= (0.11)(9.87)^2 = 10.715859 \\ &\cong 10.7\ m/s^2 \end{aligned}$$

g) The phase constant ϕ, can be calculated as:

At $t = 0$, the displacement and velocity are

$$\begin{aligned} x(0) &= 11\ cm = 0.11\ m \\ v(0) &= 0\ m/s \end{aligned}$$

Using these values in (6.3.6)

$$\begin{aligned} \phi &= \tan^{-1}\left(-\frac{v(0)}{\omega x(0)}\right) \\ &= \tan^{-1}\left(-\frac{0}{(9.87)(0.11)}\right) \\ &= \tan^{-1}(0) \\ &= 0\ rad. \end{aligned}$$

h) Using (6.2.17), the displacement function $x(t)$ for the moving block is:

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi) \\ &= 0.11\cos(9.87t + 0) \\ &= 0.11\cos(9.87t)\ m \end{aligned}$$

i) Using (6.4.2), the kinetic energy function for the moving block is:

$$\begin{aligned} K &= \frac{1}{2}k\left(x_1^2 - x^2\right) \\ &= \frac{1}{2}(65)\left((0.11)^2 - x^2\right) \\ &= 0.3933 - 32.5x^2 \end{aligned}$$

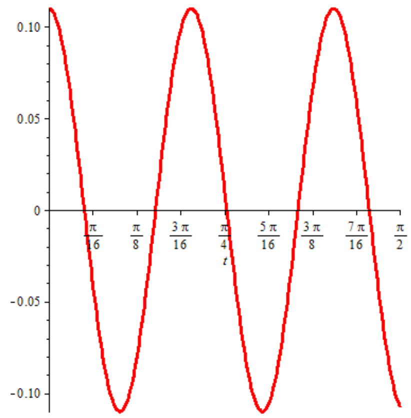


Figure 6.10: Path of motion

j) Using (6.4.3), the potential energy function for the moving block is:

$$\begin{aligned} U &= \frac{1}{2}kx^2 \\ &= \frac{1}{2}(65)x^2 \\ &= 32.5x^2 \end{aligned}$$

k) The extreme values of kinetic and potential energies exist at mean and end positions. The kinetic energy is minimum at end points (at $x = \pm 0.11$) and maximum at mean position (at $x = 0$). The values are as under:

$$\begin{aligned} K_{min} &= 0 \\ K_{max} &= 0.3933 \end{aligned}$$

The potential energy is maximum at end points (at $x = \pm 0.11$) and minimum at mean position (at $x = 0$). The values are as under:

$$\begin{aligned} U_{max} &= 0.3933 \\ U_{min} &= 0 \end{aligned}$$

l) The total energy of the system is

$$\begin{aligned} E &= K + U \\ &= (0.3933 - 32.5x^2) + 32.5x^2 \\ &= 0.3933J \end{aligned}$$

The energy of the system is constant.

6.5 Relation Between Uniform Circular Motion and Simple Harmonic Motion

There is a correspondence between simple harmonic motion and uniform circular motion. Consider a particle P moves at constant speed v around a circle of radius r. At the instant t, P has coordinates (x, y). Then the radius vector $\vec{OP} = \vec{r}$ makes an angle θ with x *axis*. Also the x coordinate in polar form is

$$x = r\cos\theta$$

The velocity component along x *axis* is

$$\begin{aligned} v_x = \frac{dx}{dt} &= -r\sin\theta\frac{d\theta}{dt} \\ &= -r\sin\theta\ \omega \end{aligned}$$

The acceleration component along x *axis* is

$$\begin{aligned} a_x = \frac{dv_x}{dt} &= -r\cos\theta\frac{d\theta}{dt}\ \omega \\ &= -x\omega^2 \end{aligned}$$

Hence the system executes simple harmonic motion.

6.5.1 Time Period of Uniform Circular Motion

The relation between linear and angular speed is

$$v = r\omega$$

Let the particle complete one rotation in time T, the distance covered is

$$s = 2\pi r$$

The time T is given by the relation

$$\begin{aligned} T &= \frac{distance}{speed} \\ &= \frac{2\pi r}{r\omega} \\ &= \frac{2\pi}{\omega} \end{aligned} \qquad (6.5.1)$$

Example 6.5.1. *The wheel of a car has a radius of* 0.30 *m and it being rotated at* 15 *revolutions per second on a tire-balancing machine. Determine the angular speed and the speed at which the wheel moves at the outer edge.*

Solution The given data is

$$\begin{aligned} r &= 0.30\ m \\ f &= 15\ Hz \end{aligned}$$

The time period is

$$\begin{aligned} T &= \frac{1}{f} = \frac{1}{15} \\ &= 0.0667\ s \end{aligned}$$

The angular speed is

$$\begin{aligned} \omega &= \frac{2\pi}{T} = \frac{2\pi}{0.0667} \\ &= 94.274\ m/s \end{aligned}$$

And the speed at which the outer edge of the wheel moving is

$$\begin{aligned} v &= \frac{2\pi r}{T} = \frac{2\pi(0.3)}{0.0667} \\ &= 28.274\ m/s \end{aligned}$$

Example 6.5.2. ***The Simple Pendulum***

From (5.4.44), the equation of motion of a simple pendulum is

$$\ddot{\theta} = -\omega^2 \sin\theta$$

with

$$\omega = \sqrt{\frac{g}{l}} \tag{6.5.2}$$

is the frequency of oscillation. The angle θ is defined with respect to the equilibrium position. When $\theta > 0$, the bob has moved to the right, and when $\theta < 0$, the bob has moved to the left. The object will move in a circular arc centered at the pivot point in the absence of any dissipation due to air resistance or frictional forces acting at the pivot. When there is small oscillation, *i.e.* θ is very very small, we can write $\sin\theta \cong \theta$, then above equation can be written as

$$\ddot{\theta} = -\omega^2\theta$$

which now represents simple harmonic motion.

Example 6.5.3. *A simple pendulum has a period of* 2.50 *s.*

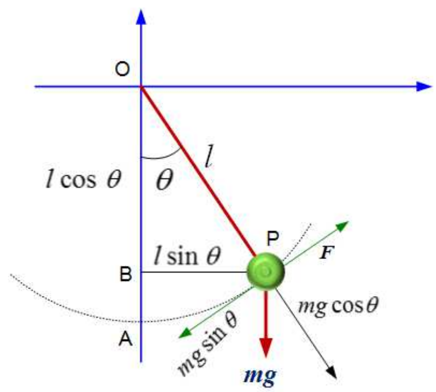


Figure 6.11: Simple Pendulum

a) *What is the angular frequency?*

b) *What is its length?*

c) *What would be its angular frequency on the Moon, if* $g_{Moon} = 1.67 m/s^2$*?*

d) *What would be its period on the Moon?*

Solution The given data is

$$T = 2.50\ s$$
$$g = 9.80\ m/s^2$$

a) The angular frequency is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2.5} = 2.5133\ rad/s$$

b) The length of the pendulum is

Since the angular frequency is

$$\omega = \sqrt{\frac{g}{l}}$$

or the length is

$$l = \frac{g}{\omega^2} = \frac{9.8}{(2.5133)^2} = 1.5515\ m$$

c) Its angular frequency on the Moon is

If we take this pendulum on the Moon, it will oscillate under Moon's acceleration of gravity with same length, so its angular frequency is

$$\begin{aligned}\omega &= \sqrt{\frac{g}{l}} = \sqrt{\frac{1.67}{(1.5515)}} \\ &= 1.0375\ rad/s\end{aligned}$$

d) Its period on the Moon is

$$\begin{aligned}T &= \frac{2\pi}{\omega} = \frac{2\pi}{1.0375} \\ &= 6.056 \cong 6.06\ s\end{aligned}$$

6.5.2 Damped Oscillatory Motion

In many real situations, there is some damping, or loss of mechanical energy, which is dissipated as heat. In our example spring mass system, damping may be caused by friction between the block and the surface. Also this damping may be that the constant bending and stretching of the spring produces heat. In any case, in addition to the restoring force kx, there is also a damping force that is proportional to a power of the instantaneous velocity of the particle. In this discussion, we shall assume that power one. Then the magnitude of this damping force is

$$F = -b\dot{x}$$

Where $b > 0$ is damping constant and negative sign indicates that the damping force acts in a direction opposite to the direction of motion. Then by Newton's second law of motion, the equation of motion is

$$\begin{aligned}F &= -kx - b\dot{x} \\ m\ddot{x} + b\dot{x} + kx &= 0 \qquad (6.5.3)\end{aligned}$$

(6.5.3) is second order homogenous linear differential equation with constant coefficients. Its linear standard form is

$$\begin{aligned}\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x &= 0 \\ \ddot{x} + \beta\dot{x} + \omega^2 x &= 0 \qquad (6.5.4)\end{aligned}$$

with $\beta = \frac{b}{m}$ is damping constant. Let

$$x = e^{mt} \qquad (6.5.5)$$

be the solution of (6.5.4). Then its characteristic equation is

$$d^2 + \beta d + \omega^2 = 0 \tag{6.5.6}$$

(6.5.6) has roots

$$d = \frac{-\beta \pm \sqrt{\beta^2 - 4\omega^2}}{2} \tag{6.5.7}$$

The nature of the roots depends on the value $\beta^2 - 4\omega^2$, and we thus classify the damping on this basis.

Case 1. If $\beta^2 - 4\omega^2 = 0$, the system has critical damping.

Case 2. If $\beta^2 - 4\omega^2 < 0$, the system is under damped has light damping.

Case 3. If $\beta^2 - 4\omega^2 > 0$, the system is over damped or has heavy damping.

Case 1. **Critical damping**

When $\beta^2 - 4\omega^2 = 0$, the characteristic equation (6.5.6) has equal roots:

$$d = -\frac{\beta}{2}$$

and the solution is

$$\begin{aligned} x(t) &= A_1 e^{-(\beta/2)t} + A_2 t e^{-(\beta/2)t} \\ &= (A_1 + A_2 t)\, e^{-(\beta/2)t} \end{aligned} \tag{6.5.8}$$

The motion is aperiodic (non-periodic) and non-oscillatory. Taking limit $t \to \infty$ of (6.5.8)

$$\begin{aligned} \lim_{t\to\infty} x(t) &= \lim_{t\to\infty} (A_1 + A_2 t)\, e^{-(\beta/2)t} \\ &= 0 \qquad \text{(by L'Hospital's rule)} \end{aligned} \tag{6.5.9}$$

Thus $x \to 0$ as $t \to \infty$. Hence the motion decays with time as shown in Fig. 6.12.

Case 2. **Light damping**

When $\beta^2 - 4\omega^2 < 0$, the characteristic equation (6.5.6) has imaginary roots:

$$d = \frac{1}{2}(-\beta \pm i\gamma)$$

with $-\gamma^2 = \beta^2 - 4\omega^2$ and the solution is

$$x(t) = e^{-(\beta/2)t}(B_1 \cos\gamma t + B_2 \sin\gamma t) \tag{6.5.10}$$

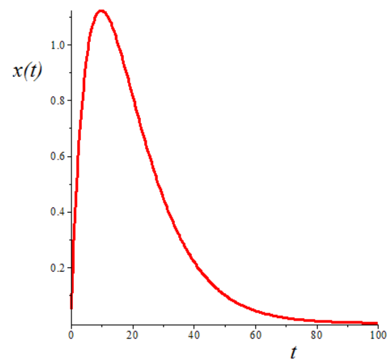


Figure 6.12: Critical damping

Set the constants $B_1 = B\sin\phi$ and $B_2 = B\cos\phi$, then (6.5.10) is

$$\begin{aligned} x(t) &= Be^{-(\beta/2)t}(\sin\phi\cos\gamma t + \cos\phi\sin\gamma t) \\ &= Be^{-(\beta/2)t}\left(\sin\left(\phi + \gamma t\right)\right) \end{aligned} \tag{6.5.11}$$

(6.5.11) represents simple harmonic motion with amplitude

$$A = Be^{-(\beta/2)t} \tag{6.5.12}$$

Taking limit $t \to \infty$ of (6.5.12)

$$\begin{aligned} \lim_{t\to\infty} A &= \lim_{t\to\infty} Be^{-(\beta/2)t} \\ &= 0 \qquad \text{as} \quad k > 0 \end{aligned} \tag{6.5.13}$$

Thus the amplitude decays with time and is damping. Consequently $x \to 0$ as $t \to \infty$. Hence the motion decays with time as shown in Fig. 6.13. This motion is called **damped oscillatory motion**.

Case 3. **Heavy damping**

When $\beta^2 - 4\omega^2 > 0$, the characteristic equation (6.5.6) has negative real and distinct roots:

$$d = -a, -b$$

with

$$\begin{aligned} -a = d_1 &= \frac{-\beta + \sqrt{\beta^2 - 4\omega^2}}{2} \\ -b = d_2 &= \frac{-\beta - \sqrt{\beta^2 - 4\omega^2}}{2} \end{aligned}$$

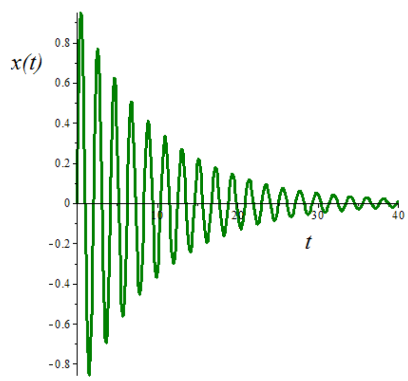


Figure 6.13: Light damping

and the solution is

$$x(t) \quad = \quad C_1 e^{-at} + C_2 e^{-bt} \tag{6.5.14}$$

This motion is also aperiodic and non-oscillatory. Taking limit $t \to \infty$ of (6.5.14)

$$\begin{aligned} \lim_{t\to\infty} x(t) &= \lim_{t\to\infty} \left(C_1 e^{-at} + C_2 e^{-bt} \right) \\ &= 0 \end{aligned} \tag{6.5.15}$$

Thus $x \to 0$ as $t \to \infty$. Hence the motion decays with time as shown in Fig. 6.14. The three damping cases, collectively are shown in Fig. 6.15.

Example 6.5.4. *A block of mass* 500 g *is fastened to a spring whose spring constant is* 2 N/m*. The block is pulled a distance one meter from its equilibrium position on a surface and is released with velocity* 1 m/s*. Assuming* 5 *is the damping constant of the surface. Find*

a) the angular frequency.

b) the restoring force.

c) the damping force function.

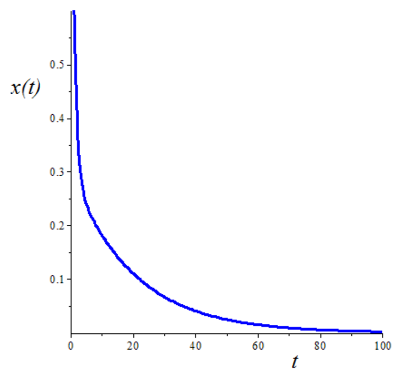


Figure 6.14: Heavy damping

d) the nature of the damping.

e) the path of motion.

f) the time for which the function has a maximum.

g) the amplitude.

Solution In this problem, we have

$$\begin{aligned} m &= 500\ g = 0.5\ kg \\ k &= 2\ N/m \\ \beta &= 5 \end{aligned}$$

And the initial data is (at $t = 0$)

$$\begin{aligned} x(0) &= 1\ m \\ v(0) = \dot{x}(0) &= 1\ m/s \end{aligned}$$

a) the angular frequency.

First of all we will find its angular frequency, for this use (6.1.3)

$$\begin{aligned} \omega &= \sqrt{\frac{k}{m}} \\ &= \sqrt{\frac{2}{0.5}} \\ &= 2\ rad/s \end{aligned}$$

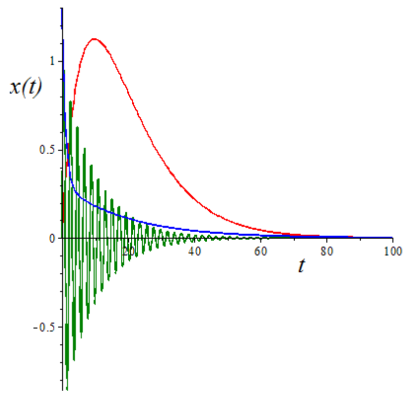


Figure 6.15: All three damping oscillations.

b) The restoring force is

$$F_r = kx = 2x$$

c) For damping function, first find b

$$\begin{aligned} b &= \beta m \\ &= 5(0.5) = 2.5 \end{aligned}$$

Now the damping force function is

$$F_d = 2.5v$$

d) The nature of the damping is

$$\begin{aligned} \beta^2 - 4\omega^2 &= 25 - 4(4) \\ &= 9 > 0 \end{aligned}$$

The damping is heavy damping.

e) The path of motion is

Using (6.5.4), the equation of motion is

$$\begin{aligned} \ddot{x} + \beta\dot{x} + \omega^2 x &= 0 \\ \ddot{x} + 5\dot{x} + 4x &= 0 \end{aligned} \quad (6.5.16)$$

Then its characteristic equation is

$$\begin{aligned} d^2 + \beta d + \omega^2 &= 0 \\ d^2 + 5d + 4 &= 0 \end{aligned} \qquad (6.5.17)$$

(6.5.17) has roots

$$d = -1 \text{ and } -4 \qquad (6.5.18)$$

Using (6.5.14), the solution is

$$x(t) = C_1 e^{-t} + C_2 e^{-4t} \qquad (6.5.19)$$

Using initial condition $x(0) = 1$, implies that

$$C_1 + C_2 = 0 \qquad (6.5.20)$$

Differentiate (6.5.19) with respect to t

$$\dot{x}(t) = -C_1 e^{-t} - 4C_2 e^{-4t} \qquad (6.5.21)$$

Using initial condition $\dot{x}(0) = 1$, (6.5.21) implies that

$$-C_1 - 4C_2 = 0 \qquad (6.5.22)$$

Solving (6.5.20) and (6.5.40), we have

$$\begin{aligned} C_1 &= \frac{5}{3} \\ C_2 &= -\frac{2}{3} \end{aligned}$$

The particular solution is

$$x(t) = \frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t} \qquad (6.5.23)$$

And the graphical representation is given in Fig. 6.16

f) the time for which the function has a maximum.

We have to find the time when the block is at the amplitude position. At that position, the velocity of the block is zero. Taking time derivative of (6.5.23), we have

$$\dot{x}(t) = -\frac{5}{3}e^{-t} + \frac{8}{3}e^{-4t} \qquad (6.5.24)$$

Setting $\dot{x}(t) = 0$, we have

$$-\frac{5}{3}e^{-t} + \frac{8}{3}e^{-4t} = 0$$

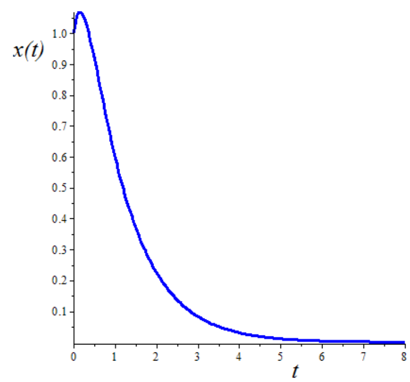


Figure 6.16: Heavy damping.

or

$$\begin{aligned} e^{3t} &= \frac{8}{5} \\ t &= \frac{1}{3}\ln\left(\frac{8}{5}\right) \\ &= 0.157s \end{aligned} \tag{6.5.25}$$

g) The amplitude

The position function is maximum at $t = 0.157$ if

$$\ddot{x}(0.157) \quad < \quad 0$$

Taking time derivative of (6.5.24), we have

$$\begin{aligned} \ddot{x}(t) &= \frac{5}{3}e^{-t} - \frac{32}{3}e^{-4t} \\ \ddot{x}(0.157) &= \frac{5}{3}e^{-0.157} - \frac{32}{3}e^{-4(0.157)} \\ &= -4.267 < 0 \end{aligned}$$

Thus position function has maxima at $t = 0.157$ (see Fig 6.16)
Using (6.5.25) in (6.5.23)

$$\begin{aligned} x(0.157) &= \frac{5}{3}e^{-0.157} - \frac{2}{3}e^{-4(0.157)} \\ x_{max} &= 1.069\ m \end{aligned} \tag{6.5.26}$$

Example 6.5.5. *A block of mass* 250 g *is fastened to a spring whose spring constant is* 4 N/m *and can move on a surface. The block is released from its equilibrium position to move towards left with a velocity* 3 m/s *. Assuming the damping force numerically equals to two times the instantaneous velocity acts on the system. Determine the extremum of its motion.*

Solution In this problem, we have

$$\begin{aligned} m &= 250\ g = 0.25\ kg \\ k &= 4\ N/m \\ F_d &= 2v = 2\dot{x} \end{aligned}$$

And the initial data is (at $t = 0$)

$$\begin{aligned} x(0) &= 0\ m \\ v(0) = \dot{x}(0) &= -3\ m/s \end{aligned}$$

The restoring force is

$$F_r = kx = 4x$$

Then by Newton's second law of motion, the equation of motion is

$$\begin{aligned} F &= -F_d - F_r \\ m\ddot{x} &= -2\dot{x} - 4x \\ 0.25\ddot{x} &= -2\dot{x} - 4x \\ \ddot{x} + 8\dot{x} + 16x &= 0 \end{aligned} \tag{6.5.27}$$

(6.5.27) is the equation of motion. From it, we have

$$\begin{aligned} \beta &= 8 \\ \omega &= 4 \end{aligned}$$

The nature of the damping is determined as

$$\begin{aligned} \beta^2 - 4\omega^2 &= 64 - 4(16) \\ &= 0 \end{aligned}$$

The system is under critical damping.
Considering (6.5.27), the characteristic equation is

$$\begin{aligned} d^2 + \beta d + \omega^2 &= 0 \\ d^2 + 8d + 16 &= 0 \\ (d+4)^2 &= 0 \end{aligned} \tag{6.5.28}$$

(6.5.28) has roots

$$d = -4 \text{ and } -4$$

Using (6.5.8), the solution is

$$\begin{aligned} x(t) &= (A_1 + A_2 t)\, e^{-(\beta/2)t} \\ &= (A_1 + A_2 t)\, e^{-4t} \end{aligned} \tag{6.5.29}$$

Using initial condition $x(0) = 0$, implies that

$$A_1 = 0$$

Then we are left with

$$x(t) = A_2 t e^{-4t} \tag{6.5.30}$$

Differentiate (6.5.30) with respect to t

$$\dot{x}(t) = A_2\, (1 - 4t)\, e^{-4t} \tag{6.5.31}$$

Using initial condition $\dot{x}(0) = -3$, (6.5.31) implies that

$$A_2 = -3 \tag{6.5.32}$$

The particular solution is

$$x(t) - 3te^{-4t} \tag{6.5.33}$$

Its graphical representation is given in Fig. 6.18.

The time for which the function has an extremum. We have to find the time when the block is at the amplitude position. At that position, the velocity of the block is zero. Taking time derivative of (6.5.33), we have

$$\dot{x}(t) = -3(1 - 4t)e^{-4t} \tag{6.5.34}$$

Setting $\dot{x}(t) = 0$, we have

$$-3(1 - 4t)e^{-4t} = 0$$

or

$$t = \frac{1}{4} = 0.25s \tag{6.5.35}$$

The position function has an extremum at $t = 0.25$. For maxima, it has to follow

$$\ddot{x}(0.25) < 0$$

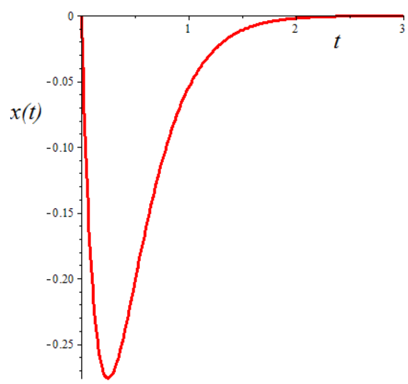


Figure 6.17: Critical damping.

and for minima it has to follow

$$\ddot{x}(0.25) \quad > \quad 0$$

Taking time derivative of (6.5.34), we have

$$\begin{aligned}\ddot{x}(t) &= 24(1-2t)e^{-4t}\\ \ddot{x}(0.25) &= 24(1-2(0.25))e^{-4(0.25)}\\ &= 4.4146 > 0\end{aligned}$$

Thus position function has minima at $t = 0.25$ (see Fig 6.18)
Using (6.5.35) in (6.5.33)

$$\begin{aligned}x(0.25) &= -3te^{-4(0.25)}\\ x_{min} &= -0.276\ m\end{aligned}$$

The particle is towards left, at a distance $0.276m$, from equilibrium position, and the amplitude of the motion is

$$x_{max} \quad = \quad 0.276\ m$$

Example 6.5.6. *A block of mass* 500 g *is fastened to a spring whose spring constant is* 5 N/m *and can move on a surface. The block is pulled a distance* 2 cm *towards left from its equilibrium position on a surface and released from rest. Assuming the damping force* *numerically equals to the instantaneous velocity acts on the system. Determine its path of motion.*

Solution In this problem, we have

$$\begin{aligned} m &= 500\ g = 0.5\ kg \\ k &= 5\ N/m \\ F_d &= v = \dot{x} \end{aligned}$$

And the initial data is (at $t = 0$)

$$\begin{aligned} x(0) &= -2\ m \\ v(0) = \dot{x}(0) &= 0m/s \end{aligned}$$

The restoring force is

$$F_r = kx = 5x$$

Then by Newton's second law of motion, the equation of motion is

$$\begin{aligned} F &= -F_d - F_r \\ m\ddot{x} &= -\dot{x} - 5x \\ 0.5\ddot{x} &= -\dot{x} - 5x \\ \ddot{x} + 2\dot{x} + 10x &= 0 \end{aligned} \tag{6.5.36}$$

(6.5.36) is the equation of motion. From it, we have

$$\begin{aligned} \beta &= 2 \\ \omega &= \sqrt{10}\ rad/s \end{aligned}$$

The nature of the damping is determined as

$$\begin{aligned} \beta^2 - 4\omega^2 &= 4 - 4(10) \\ &= -36 < 0 \end{aligned}$$

The system is under light damping.
Considering (6.5.36), the characteristic equation is

$$\begin{aligned} d^2 + \beta d + \omega^2 &= 0 \\ d^2 + 2d + 10 &= 0 \end{aligned}$$

(6.5.37) has roots

$$d = -1 \pm 3i$$

Using (6.5.10), the solution is

$$\begin{aligned} x(t) &= e^{-(\beta/2)t}(B_1 \cos \gamma t + B_2 \sin \gamma t) \\ &= e^{-t}(B_1 \cos 3t + B_2 \sin 3t) \end{aligned} \tag{6.5.37}$$

Using initial condition $x(0) = -2$, implies that

$$B_1 = -2$$

Then (6.5.37) takes the form

$$x(t) = e^{-t}(-2\cos 3t + B_2 \sin 3t) \tag{6.5.38}$$

Differentiate (6.5.38) with respect to t

$$\dot{x}(t) = e^{-t}(2\cos 3t - B_2 \sin 3t + 6\sin 3t + 3B_2 \cos 3t) \tag{6.5.39}$$

Using initial condition $\dot{x}(0) = 0$, (6.5.39) implies that

$$B_2 = -\frac{2}{3}$$

Hence the solution (particular solution) is

$$x(t) = e^{-t}(-2\cos 3t - \frac{2}{3}\sin 3t) \tag{6.5.40}$$

Its graphical representation is given in Fig. 6.18.

6.5.3 Driven or Forced Oscillatory Motion

If an external force $f(t)$ is acting on a vibrating mass, executing harmonic motion, the motion is driven or forced oscillatory motion. It has the following two cases.

a) Undamped Forced Oscillatory Motion.

b) Damped Forced Oscillatory Motion.

a) **Undamped Forced Oscillatory Motion.**

Consider spring mass system with $f(t)$, a driving force causing an oscillatory motion of the support of the spring. Let there is no damping force, then by Newton's second law of motion, its equation of motion is

$$\begin{aligned} F &= -F_r + f(t) \\ m\ddot{x} + kx &= f(t) \\ \ddot{x} + \frac{k}{m}x &= \frac{f(t)}{m} \\ \ddot{x} + \omega^2 x &= F(t) \end{aligned} \tag{6.5.41}$$

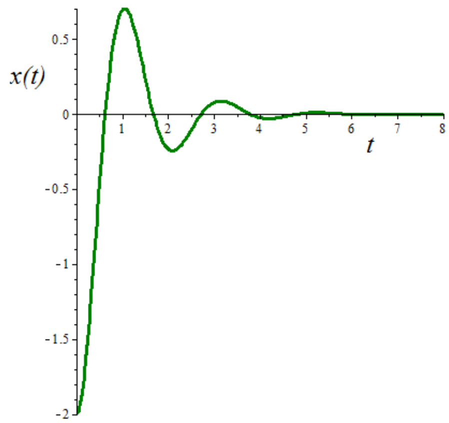


Figure 6.18: Light damping.

With $\omega^2 = \frac{k}{m}$ and $F(t) = \frac{f(t)}{m}$
We usually use a sinusoidal driving force.

$$F(t) = F_0 \sin \gamma t \quad \text{or} \quad = F_0 \cos \gamma t$$

with frequency $\gamma \neq \omega$.
Initially the system is resting at equilibrium position. Next an external force $F(t) = F_0 \sin \gamma t$ is applied and it executes oscillatory motion. Then (6.5.41) takes the form

$$\ddot{x} + \omega^2 x = F_0 \sin \gamma t \tag{6.5.42}$$

Also we have initial conditions:

$$\begin{aligned} x(0) &= 0 \\ \dot{x}(0) &= 0 \end{aligned}$$

(6.5.42) is second order linear nonhomogeneous differential equation and has solution

$$x(t) = x_c(t) + x_p(t) \tag{6.5.43}$$

The complementary function $x_c(t)$ is given in (6.2.16) as

$$x_c(t) = C_1 \cos \omega t + C_2 \sin \omega t$$

For Particular Integral x_p, we proceed as follows

$$x_p(t) = A_1 \cos \gamma t + A_2 \sin \gamma t \tag{6.5.44}$$

Taking first and second time derivatives of (6.5.55)

$$\begin{aligned}\dot{x}_p(t) &= -A_1\gamma\sin\gamma t + A_2\gamma\cos\gamma t \\ \ddot{x}_p(t) &= -\gamma^2\left(A_1\cos\gamma t + A_2\sin\gamma t\right)\end{aligned} \tag{6.5.45}$$

Using (6.5.55) and (6.5.45) in (6.5.42), we have

$$-\gamma^2\left(A_1\cos\gamma t + A_2\sin\gamma t\right) + \omega^2\left(A_1\cos\gamma t + A_2\sin\gamma t\right) = F_0\sin\gamma t$$

$$A_1\left(\omega^2-\gamma^2\right)\cos\gamma t + A_2\left(\omega^2-\gamma^2\right)\sin\gamma t = F_0\sin\gamma t \tag{6.5.46}$$

Comparing coefficients of $\cos\gamma t$ and $\sin\gamma t$, we have

$$A_1\left(\omega^2-\gamma^2\right) = 0 \tag{6.5.47}$$

$$A_2\left(\omega^2-\gamma^2\right) = F_0 \tag{6.5.48}$$

Since $\gamma \neq \omega$, (6.5.47) implies that

$$A_1 = 0$$

and (6.5.48) implies that

$$A_2 = \frac{F_0}{\omega^2-\gamma^2}$$

Using these coefficients, (6.5.55) implies that

$$x_p(t) = \frac{F_0}{\omega^2-\gamma^2}\sin\gamma t \tag{6.5.49}$$

Using (6.2.16) and (6.5.49) in (6.5.64), the general solution is

$$x(t) = C_1\cos\omega t + C_2\sin\omega t + \frac{F_0}{\omega^2-\gamma^2}\sin\gamma t \tag{6.5.50}$$

Using initial conditions, a particular solution is calculated. Using initial condition $x(0) = 0$, (6.5.52) implies that

$$C_1 = 0$$

The time derivative of (6.5.52) is

$$\dot{x}(t) = -C_1\omega\sin\omega t + C_2\omega\cos\omega t + \frac{F_0\gamma}{\omega^2-\gamma^2}\cos\gamma t \tag{6.5.51}$$

Using initial condition $\dot{x}(0) = 0$, (6.5.51) implies that

$$C_2 = -\frac{F_0\gamma}{\omega\left(\omega^2-\gamma^2\right)}$$

Using these coefficients, a particular solution is

$$\begin{aligned} x(t) &= -\frac{F_0\gamma}{\omega\left(\omega^2-\gamma^2\right)}\sin\omega t+\frac{F_0}{\omega^2-\gamma^2}\sin\gamma t \\ &= -\frac{F_0}{\omega\left(\omega^2-\gamma^2\right)}\left(\gamma\sin\omega t+\omega\sin\gamma t\right) \end{aligned} \tag{6.5.52}$$

Example 6.5.7. *A* 128 *lb weight is attached to a spring having a spring constant of* $64lb/ft$*. The weight is pulled a distance* 6 *inches towards left, from its equilibrium position. From there, it is started in motion with no initial velocity by applying an external force* $F(t) = 8sin4t$ *to the weight. Assuming no air resistance, find the subsequent motion of the weight.*

Solution In this problem, we have

$$\begin{aligned} W &= 128\ lb \\ k &= 64lb/ft \\ F(t) &= 8\sin 4t \end{aligned}$$

And the initial data is (at $t = 0$)

$$\begin{aligned} x(0) &= -6\ in = -0.5\ ft \\ v(0) = \dot{x}(0) &= 0ft/s \end{aligned}$$

The mass of the body is

$$\begin{aligned} m &= \frac{W}{g} = \frac{128}{32} \\ &= 4\ slug \end{aligned}$$

Then

$$F(t) = \frac{8}{4}\sin 4t = 2\sin 4t$$

And the angular frequency frequency is calculated by using (6.1.3)

$$\begin{aligned} \omega &= \sqrt{\frac{k}{m}} \\ &= \sqrt{\frac{64}{4}} = \sqrt{16} \\ &= 4\ rad/s \end{aligned}$$

Using (6.5.42), the equation of motion is

$$\begin{aligned} \ddot{x}+\omega^2 x &= F_0\sin\gamma t \\ \ddot{x}+16x &= 2\sin 4t \end{aligned} \tag{6.5.53}$$

(6.5.53) is second order linear nonhomogeneous differential equation and has solution

$$x(t) = x_c(t) + x_p(t)$$

The complementary function $x_c(t)$ is given in (6.2.16) as

$$\begin{aligned} x_c(t) &= C_1 \cos \omega t + C_2 \sin \omega t \\ &= C_1 \cos 4t + C_2 \sin 4t \end{aligned} \tag{6.5.54}$$

The assumption for Particular Integral

$$x_p(t) = A_1 \cos 4t + A_2 \sin 4t$$

will fail, since it is already appearing in x_c. Next, we will try

$$x_p(t) = A_1 t \cos 4t + A_2 t \sin 4t \tag{6.5.55}$$

Taking first and second time derivatives of (6.5.55)

$$\begin{aligned} \dot{x}_p(t) &= A_1 \cos 4t + A_2 \sin 4t - 4A_1 t \sin 4t + 4A_2 t \cos 4t \\ \ddot{x}_p(t) &= -8A_1 \sin 4t + 8A_2 \cos 4t - 16t\,(A_1 \cos 4t + A_2 \sin 4t) \end{aligned} \tag{6.5.56}$$

Using (6.5.55) and (6.5.56) in (6.5.53), we have

$$\begin{aligned} -8A_1 \sin 4t + 8A_2 \cos 4t &- 16t\,(A_1 \cos 4t + A_2 \sin 4t) + \\ 16t\,(A_1 \cos 4t + A_2 \sin 4t) &= 2 \sin 4t \end{aligned}$$

$$-8A_1 \sin 4t + 8A_2 \cos 4t = 2 \sin 4t \tag{6.5.57}$$

Comparing coefficients of $\cos 4t$ and $\sin 4t$, we have

$$\begin{aligned} -\frac{1}{8} A_1 &= 2 \\ A_2 &= 0 \end{aligned} \tag{6.5.58}$$

(6.5.58) implies that

$$A_1 = -\frac{1}{4}$$

Using these coefficients, (6.5.55) implies that

$$x_p(t) = -\frac{1}{4} t \cos 4t \tag{6.5.59}$$

Using (6.5.54) and (6.5.59), the general solution is

$$x(t) = C_1 \cos 4t + C_2 \sin 4t - \frac{1}{4} t \cos 4t \tag{6.5.60}$$

Using initial conditions, a particular solution is calculated. Using initial condition $x(0) = -0.5$, (6.5.60) implies that

$$C_1 = -0.5$$

The time derivative of (6.5.60) is

$$\dot{x}(t) = 0.5\sin 4t + 4C_2\cos 4t + t\sin 4t - \frac{1}{4}\cos 4t \qquad (6.5.61)$$

Using initial condition $\dot{x}(0) = 0$, (6.5.61) implies that

$$C_2 = \frac{1}{16}$$

Using these coefficients, a particular solution is

$$x(t) = -\frac{1}{2}\cos 4t + \frac{1}{16}\sin 4t - \frac{1}{4}t\cos 4t \qquad (6.5.62)$$

Its graphical representation is given in Fig. 6.19.

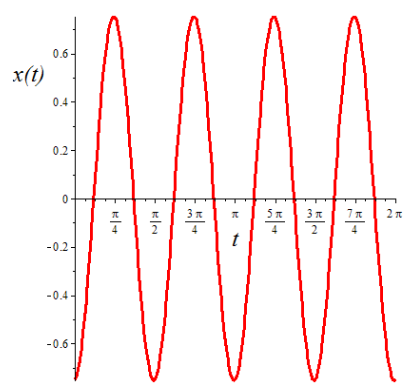


Figure 6.19: Undamped Forced oscillatory motion.

b) **Damped Forced Oscillatory Motion.**

During damped oscillatory motion, the oscillations eventually die away due to frictional energy losses. To maintain the motion of a damped oscillator, an external force $f(t)$ is applied to the system. Let the damping force is proportional to instantaneous velocity, then by Newton's second law of motion, its equation of motion is

$$\begin{aligned} F &= -F_r - F_d + f(t) \\ m\ddot{x} + b\dot{x} + kx &= f(t) \\ \ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x &= \frac{f(t)}{m} \\ \ddot{x} + \beta\dot{x} + \omega^2 x &= F(t) \end{aligned}$$

Example 6.5.8. *A body* 200 *g mass is attached to a spring having a spring constant of* $2N/m$*. The weight is pulled a distance* 50 *cm towards right, from its equilibrium position. From there, it is started in motion with no initial velocity by applying an external force* $f(t) = 5cos4t$ *to the mass. Assuming there is* 1.2 *v m/s a resistance force, find the subsequent motion of the mass.*

Solution In this problem, we have

$$\begin{aligned} m &= 200\ g = \frac{1}{5}kg \\ k &= 2N/m \\ f(t) &= 5\cos 4t \end{aligned}$$

And the initial data is (at $t = 0$)

$$\begin{aligned} x(0) &= 50\ cm = 0.5\ m \\ v(0) = \dot{x}(0) &= 0m/s \end{aligned}$$

Three forces are acting on the body. These forces are

$$\begin{aligned} F_r &= kx = 2x\ m \\ F_r &= k\dot{x} = 1.2\dot{x}\ m/s \\ f(t) &= 5\cos 4t \end{aligned}$$

Then by Newton's second law of motion, its equation of motion is

$$\begin{aligned} F &= -F_r - F_d + f(t) \\ m\ddot{x} + b\dot{x} + kx &= f(t) \\ \frac{1}{5}\ddot{x} + 1.2\dot{x} + 2x &= 5\cos 4t \\ \ddot{x} + 6\dot{x} + 10x &= 25\cos 4t \end{aligned} \tag{6.5.63}$$

(6.5.63) is second order linear nonhomogeneous differential equation and has solution

$$x(t) = x_c(t) + x_p(t)$$

The complementary function $x_c(t)$ is given by considering homogeneous part of (6.5.63)

$$\ddot{x} + 6\dot{x} + 10x = 0 \qquad (6.5.64)$$

(6.5.64) has characteristic equation

$$d^2 + 6d + 10 = 0 \qquad (6.5.65)$$

(6.5.65) has roots

$$d = -3 \pm i$$

The complementary function $x_c(t)$ is

$$x_c(t) = e^{-3t}\left(C_1 \cos t + C_2 \sin t\right) \qquad (6.5.66)$$

Using undetermined coefficients method, the assumption for Particular Integral is

$$x_p(t) = A_1 \cos 4t + A_2 \sin 4t \qquad (6.5.67)$$

Taking first and second time derivatives of (6.5.67)

$$\dot{x}_p(t) = -4A_1 \sin 4t + 4A_2 \cos 4t \qquad (6.5.68)$$
$$\ddot{x}_p(t) = -16\left(A_1 \cos 4t + A_2 \sin 4t\right) \qquad (6.5.69)$$

Using (6.5.67), (6.5.68) and (6.5.69) in (6.5.63), we have

$$\begin{aligned} -16\left(A_1 \cos 4t + A_2 \sin 4t\right) &- 4A_1 \sin 4t + 4A_2 \cos 4t + \\ 10\left(A_1 \cos 4t + A_2 \sin 4t\right) &= 25 \sin 4t \end{aligned}$$

$$\left(-6A_1 + 24A_2\right)\cos 4t + \left(-24A_1 - 6A_2\right)\sin 4t = 25 \sin 4t \qquad (6.5.70)$$

Comparing coefficients of $\cos 4t$ and $\sin 4t$, we have

$$-6A_1 + 24A_2 = 25 \qquad (6.5.71)$$
$$-24A_1 - 6A_2 = 0 \qquad (6.5.72)$$

Solving (6.5.71) and (6.5.72)

$$A_1 = -\frac{25}{102}$$

$$A_2 = \frac{50}{51}$$

Using these coefficients, (6.5.67) implies that

$$x_p(t) = -\frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t \tag{6.5.73}$$

Using (6.5.66) and (6.5.73), the general solution is

$$x(t) = e^{-3t}\left(C_1 \cos t + C_2 \sin t\right) - \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t \tag{6.5.74}$$

Using initial conditions, a particular solution is calculated. Using initial condition $x(0) = 0.5$, (6.5.74) implies that

$$C_1 = \frac{38}{51}$$

The time derivative of (6.5.74) is

$$\begin{aligned} \dot{x}(t) &= -3e^{-3t}\left(C_1 \cos t + C_2 \sin t\right) + e^{-3t}\left(-C_1 \sin t + C_2 \cos t\right) \\ &+ \frac{50}{51}\sin 4t + \frac{200}{51}\cos 4t \end{aligned} \tag{6.5.75}$$

Using initial condition $\dot{x}(0) = 0$, (6.5.61) implies that

$$C_2 = -\frac{86}{51}$$

Using these coefficients, a particular solution is

$$x(t) = e^{-3t}\left(\frac{38}{51}\cos t - \frac{86}{51}\sin t\right) - \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t \tag{6.5.76}$$

Its graphical representation is given in Fig. 6.20.

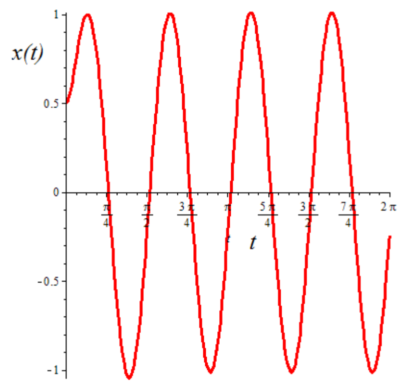


Figure 6.20: Damped Forced oscillatory motion.

Exercises

1. A block of weight 490 N is fastened to a spring whose spring constant is 1.2 N/m. The block is pulled a distance one meter from its equilibrium position on a surface and is released with velocity 3 m/s. Assuming 2 is the damping constant of the surface. Find

   a) the angular frequency.
   b) the restoring force.
   c) the damping force function.
   d) the nature of the damping.
   e) the path of motion.
   f) the time for which the function has a maximum.
   g) the amplitude.

2. A particle describing simple harmonic motion has speeds 6 m/s and 4 m/s when its distances from the center are 4 m and 4.5 m respectively. Find

   a) the angular frequency.
   b) its acceleration at these distances.
   c) the time period of motion.
   d) the amplitude of motion.

3. A particle describing simple harmonic motion has velocities $5ft/sec$ and $4ft/sec$.When its distances from the centre are $12ft$ and $13ft$ respectively find the time period of the motion

4. The maximum velocity that a particle executing simple harmonic motion of amplitude a attains.is v if it is disturbed in such a way its maximum velocity becomes nv find the change in the amplitude and the time period of the motion.

5. A point describes simple harmonic motion in such a way that its velocity and acceleration at point P are u and f respectively and the corresponding quantities at another point Q are v and g find the distance PQ.

6. If a point P moves with a velocity v given by

$$v^2 = n^2 \left(ax^2 + 2bx + c\right)$$

show that P executes a simple harmonic motion. Find the centre, the amplitude and the time period of the motion.

7. A particle describes simple harmonic motion with frequency N. If the greatest velocity is V, find the amplitude and the maximum value of the acceleration of the particle. Also show that the velocity v at a distance x from the centre of motion is given by $v = 2\pi\sqrt{a^2 - x^2}$, where a is the amplitude.

Chapter 7

Two Dimensional Projectile Motion

Two dimensional projectile motion is namely horizontal motion and vertical motion. These two motions are independent of each other. In cricket a bowler bowls a ball, a batsman hit a ball by bat and fielder throw a ball. In all activities we observe two dimensional projectile motion. This motion is simply known as projectile motion. The ball (object) executing this motion is called projectile. Here we will discuss this motion as:

a) Free projectile motion

b) Resisted projectile motion

First consider free projectile motion.

7.1 Free Projectile Motion

In this projectile motion, we assume that air resistance is negligible. The projectile is projected from a reference point making an oblique angle with some axis and is considered to move under gravity, . Path followed by a projectile is known as a trajectory. If the gravity is not present, the trajectory is a constant straight line. However, if the gravity is present, the trajectory is an arc of a parabola, thus gravity accelerates objects downwards (see Fig. 7.1).

The trajectory depends on the following factors:

a) Angle of projection

b) Projection velocity

c) Relative height of projection

We will discuss this motion in the following three categories.

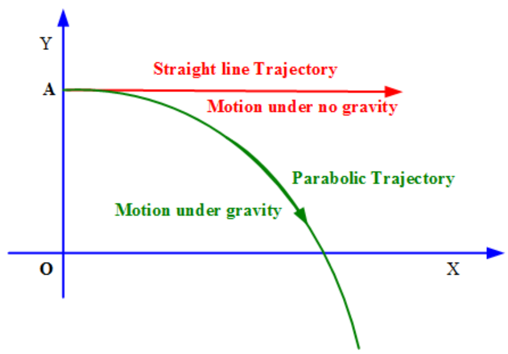


Figure 7.1: Projectile motion.

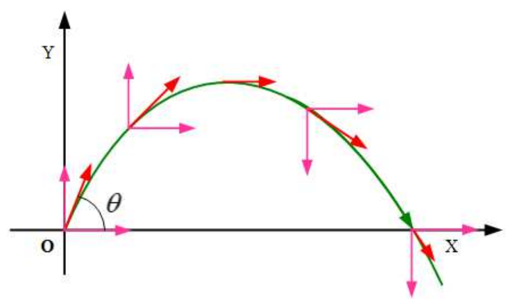


Figure 7.2: Horizontal and vertical components of projectile motion.

1. The projectile is projected from origin.
2. The projectile is projected from relative height from origin.
3. The projectile is projected from origin along inclined plane

7.2 Projectile Motion of a Projectile Projected from Origin

In this projectile motion the reference point is the origin. We will discuss it as following.

7.2.1 Path or Trajectory of a Projectile

Consider a cartesian plane as the vertical plane with x axis along horizontal and y axis along vertical. A particle of mass m is projected from origin O with a velocity $\vec{v}_0$, making an angle α with the horizontal. The point O is named as point of projection, the velocity $\vec{v}_0$ is the velocity of projection and the angle α is called angle of projection. In the absence of air resistance, the particle executes projectile motion. The initial velocity $\vec{v}_0$ can be written as

$$\vec{v}_0 = \vec{v}(0) = v_0 \cos\alpha\hat{i} + v_0 \sin\alpha\hat{j} \tag{7.2.1}$$

After time t, the particle is at position P, as shown in Fig. 7.15. Let

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

be its position vector. Clearly at $t = 0$

$$\begin{aligned} \vec{r}(0) &= \vec{0} \\ x(0)\hat{i} + y(0)\hat{j} &= 0\hat{i} + 0\hat{j} \end{aligned}$$

implies that

$$x(0) = 0 \tag{7.2.2}$$

and

$$y(0) = 0 \tag{7.2.3}$$

Next its velocity is

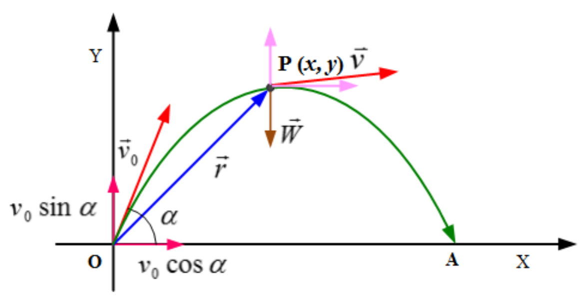


Figure 7.3: Projectile motion.

$$\vec{v}(t) = \dot{\vec{r}}(t) = \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j}$$

at $t = 0$

$$\vec{v}(0) = \dot{x}(0)\hat{i} + \dot{y}(0)\hat{j} \tag{7.2.4}$$

From (7.2.1) and (7.2.4), we can write

$$\dot{x}(0)\hat{i} + \dot{y}(0)\hat{j} = v_0 \cos\alpha\hat{i} + v_0 \sin\alpha\hat{j}$$

Then the initial horizontal scalar component of velocity is

$$\dot{x}(0) = v_0 \cos\alpha \tag{7.2.5}$$

And the initial vertical scalar component of velocity is

$$\dot{y}(0) = v_0 \sin\alpha \tag{7.2.6}$$

Finally its acceleration is

$$\vec{a}(t) = \ddot{\vec{r}}(t) = \ddot{x}(t)\hat{i} + \ddot{y}(t)\hat{j}$$

At P, the only force acting on the particle is force of gravity acting in the downward direction. Then by Newton's second law of motion its equation of motion is

$$\begin{aligned} \vec{F} &= \vec{W} \\ m\vec{a} &= -m\vec{g} \\ \ddot{x}\hat{i} + \ddot{y}\hat{j} &= -g\hat{j} \end{aligned} \tag{7.2.7}$$

From (7.2.7), the horizontal component of equation of motion is

$$\ddot{x}(t) = 0 \tag{7.2.8}$$

and the vertical component is

$$\ddot{y}(t) = -g \tag{7.2.9}$$

Integrating (7.2.8) with respect to t

$$\dot{x}(t) = A_1 \tag{7.2.10}$$

At $t = 0$, (7.2.10) becomes

$$\dot{x}(0) = A_1 \tag{7.2.11}$$

Using (7.2.5), (7.6.10) implies that

$$A_1 = v_0 \cos\alpha \tag{7.2.12}$$

Using (7.2.12), (7.2.10) becomes

$$\dot{x}(t) = v_0 \cos\alpha \tag{7.2.13}$$

(7.2.13) gives the horizontal scalar component of velocity of the particle at any time t. Integrating it with respect to t

$$x(t) = (v_0 \cos\alpha)t + B_1 \tag{7.2.14}$$

At $t = 0$, (7.2.14) becomes

$$x(0) = (v_0 \cos\alpha)(0) + B_1 \tag{7.2.15}$$

Using (7.2.2), (7.2.15) becomes

$$B_1 = 0 \tag{7.2.16}$$

Using (7.2.16), (7.2.14) becomes

$$x(t) = (v_0 \cos\alpha)t \tag{7.2.17}$$

(7.2.17) gives the horizontal component of position of the particle at any time t.
Next for vertical component integrate (7.2.9) with respect to t

$$\dot{y}(t) = -gt + A_2 \tag{7.2.18}$$

At $t = 0$, (7.2.18) becomes

$$\dot{y}(0) = A_2 \tag{7.2.19}$$

Using (7.2.6), (7.2.19) implies that

$$A_2 = v_0 \sin\alpha \tag{7.2.20}$$

Using (7.2.20), (7.2.18) becomes

$$\dot{y}(t) = -gt + v_0 \sin\alpha \tag{7.2.21}$$

(7.2.21) gives the vertical scalar component of velocity of the particle at any time t.
Integrating (7.2.21) with respect to t

$$y(t) = -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t + B_2 \tag{7.2.22}$$

At $t = 0$, (7.2.22) becomes

$$y(0) = -\frac{1}{2}g(0)^2 + (v_0 \sin\alpha)(0) + B_2 \tag{7.2.23}$$

Using (7.2.3), (7.2.23) becomes

$$B_2 = 0 \tag{7.2.24}$$

Using (7.2.24), (7.2.22) becomes

$$y(t) = -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t \tag{7.2.25}$$

(7.2.25) gives the vertical component of position of the particle at any time t.
From (7.2.17), we can find the time required to reach the particle at P as

$$t = \frac{x(t)}{v_0 \cos\alpha} \tag{7.2.26}$$

Using (7.2.26), (7.2.25) becomes

$$\begin{aligned} y(x) &= -\frac{1}{2}g\left(\frac{x}{v_0\cos\alpha}\right)^2 + x\tan\alpha \\ &= -\frac{g\sec^2\alpha}{2v_0^2}x^2 + x\tan\alpha \end{aligned} \tag{7.2.27}$$

(7.2.27) gives the path of the projectile of the particle at any time t.

7.2.2 Parabolic Trajectory

(7.2.27) can be written as

$$x^2 - \frac{2v_0^2\cos\alpha\sin\alpha}{g}x = -\frac{2v_0^2\cos^2\alpha}{g}y \tag{7.2.28}$$

Adding $\left(\frac{v_0^2\cos\alpha\sin\alpha}{g}\right)^2$ on both sides, then left hand side of (7.2.28) will become a complete square.

$$\left(x - \frac{v_0^2\cos\alpha\sin\alpha}{g}\right)^2 = -\frac{2v_0^2\cos^2\alpha}{g}\left(y - \frac{v_0^2\sin^2\alpha}{2g}\right) \tag{7.2.29}$$

Comparing (7.2.29) with the equation of the parabola

$$(x-h)^2 = -4p(y-k)$$

with (h,k) is the vertex, p is the distance from the vertex to the focus and the vertex to the directrix. The axis of the parabola is

$$x = h$$

equation of the directrix is

$$y = k$$

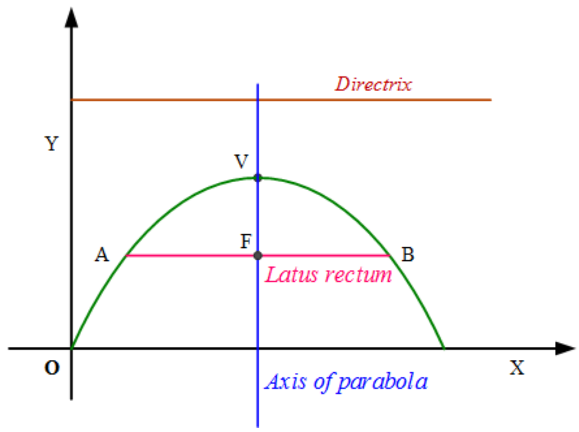


Figure 7.4: Projectile motion.

and the length of the latus rectum is $|4p|$.
We observe that (7.2.29) represents a parabola as shown in Fig. 7.4, therefore the path of the projectile is parabola with vertex

$$V = \left(\frac{v_0^2 \cos\alpha \sin\alpha}{g}, \frac{v_0^2 \sin^2\alpha}{2g}\right).$$

The axis of the parabola is

$$x = \frac{v_0^2 \cos\alpha \sin\alpha}{g},$$

equation of the directrix is

$$\begin{aligned} y &= \frac{v_0^2 \sin^2\alpha}{2g} + \frac{v_0^2 \cos^2\alpha}{2g} \\ &= \frac{v_0^2}{2g} \end{aligned}$$

and the length of the latus rectum is $\frac{2v_0^2 \cos^2\alpha}{g}$. The focus is a point $F(x_1, y_1)$ on the axis of the parabola with

$$\begin{aligned} x_1 &= \text{abscissa of the vertex } = \frac{v_0^2 \cos\alpha \sin\alpha}{g} \\ &= \frac{v_0^2}{2g} \sin 2\alpha \end{aligned}$$

and

$$\begin{aligned}
y_1 &= \text{ordinate of the vertex} - \frac{1}{4}(\text{length of the latus rectum}) \\
&= \frac{v_0^2 \sin^2\alpha}{2g} - \frac{1}{4}\left(\frac{2v_0^2\cos^2\alpha}{g}\right) \\
&= -\frac{v_0^2}{2g}\left(\cos^2\alpha - \sin^2\alpha\right) \\
&= -\frac{v_0^2}{2g}\cos 2\alpha
\end{aligned}$$

Hence the focus is

$$F = \left(\frac{v_0^2}{2g}\sin 2\alpha, -\frac{v_0^2}{2g}\cos 2\alpha\right)$$

Height of Directrix The directrix of a parabola is a line perpendicular to the axis of the parabola and is given by

$$\begin{aligned}
y &= \text{height of the vertex} + \frac{1}{4}(\text{length of the latus rectum}) \\
&= \frac{v_0^2 \sin^2\alpha}{2g} + \frac{1}{4}\left(\frac{2v_0^2\cos^2\alpha}{g}\right) \\
&= \frac{v_0^2}{2g}\left(\sin^2\alpha + \cos^2\alpha\right)
\end{aligned}$$

or

$$y_2 = \frac{v_0^2}{2g} \tag{7.2.30}$$

(7.2.30) gives the height of directrix of parabola.

Time of Flight As the particle is moving under gravity, so it will strike horizontal axis (or plane) after time t. At that time we have

$$\begin{aligned}
y &= 0 \\
-\frac{1}{2}gt^2 + (v_0\sin\alpha)t &= 0 \\
t\left(-\frac{1}{2}gt + (v_0\sin\alpha)\right) &= 0
\end{aligned}$$

Since $t \neq 0$, then

$$\left(-\frac{1}{2}gt + (v_0\sin\alpha)\right) = 0$$

or

$$t = t_r = \frac{2v_0\sin\alpha}{g} \tag{7.2.31}$$

(7.2.31) gives the time of flight of the projectile.
Horizontal Range Let the particle hits the x axis at A after projected from O. Then the distance $|\overline{OA}|$ is known as horizontal range and is calculated by using (7.2.31) in (7.2.17)

$$
\begin{aligned}
x(t) &= (v_0 \cos\alpha)\frac{2v_0 \sin\alpha}{g} \\
x_R &= \frac{v_0^2}{g}\sin 2\alpha
\end{aligned} \tag{7.2.32}
$$

(7.2.32) gives the horizontal range of the projectile.
Using trigonometric relation

$$\sin 2\alpha = \sin(\pi - 2\alpha),$$

(7.2.32) can also be written as

$$
\begin{aligned}
x_R &= \frac{v_0^2}{g}\sin(\pi - 2\alpha) \\
&= \frac{v_0^2}{g}\sin\left[2\left(\frac{\pi}{2} - \alpha\right)\right] \\
&= \frac{v_0^2}{g}\sin(\pi - 2\alpha) \\
&= \frac{v_0^2}{g}\sin 2\alpha
\end{aligned} \tag{7.2.33}
$$

(7.2.32) and (7.2.33) implies that the horizontal range is the same for both the angles of projection α and $\frac{\pi}{2} - \alpha$ as shown in Fig. 7.5.
Maximum Horizontal Range The maximum horizontal range is given by the maximum

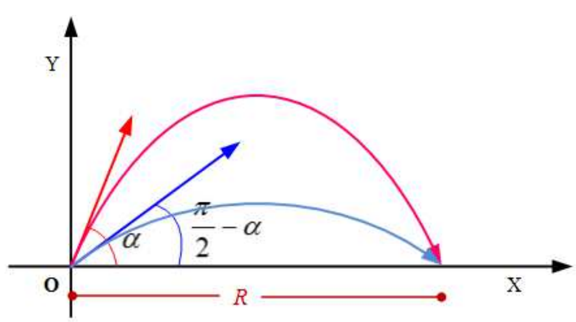


Figure 7.5: Range of Projectile motion.

of right hand side of (7.2.32). This maximum is accompanied with the maximum of $\sin 2\alpha$

and maximum of $\sin 2\alpha$ is 1, which implies that

$$2\alpha = \frac{\pi}{2}$$

or

$$\alpha = \frac{\pi}{4}$$

Hence the projectile has maximum horizontal range when its angle of projection is $\frac{\pi}{4}$ and the maximum horizontal range is

$$x_{R-max} = \frac{v_0^2}{g} \tag{7.2.34}$$

Hight of the Projectile The vertex of the parabola is the highest point of the trajectory, so ordinate of the vertex gives the hight attained by the projectile.

$$y_H = \frac{v_0^2 \sin^2 \alpha}{2g} \tag{7.2.35}$$

When the projectile has maximum horizontal range, angle of projection is $\frac{\pi}{4}$. In this case height of the projectile is

$$\begin{aligned} y_3 &= \frac{v_0^2}{2g} \sin^2\left(\frac{\pi}{4}\right) \\ &= \frac{v_0^2}{2g}\left(\frac{1}{2}\right) \\ &= \frac{v_0^2}{4g} \end{aligned} \tag{7.2.36}$$

Maximum Hight of the Projectile The projectile can attain maximum hight if

$$\sin^2 \alpha = 1$$

which implies that $\alpha = \frac{\pi}{2}$, then the projectile will be projected vertically upward and the motion will be one dimensional motion. Hence the height will be

$$y_{max} = \frac{v_0^2}{2g} \tag{7.2.37}$$

(7.2.37) gives the maximum hight of the projectile.

Example 7.2.1. *A golfer hit a ball and it is projected with a speed of* 10 m/s *at an inclination* $\frac{\pi}{6}$ *with the ground.*

(a) Write an expression for its horizontal component of position.

(b) *Write an expression for its vertical component of position.*

(c) *Find its horizontal range.*

(d) *What will be its maximum horizontal range.*

(e) *Calculate its time of flight.*

(f) *Calculate height attained by the ball.*

Solution

(a) Using (7.2.17) an expression for its horizontal component of position is

$$\begin{aligned} x(t) &= (v_0 \cos\alpha)t \\ &= 10(\cos\frac{\pi}{6})t \\ &= 8.6601t\ m \end{aligned}$$

(b) Using (7.2.25) an expression for its vertical component of position is

$$\begin{aligned} y(t) &= -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t \\ &= -\frac{1}{2}9.8\ t^2 + 10(\sin\frac{\pi}{6})t \\ &= (-4.9\ t^2 + 5t)\ m \end{aligned}$$

(c) Using (7.2.32) horizontal range is

$$\begin{aligned} x_R &= \frac{v_0^2}{g}\sin 2\alpha \\ &= \frac{(10)^2}{9.8}\sin 2\left(\frac{\pi}{6}\right) \\ &= 8.837\ m \end{aligned}$$

(d) Using (7.2.34), its maximum horizontal range is

$$\begin{aligned} x_{R-max} &= \frac{v_0^2}{g} \\ &= \frac{(10)^2}{9.8} \\ &= 10.2041\ m \end{aligned}$$

(e) Using (7.2.31), its time of flight is

$$\begin{aligned} t_1 &= \frac{2v_0 \sin\alpha}{g} \\ &= \frac{2(10)\sin\left(\frac{\pi}{6}\right)}{9.8} \\ &= 1.0204\ s \end{aligned}$$

(f) Using (7.2.35), height attained by the ball is.

$$\begin{aligned} y_1 &= \frac{v_0^2 \sin^2\alpha}{2g} \\ &= \frac{(10)^2 \sin^2\left(\frac{\pi}{6}\right)}{2(9.8)} \\ &= 1.2755\ m \end{aligned}$$

Example 7.2.2. *A built is fired from a cannon having muzzle velocity* 1 *mile*/*s*.

(a) What will be its maximum horizontal range.

(b) What will be the height attained in this case.

Solution Muzzle velocity The velocity with which a bullet or shell leaves the muzzle of a gun is called muzzle velocity.
Here the initial speed is in $mile/s$. It should be in $mile/h$ or ft/s. Let it be in ft/s. As

$$\begin{aligned} 1\ mile &= 1760\ yards \\ 1\ yard &= 3\ feet \\ \text{then} \quad 1\ mile &= 5280\ feet \end{aligned}$$

the initial speed is

$$v_0 = 5280\ ft/s$$

(a) Using (7.2.34), its maximum horizontal range is

$$\begin{aligned} x_{R-max} &= \frac{v_0^2}{g} \\ &= \frac{(5280)^2}{32} \\ &= 871200\ ft \end{aligned}$$

(b) Using (7.2.36), its maximum height in this case is

$$\begin{aligned} y_r &= \frac{v_0^2}{4g} \\ &= \frac{(5280)^2}{4(32)} \\ &= 217800\ ft \end{aligned}$$

Corollary 7.2.1. *A projectile projected from origin with initial speed v_0 with angle of projection α (moving under gravity only), has horizontal range x_R and maximum height y_H, then show that initial speed v_0 has expression*

$$v_0 = \sqrt{g\frac{x_R^2 + 16y_H^2}{8y_H}} \tag{7.2.38}$$

and angle of projection α has expression

$$\alpha = \arccos\left(\frac{x_R}{\sqrt{x_R^2 + 16y_H^2}}\right) \tag{7.2.39}$$

$$\text{and} \quad \alpha = \arcsin\left(\frac{4y_H}{\sqrt{x_R^2 + 16y_H^2}}\right) \tag{7.2.40}$$

Solution: Horizontal range x_R of a projectile projected from origin with initial speed v_0 with angle of projection α (moving under gravity only), is given by (7.2.32)

$$x_R = \frac{v_0^2}{g}\sin 2\alpha$$

and maximum height y_H is given by (7.2.35)

$$y_H = \frac{v_0^2 \sin^2\alpha}{2g}$$

Here the goal is to express v_0 in terms of x_R, y_H and g. From (7.2.32) we can write

$$x_R^2 = 4\frac{v_0^2}{g}\sin^2\alpha\frac{v_0^2}{g}\cos^2\alpha$$

Using (7.2.35), we can write

$$x_R^2 = 8y_H\frac{v_0^2}{g}\cos^2\alpha$$

or

$$v_0^2 \cos^2 \alpha = g \frac{x_R^2}{8y_H} \tag{7.2.41}$$

Also from (7.2.35), we can write

$$v_0^2 \sin^2 \alpha = 2gy_H \tag{7.2.42}$$

Adding (7.2.41) and (7.2.43)

$$v_0^2 \left(\cos^2 \alpha + \sin^2 \alpha\right) = g \frac{x_R^2}{8y_H} + 2gy_H \tag{7.2.43}$$

$$v_0^2 = g \frac{x_R^2 + 16y_H^2}{8y_H} \tag{7.2.44}$$

Taking square root we have

$$v_0 = \sqrt{g \frac{x_R^2 + 16y_H^2}{8y_H}}$$

Using (7.2.38) in (7.2.41) we can write

$$\cos^2 \alpha = \frac{x^2}{x_R^2 + 16y_H^2}$$

or

$$\cos \alpha = \frac{x_R}{\sqrt{x_R^2 + 16y_H^2}}$$

i.e.

$$\alpha = \arccos \left(\frac{x_R}{\sqrt{x_R^2 + 16y_H^2}} \right)$$

Using (7.2.38) in (7.2.43) we can write

$$\sin^2 \alpha = \frac{16y_H^2}{x_R^2 + 16y_H^2}$$

or

$$\sin \alpha = \frac{4y_H}{\sqrt{x_R^2 + 16y_H^2}}$$

i.e.

$$\alpha = \arcsin \left(\frac{4y_H}{\sqrt{x_R^2 + 16y_H^2}} \right)$$

Corollary 7.2.2. *A projectile projected from origin with initial speed* v_0 *(moving under gravity only) has same horizontal range* x_R *for two angles of projections, namely* α *and* $\frac{\pi}{2}-\alpha$*. If* y_1 *is the maximum height with angle* α *and* y_2 *is the maximum height with angle* $\frac{\pi}{2}-\alpha$*, then show that*

$$x_R = 4\sqrt{y_1 y_2}$$

Solution: For angle of projection α, the horizontal range x_R is given by (7.2.32)

$$x_R = \frac{v_0^2}{g}\sin 2\alpha$$

and maximum height y_1 is given by (7.2.35)

$$y_1 = \frac{v_0^2 \sin^2\alpha}{2g}$$

And for angle of projection $\frac{\pi}{2}-\alpha$, the horizontal range x_R is same given by (7.2.32)

$$x_R = \frac{v_0^2}{g}\sin 2\alpha$$

and maximum height y_2 is given by (7.2.35)

$$\begin{aligned} y_2 &= \frac{v_0^2}{2g}\sin^2\left(\frac{\pi}{2}-\alpha\right) \\ &= \frac{v_0^2\cos^2\alpha}{2g} \end{aligned}$$

The product of these two heights is

$$\begin{aligned} y_1 y_2 &= \left(\frac{v_0^2}{2g}\right)^2 \sin^2\alpha\cos^2\alpha \\ &= \frac{1}{16}\left(\frac{v_0^2}{2g}\right)^2 (2\sin\alpha\cos\alpha)^2 \\ &= \frac{1}{16}\left(\frac{v_0^2}{2g}\sin 2\alpha\right)^2 \end{aligned}$$

Using (7.2.32), we have

$$x_R = 4\sqrt{y_1 y_2}$$

Corollary 7.2.3. *A projectile projected from origin with initial speed v_0 (moving under gravity only) has maximum horizontal range x_{R-max}. If y is its maximum height, then show that*

$$\text{Its maximum height is} \quad y = \frac{x_{R-max}}{4} \tag{7.2.45}$$

$$\text{Its initial speed is} \quad v_0 = \sqrt{x_{R-max}g} \tag{7.2.46}$$

$$\text{Its time of flight is} \quad t_r = \sqrt{\frac{2x_{R-max}}{g}} \tag{7.2.47}$$

Solution: When a projectile has maximum horizontal range its angle of projection is $\frac{\pi}{4}$. In this case maximum horizontal range x_{R-max} is given by (7.2.34)

$$x_{R-max} = \frac{v_0^2}{g}$$

and maximum hight attained by the projectile is given by (7.2.36)

$$y = \frac{v_0^2}{4g}$$

Using (7.2.34), we have

$$y = \frac{x_{R-max}}{4}$$

From (7.2.34), initial speed of a projectile can be written as

$$v_0 = \sqrt{x_{R-max}g}$$

Time of flight of the projectile is given by (7.2.31)

$$t = \frac{2v_0 \sin\alpha}{g}$$

For maximum horizontal range, α is $\frac{\pi}{4}$ and v_0 is given by (7.2.46), so the above relation can be written as

$$\begin{aligned} t_r &= \frac{2}{g}\left(\sqrt{x_{R-max}g}\right)\frac{1}{\sqrt{2}} \\ &= \sqrt{\frac{2x_{R-max}}{g}} \end{aligned}$$

Corollary 7.2.4. *A particle of mass m is projected from origin with initial speed v_0 and angle of projection α. At any time t the particle is at $A(x, y)$. Another particle of same*

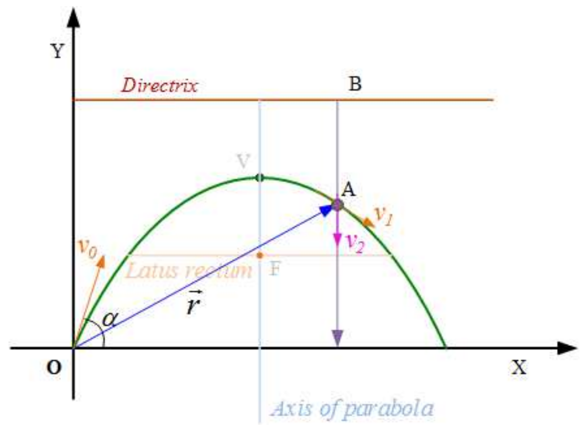


Figure 7.6: Projectile motion.

mass is dropped from B, a point on the directrix of the trajectory, vertically above A as shown in Fig. 7.6 . Neglecting air resistance in both motions, show that both particles have same speed at A.

Solution: This can be shown by the law of conservation of energy. First consider the particle executing two dimensional projectile motion.
At O its speed is v_0, so its kinetic energy is

$$T_1 = \frac{1}{2}mv_0^2$$

And its height is $y = 0$, so its potential energy is

$$U_1 = mgh = 0$$

Total energy at O is

$$\begin{aligned} E = T_1 + U_1 &= \frac{1}{2}mv_0^2 + 0 \\ &= \frac{1}{2}mv_0^2 \end{aligned}$$

Next at A its speed is v_1, so its kinetic energy is

$$T_2 = \frac{1}{2}mv_1^2$$

And its height is y, so its potential energy is

$$U_2 = mgh = mgy$$

Total energy at A is

$$E = T_2 + U_2 = \frac{1}{2}mv_1^2 + mgy$$

By law of conservation of energy, we have

$$\begin{aligned} \text{Total energy at O} &= \text{Total energy at A} \\ \frac{1}{2}mv_0^2 &= \frac{1}{2}mv_1^2 + mgy \\ v_0^2 &= v_1^2 + 2gy \end{aligned} \tag{7.2.48}$$

Next we consider the particle executing one dimensional projectile motion.
At B its speed is $v_b = 0$, so its kinetic energy is

$$T_3 = \frac{1}{2}mv_b^2 = 0$$

Since B lies on the directrix of the parabolic trajectory, so its height is $y = \frac{v_0^2}{2g}$, and its potential energy is

$$\begin{aligned} U_3 &= mgh = mg\frac{v_0^2}{2g} \\ &= m\frac{v_0^2}{2} \end{aligned}$$

Total energy at B is

$$\begin{aligned} E = T_3 + U_3 &= 0 + \frac{1}{2}mv_0^2 \\ &= \frac{1}{2}mv_0^2 \end{aligned}$$

As the particle is dropped, at A it gains a speed v_2, so its kinetic energy is

$$T_4 = \frac{1}{2}mv_2^2$$

And its height is y, so its potential energy is

$$U_2 = mgh = mgy$$

Total energy at P is

$$E = T_4 + U_4 = \frac{1}{2}mv_2^2 + mgy$$

By law of conservation of energy, we have

$$\begin{aligned} \text{Total energy at B} &= \text{Total energy at A} \\ \frac{1}{2}mv_0^2 &= \frac{1}{2}mv_2^2 + mgy \\ v_0^2 &= v_2^2 + 2gy \end{aligned} \tag{7.2.49}$$

From (7.2.48) and (7.2.49), we can write

$$\begin{aligned} v_1^2 &= v_2^2 \\ \text{or} \quad v_1 &= v_2 \end{aligned}$$

Hence proved.

Corollary 7.2.5. *A particle of mass m is projected from origin with initial speed v_0 and angle of projection α. Neglecting air resistance, find the least speed v_0, so that it passes through two points P and Q at heights y_1 and y_2 respectively.*

Solution: A particle of mass m is projected from origin with initial speed v_0 and angle of projection α. At any time t the particle is at $A(x, y)$. Let $\vec{r}$ be its position vector

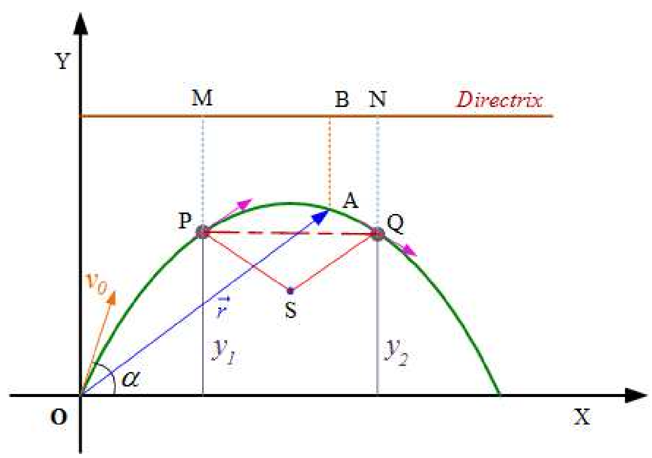


Figure 7.7: Projectile motion.

$$\vec{r} = x\hat{i} + y\hat{j}$$

then its velocity is

$$\begin{aligned} \vec{v} = \frac{d\vec{r}}{dt} &= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} \\ &= v_0 \cos\alpha\hat{i} + (v_0 \sin\alpha - gt)\,\hat{j} \end{aligned}$$

and its speed is

$$\begin{aligned} v = |\vec{v}| &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \\ &= \sqrt{(v_0 \cos\alpha)^2 + (v_0 \sin\alpha - gt)^2} \\ &= \sqrt{v_0^2 \cos^2\alpha + v_0^2 \sin^2\alpha - 2v_0 \sin\alpha gt + g^2t^2} \\ &= \sqrt{v_0^2 - 2g\left(v_0 \sin\alpha t - \frac{1}{2}gt^2\right)} \end{aligned}$$

Since

$$y = v_0 \sin\alpha t - \frac{1}{2}gt^2$$

so the speed of the particle at A is

$$\begin{aligned} v &= \sqrt{v_0^2 - 2gy} \\ &= \sqrt{2g\left(\frac{v_0^2}{2g} - y\right)} \\ &= \sqrt{2g\,(\text{height of the directrix} - \text{ordinate of A})} \qquad (7.2.50) \end{aligned}$$

If B is a point on the directrix of the trajectory, vertically above A as shown in Fig. 7.7, then speed of the particle at A is

$$v = \sqrt{2g\,|AB|} \qquad (7.2.51)$$

If the particle passes through P with speed v_1, then by (7.2.50), its speed is

$$\begin{aligned} v_1 &= \sqrt{2g\,(\text{height of the directrix} - \text{ordinate of P})} \\ &= \sqrt{2g\left(\frac{v_0^2}{2g} - y_1\right)} \qquad (7.2.52) \end{aligned}$$

If M is a point on the directrix of the trajectory, vertically above P as shown in Fig. 7.7, then using (7.2.51) speed of the particle at P is

$$v_1 = \sqrt{2g\,|MP|} \qquad (7.2.53)$$

From (7.2.52) and (7.2.53), we can write

$$2g\,|MP| = 2g\left(\frac{v_0^2}{2g} - y_1\right)$$

or

$$v_0^2 = 2g\left(|MP| + y_1\right) \tag{7.2.54}$$

If the particle passes through Q with speed v_2, then by (7.2.50), its speed is

$$\begin{aligned} v_2 &= \sqrt{2g\,(\text{height of the directrix} - \text{ordinate of Q})} \\ &= \sqrt{2g\left(\frac{v_0^2}{2g} - y_2\right)} \end{aligned} \tag{7.2.55}$$

If N is a point on the directrix of the trajectory, vertically above Q as shown in Fig. 7.7, then using (7.2.51) speed of the particle at Q is

$$v_2 = \sqrt{2g\,|NQ|} \tag{7.2.56}$$

From (7.2.55) and (7.2.56), we can write

$$2g\,|NQ| = 2g\left(\frac{v_0^2}{2g} - y_2\right)$$

or

$$v_0^2 = 2g\left(|NQ| + y_2\right) \tag{7.2.57}$$

Adding (7.2.54) and (7.2.57), we have

$$\begin{aligned} 2v_0^2 &= 2g\left(|MP| + y_1 + |NQ| + y_2\right) \\ \text{or} \quad v_0^2 &= g\left(y_1 + y_2 + |MP| + |NQ|\right) \end{aligned} \tag{7.2.58}$$

If S is the focus of parabolic trajectory, then using the focus-directrix property of the parabola, $|MP| = |PS|$ and $|NQ| = |SQ|$ as shown in Fig. 7.7. Then (7.2.58) can be written as

$$v_0^2 = g\left(y_1 + y_2 + |PS| + |SQ|\right)$$

Now v_0^2 is least when $|PS| + |SQ|$ is least, which is least when S lies on PQ, *i.e.*, when

$$\begin{aligned} |PS| + |SQ| &= |PQ| \\ \text{or} \quad |MP| + |NQ| &= |PQ| \end{aligned}$$

Hence the least speed v_0, so that it passes through two points P and Q at heights y_1 and y_2 respectively is

$$(v_0)_{min} = \sqrt{g\left(y_1 + y_2 + |PQ|\right)} \tag{7.2.59}$$

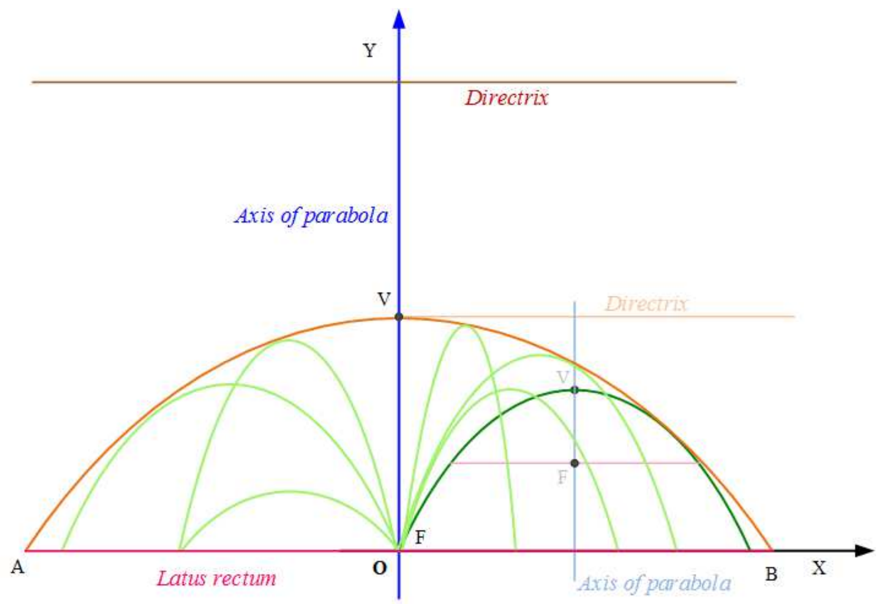


Figure 7.8: Parabola of Safety.

7.3 Parabola of Safety

Parabola of safety in a vertical plane is a boundary curve which includes all possible paths of projectiles, that are projected with same speed v_0 in different directions from same point.

7.3.1 Equation of Parabola of Safety

Consider a particle of mass m is projected from origin with initial speed v_0 and angle of projection α. Neglecting air resistance, its path is given by (7.2.27)

$$y(x) \quad = \quad -\frac{g\sec^2\alpha}{2v_0^2}x^2 + x\tan\alpha$$

Using trigonometric relation

$$\sec^2\alpha \quad = \quad 1+\tan^2\alpha$$

(7.2.27) can be written as

$$\frac{gx^2}{2v_0^2}\tan^2\alpha - x\tan\alpha + \frac{g}{2v_0^2}x^2 + y \quad = \quad 0 \tag{7.3.1}$$

If α is regarded as parameter, then (7.3.1) represents a family of projectile trajectories with the same initial speed v_0. Hence (7.3.1) is quadratic equation in $\tan\alpha$. For real and equal roots, the discernment must be equal to zero. *i.e.*

$$\begin{aligned} x^2 - 4\left(\frac{gx^2}{2v_0^2}\right)\left(\frac{g}{2v_0^2}x^2 + y\right) &= 0 \\ x^2\left(1 - \frac{2g}{v_0^2}\left(\frac{g}{2v_0^2}x^2 + y\right)\right) &= 0 \end{aligned}$$

Since $x^2 \neq 0$, implies that

$$\left(1 - \frac{2g}{v_0^2}\left(\frac{g}{2v_0^2}x^2 + y\right)\right) = 0$$

or we can write

$$\begin{aligned} \frac{2g}{v_0^2}\left(\frac{g}{2v_0^2}x^2 + y\right) &= 1 \\ \left(\frac{g}{2v_0^2}x^2 + y\right) &= \frac{v_0^2}{2g} \end{aligned}$$

or

$$x^2 = -\frac{2v_0^2}{g}\left(y - \frac{v_0^2}{2g}\right) \tag{7.3.2}$$

(7.3.2) represents a parabola, known as parabola of safety as shown in Fig. 7.8. Its vertex is

$$V = \left(0, \frac{v_0^2}{2g}\right),$$

the axis of the parabola is the y axis, equation of the directrix is

$$y_1 = \frac{v_0^2}{g},$$

the length of the latus rectum is $\frac{2v_0^2}{g}$ and origin is the focus

$$F = O = (0,0)$$

Example 7.3.1. *A shell bursts on contact with the ground and pieces from it fly in all directions with all speeds up-to* 80 ft/s. *Prove that a man standing* 100 ft *away is in danger for* $\frac{5}{\sqrt{2}}$ *seconds.*

Solution The point of contact with the ground may be taken as origin O. From O all pieces fly in different directions with all speeds up-to 80 ft/s. We will discuss projectile motion for one of the pieces with initial speed $v_0 = 80$ ft/s. We will discuss projectile motion for one of the pieces that has 100 ft horizontal range. Here we have two cases:

(a) The horizontal range is 100 ft.

(b) The maximum horizontal range is 100 ft.

(a) The horizontal range is 100 ft.

This horizontal range may have two angles of projections. If first angle of projection is α, then the other angle of projection is $\frac{\pi}{2} - \alpha$. These angles may be calculated using (7.2.32), describing the horizontal range of a projectile.

$$x_R = \frac{v_0^2}{g} \sin 2\alpha$$

and α is

$$\begin{aligned}
\alpha &= \frac{1}{2} \arcsin\left(\frac{g x_R}{v_0^2}\right) \\
&= \frac{1}{2} \arcsin\left(\frac{32(100)}{(80)^2}\right) \\
&= \frac{1}{2} \arcsin\left(\frac{1}{2}\right) \\
&= \frac{1}{2}\left(\frac{\pi}{6}\right) \\
&= \frac{\pi}{12} = 15^\circ
\end{aligned}$$

the other angle $\frac{\pi}{2} - \alpha$ is

$$\begin{aligned}
\frac{\pi}{2} - \alpha &= \frac{\pi}{2} - \frac{\pi}{12} \\
&= \frac{5\pi}{12} = 75^\circ
\end{aligned}$$

Time of flight of the projectile is given by (7.2.31)

$$t = \frac{2v_0 \sin\alpha}{g}$$

Let t_1 be the time of flight with elevation $\alpha = \frac{\pi}{12}$

$$t_1 = \frac{2(100)\sin\left(\frac{\pi}{12}\right)}{32}$$

t_2 be the time of flight with elevation $\alpha = \frac{5\pi}{12}$

$$t_2 = \frac{2(100)\sin\left(\frac{5\pi}{12}\right)}{32}$$

The man is danger for the time $t_1 - t_2$

(b) The maximum horizontal range is 100 ft.

In this case angle of projection is $\alpha = \frac{\pi}{4}$ and t_3 the time of flight is

$$\begin{aligned} t_3 &= \frac{2(100)\sin\left(\frac{\pi}{4}\right)}{32} \\ &= \frac{5}{\sqrt{2}} \end{aligned}$$

7.4 Projectile Motion of a Projectile Projected from a Relative Height from Origin

In this projectile motion the reference point is other than origin. We will discuss it as following.

7.4.1 Path of a Projectile with Oblique Angle of Projection

Consider a cartesian plane as the vertical plane with x axis along horizontal and y axis along vertical. A particle of mass m is projected from point A at a height y_0 from origin O with a projection velocity $\vec{v}_0$, making an angle α with the horizontal. The coordinates of A are $A(0, y_0)$. In the absence of air resistance, the particle executes projectile motion.
Here the velocity and horizontal component of position of projectile will be same as discussed as above. The vertical component of position will be different due to its relative height. For it we can proceed as follows.
After time t, the particle is at position P, as shown in Fig. 7.9. The position vector of P is

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

The particle starts from A at $t = 0$, so its position vector is

$$\vec{r}(0) = x(0)\hat{i} + y(0)\hat{j} = 0\hat{i} + y_0\hat{j}$$

separating vector components we have

$$x(0) = 0 \tag{7.4.1}$$

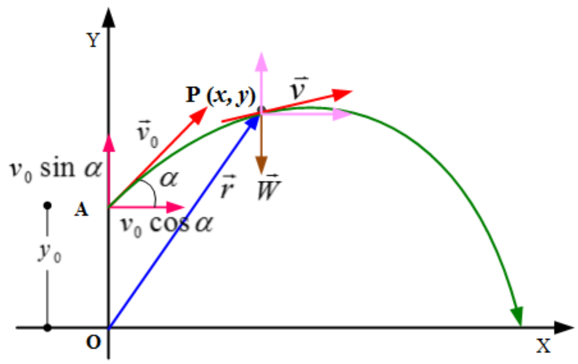


Figure 7.9: Projectile motion.

and

$$y(0) = y_0 \tag{7.4.2}$$

Next we can continue considering (7.2.22)

$$y(t) = -\frac{1}{2}gt^2 + (v_0 \sin \alpha)t + B_3 \tag{7.4.3}$$

At $t = 0$, (7.4.3) becomes

$$y(0) = -\frac{1}{2}g(0)^2 + (v_0 \sin \alpha)(0) + B_3 \tag{7.4.4}$$

Using (7.4.2), (7.4.4) becomes

$$B_3 = y_0 \tag{7.4.5}$$

Using (7.4.5), (7.4.3) becomes

$$y(t) = y_0 - \frac{1}{2}gt^2 + (v_0 \sin \alpha)t \tag{7.4.6}$$

(7.4.8) gives the vertical component of position of the particle at any time t.
Using (7.2.26), (7.4.8) becomes

$$\begin{aligned} y(x) &= y_0 - \frac{1}{2}g\left(\frac{x(t)}{v_0 \cos \alpha}\right)^2 + x(t) \tan \alpha \\ &= y_0 - \frac{g \sec^2 \alpha}{2v_0^2}x^2 + x \tan \alpha \end{aligned} \tag{7.4.7}$$

(7.4.7) gives the path of the projectile.

Example 7.4.1. *A flying squirrel launches itself from the top of a* 10 m *high tree. The squirrel leaves the tree with a velocity of* 5 m/s *making an inclination* 30° *relative to a level above the ground and approaches a shorter tree of height of* 5 m *located* 8.78 m *away from the first tree.*

(a) Calculate the time of flight of the squirrel.

(b) Write an expression for its horizontal component of position.

(c) Write an expression for its vertical component of position.

(d) Find the path of its projectile motion.

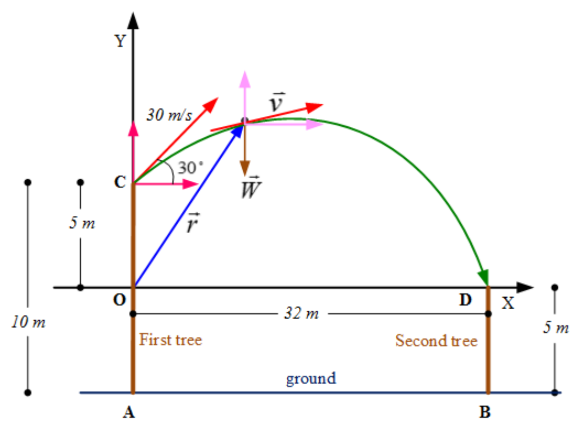


Figure 7.10: Squirrel projectile motion

Solution The height of the shorter tree can be set as zero level above the ground and the horizontal line from the mid of longer tree to the top of small tree can be considered as x $axis$ and y $axis$ along the longer tree. Its center can be considered as the origin. The system is illustrated in Fig. 7.12. Here C is the top of longer tree from where the squirrel jumps and approaches to D, executing projectile motion. The given data is

$$\begin{aligned} v_0 &= 5\ m/s \\ x_R &= 8.78\ m \\ Y_0 &= 10 - 5 = 5\ m \\ \alpha &= 30^\circ \end{aligned}$$

(a) The time of flight can be calculated as

The horizontal component of velocity is

$$\begin{aligned} v_x &= v_0 \cos\alpha \\ &= 5\cos 30^\circ \\ &= 4.33\ m/s \end{aligned}$$

Hence the time of flight is

$$\begin{aligned} t &= \frac{x_R}{v_x} \\ &= \frac{8.78}{4.33} \\ &= 2.03\ s \simeq 2\ s \end{aligned}$$

(b) Using (7.2.17) an expression for its horizontal component of position is

$$\begin{aligned} x(t) &= (v_0 \cos\alpha)t \\ &= 4.33t\ m \end{aligned}$$

(c) Using (7.4.33) an expression for its vertical component of position is

$$\begin{aligned} y(t) &= y_0 - \frac{1}{2}gt^2 + (v_0 \sin\alpha)t \\ &= 5 - \frac{1}{2}(9.8)t^2 + (5\sin 30^\circ)t \\ &= -4.9t^2 + 2.5t + 5 \end{aligned}$$

(d) The path of flight can be calculated by using (7.4.7)

$$\begin{aligned} y(x) &= y_0 - \frac{g\sec^2\alpha}{2v_0^2}x^2 + x\tan\alpha \\ &= 5 - \frac{9.81\sec^2(30)}{25^2}x^2 + x\tan(30) \\ &= 5 - 0.1307x^2 + 0.5773x \\ &= -0.13x^2 + 0.58x + 5 \end{aligned}$$

is the path of the squirrel.

7.4.2 Path of a Projectile with Zero Degree Angle of Projection

Consider polar coordinate system as the vertical plane with t as its axis. A particle of mass m is projected from point A at a height y_0 from origin O with a projection velocity $\vec{v}_0$ along the horizontal. In the absence of air resistance, the particle executes projectile motion. The initial velocity $\vec{v}_0$ can be written as

$$\vec{v}_0 = \vec{v}(0) = v_0\hat{i} + 0\hat{j} \tag{7.4.8}$$

After time t, the particle is at position P, as shown in Fig. 7.11. Let

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

be its position vector. Clearly at $t = 0$, the particle is at A, so its position vector is

$$\begin{aligned} \vec{r}(0) &= \vec{r}_0 \\ x(0)\hat{i} + y(0)\hat{j} &= 0\hat{i} + y_0\hat{j} \end{aligned}$$

implies that

$$x(0) = 0 \tag{7.4.9}$$

and

$$y(0) = y_0 \tag{7.4.10}$$

Next its velocity is

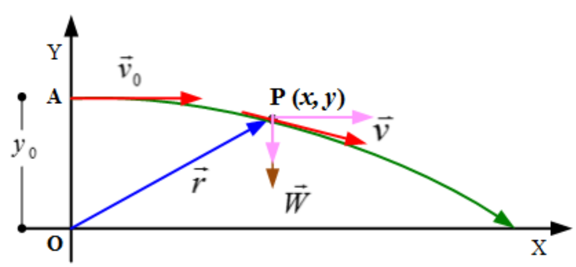


Figure 7.11: Projectile motion.

$$\vec{v}(t) = \vec{\dot{r}}(t) = \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j}$$

at $t = 0$

$$\vec{v}(0) = \dot{x}(0)\hat{i} + \dot{y}(0)\hat{j} \tag{7.4.11}$$

From (7.4.8) and (7.4.11), we can write

$$\dot{x}(0)\hat{i} + \dot{y}(0)\hat{j} = v_0\hat{i} + 0\hat{j}$$

Then the initial horizontal scalar component of velocity is

$$\dot{x}(0) = v_0 \tag{7.4.12}$$

And the initial vertical scalar component of velocity is

$$\dot{y}(0) = 0 \tag{7.4.13}$$

Finally its acceleration is

$$\vec{a}(t) = \ddot{\vec{r}}(t) = \ddot{x}(t)\hat{i} + \ddot{y}(t)\hat{j}$$

At P, the only force acting on the particle is force of gravity acting in the downward direction. Then by Newton's second law of motion its equation of motion is

$$\begin{aligned} \vec{F} &= \vec{W} \\ m\vec{a} &= -m\vec{g} \\ \ddot{x}\hat{i} + \ddot{y}\hat{j} &= -g\hat{j} \end{aligned} \tag{7.4.14}$$

From (7.4.14), the horizontal component of equation of motion is

$$\ddot{x}(t) = 0 \tag{7.4.15}$$

and the vertical component is

$$\ddot{y}(t) = -g \tag{7.4.16}$$

Integrating (7.4.15) with respect to t

$$\dot{x}(t) = A_1 \tag{7.4.17}$$

At $t = 0$, (7.4.17) becomes

$$\dot{x}(0) = A_1 \tag{7.4.18}$$

Using (7.4.12), (7.4.18) implies that

$$A_1 = v_0 \tag{7.4.19}$$

Using (7.4.19), (7.4.17) becomes

$$\dot{x}(t) = v_0 \tag{7.4.20}$$

(7.4.20) gives the horizontal scalar component of velocity of the particle at any time t. Integrating it with respect to t

$$x(t) = v_0 t + B_1 \tag{7.4.21}$$

At $t = 0$, (7.4.21) becomes

$$x(0) = v_0(0) + B_1 \tag{7.4.22}$$

Using (7.4.9), (7.4.22) becomes

$$B_1 = 0 \tag{7.4.23}$$

Using (7.4.23), (7.4.21) becomes

$$x(t) = v_0 t \tag{7.4.24}$$

(7.4.24) gives the horizontal component of position of the particle at any time t.
Next for vertical component integrate (7.4.16) with respect to t

$$\dot{y}(t) = -gt + A_2 \tag{7.4.25}$$

At $t = 0$, (7.4.25) becomes

$$\dot{y}(0) = A_2 \tag{7.4.26}$$

Using (7.4.13), (7.4.26) implies that

$$A_2 = 0 \tag{7.4.27}$$

Using (7.4.27), (7.4.25) becomes

$$\dot{y}(t) = -gt \tag{7.4.28}$$

(7.4.28) gives the vertical scalar component of velocity of the particle at any time t.
Hence the velocity of the particle at any time t is

$$\vec{v}(t) = v_0\hat{\imath} + -gt\hat{\jmath} \tag{7.4.29}$$

Integrating (7.4.28) with respect to t

$$y(t) = -\frac{1}{2}gt^2 + B_2 \tag{7.4.30}$$

At $t = 0$, (7.4.30) becomes

$$y(0) = -\frac{1}{2}g(0)^2 + B_2 \tag{7.4.31}$$

Using (7.4.10), (7.4.31) becomes

$$B_2 = y_0 \tag{7.4.32}$$

Using (7.4.32), (7.4.30) becomes

$$y(t) = y_0 - \frac{1}{2}gt^2 \tag{7.4.33}$$

(7.4.33) gives the vertical component of position of the particle at any time t.
When the projectile hit the ground, then $y = 0$, so the time of flight can be calculated by equating (7.4.33) zero.

$$0 = y_0 - \frac{1}{2}gt^2$$

or

$$t = \sqrt{\frac{2y_0}{g}} \tag{7.4.34}$$

Using (7.4.34), (7.4.29) will give the velocity with which the projectile will hit the ground.

$$\vec{v}(t) = v_0\hat{i} - g\sqrt{\frac{2y_0}{g}}\hat{j} \tag{7.4.35}$$

From (7.4.24), we can find the time required to reach the particle at P as

$$t = \frac{x(t)}{v_0} \tag{7.4.36}$$

Using (7.4.36), (7.4.33) becomes

$$y(t) = y_0 - \frac{1}{2}g\left(\frac{x(t)}{v_0}\right)^2 \tag{7.4.37}$$

(7.4.37) gives the path of the projectile of the particle at any time t.

Example 7.4.2. *A cannonball is fixed at the top of a cliff. The height of the cliff is* 20 m *from ground level. A ball is fired from the cannon horizontally with a speed* 100 m/s.

(a) Write an expression for its horizontal component of position.

(b) Write an expression for its vertical component of position.

(c) Calculate its time of flight.

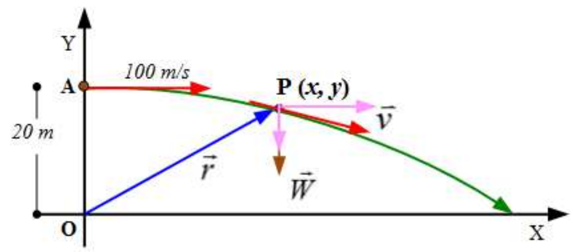


Figure 7.12: System of canon ball.

(d) Find its horizontal range.

Solution The system can be illustrated in Fig. 7.12

(a) Using (7.4.24) an expression for its horizontal component of position is

$$\begin{aligned} x(t) &= v_0 t \\ &= 100t \ m \end{aligned} \quad (7.4.38)$$

(b) Using (7.4.33) an expression for its vertical component of position is

$$\begin{aligned} y(t) &= y_0 - \frac{1}{2}gt^2 \\ &= 20 - 4.9t^2 \ m \end{aligned}$$

(c) Using (7.4.34) , its time of flight is

$$\begin{aligned} t &= \sqrt{\frac{2y_0}{g}} \\ &= \sqrt{\frac{2(10)}{9.8}} \\ &= 1.4 \ s \end{aligned}$$

(d) Using (7.4.39), horizontal range is

$$\begin{aligned} x(t) &= 100(1.4) \\ &= 140 \ m \end{aligned} \quad (7.4.39)$$

7.4.3 Projectile Motion of a Projectile projected from Origin along an Inclined Plane

Consider a vertical plane (xy plane). Let a projectile of mass m be projected from origin O with a velocity $\vec{v}_0$, at an angle of inclination α. Let a plane p be inclined at an angle $\beta(< \alpha)$ to the horizontal so that its intersection with the vertical plane is the line with slope

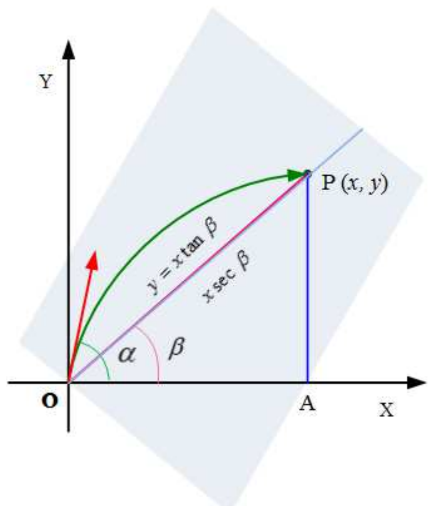


Figure 7.13: Range on an inclined plane.

$$\tan\beta = \frac{y}{x}$$

with $P(x, y)$ is a point common to both planes (see Fig. 7.13). Clearly this line passes through the origin, so its equation is

$$y = x\tan\beta \tag{7.4.40}$$

Also by (7.2.27), the path of the projectile is

$$y = -\frac{g\sec^2\alpha}{2v_0^2}x^2 + x\tan\alpha$$

Using (7.4.40), (7.2.27) can be written as

$$x\tan\beta = -\frac{g\sec^2\alpha}{2v_0^2}x^2 + x\tan\alpha$$

Since $x \neq 0$, we can write

$$\frac{g\sec^2\alpha}{2v_0^2}\,x = \tan\alpha - \tan\beta$$

or

$$\begin{aligned}
x &= \frac{2v_0^2}{g\sec^2\alpha}(\tan\alpha - \tan\beta) \\
&= \frac{2v_0^2}{g}\cos^2\alpha\left(\frac{\sin\alpha}{\cos\alpha} - \frac{\sin\beta}{\cos\beta}\right) \\
&= \frac{2v_0^2}{g}\frac{\cos\alpha}{\cos\beta}(\sin\alpha\cos\beta - \sin\beta\cos\alpha) \\
&= \frac{v_0^2\sec\beta}{g}2\cos\alpha\sin(\alpha-\beta) \qquad (7.4.41)
\end{aligned}$$

Using trigonometric relation

$$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$

(7.4.41) becomes

$$\begin{aligned}
x &= \frac{v_0^2\sec\beta}{g}\left[\sin(\alpha+(\alpha-\beta)) - \sin(\alpha-(\alpha-\beta))\right] \\
&= \frac{v_0^2\sec\beta}{g}(\sin(2\alpha-\beta) - \sin\beta) \qquad (7.4.42)
\end{aligned}$$

As the projectile hit the point P, then $|\overline{OP}|$ is its range. Consider right angle triangle OAP as shown in Fig. 7.14

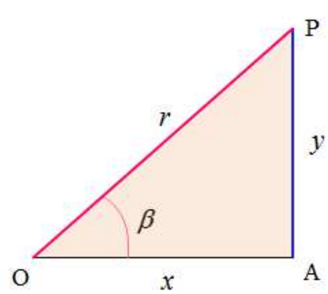


Figure 7.14: Right angle triangle OAP

$$\begin{aligned}
|\overline{OP}| = r &= \frac{x}{\cos\beta} \\
&= x\sec\beta
\end{aligned}$$

(7.4.42) can be written as

$$x_{Ru} = x \sec\beta \quad = \quad \frac{v_0^2 \sec^2\beta}{g}\left(\sin\left(2\alpha - \beta\right) - \sin\beta\right) \tag{7.4.43}$$

If the plane p is inclined downward to the horizontal, then the angle β is replaced with $-\beta$ and its intersection with the vertical plane is the line

$$y \quad = \quad -x\tan\beta \tag{7.4.44}$$

The x coordinate of the point where the projectile hits the plane p is given by replacing β by $-\beta$ in (7.4.42)

$$x \quad = \quad \frac{v_0^2 \sec(-\beta)}{g}\left(\sin\left(2\alpha - (-\beta)\right) - \sin(-\beta)\right)$$

$$x_{Rd} = \quad = \quad \frac{v_0^2 \sec\beta}{g}\left(\sin\left(2\alpha + \beta\right) + \sin\beta\right) \tag{7.4.45}$$

Time of Flight on an Inclined Plane The time of flight up the inclined plane p is defined as the time taken by the projectile to achieve the full range. This time can be calculated as

$$time \quad = \quad \frac{distance}{speed}$$

The distance is the horizontal distance x is given by (7.4.41) and speed is horizontal speed $v_0 \cos\alpha$

$$t \quad = \quad \frac{1}{v_0\cos\alpha}\frac{v_0^2\sec\beta}{g}2\cos\alpha\sin\left(\alpha - \beta\right)$$

$$= \quad \frac{2v_0}{g}\sec\beta\sin\left(\alpha - \beta\right) \tag{7.4.46}$$

Maximum Range on an Inclined Plane Since β is a fixed angle, then for a given value of v_0, the range up the inclined plane is maximum if

$$\sin\left(2\alpha - \beta\right) \quad = \quad 1 \tag{7.4.47}$$

which implies that

$$2\alpha - \beta \quad = \quad \frac{\pi}{2}$$

or

$$\alpha \quad = \quad \frac{\pi}{4} + \frac{\beta}{2}$$

Hence maximum range up the inclined plane is obtained by using (7.4.47) in (7.4.43)

$$\begin{aligned} x_{Ru-max} &= \frac{v_0^2}{g\cos^2\beta}(1-\sin\beta) \\ &= \frac{v_0^2}{g\left(1-\sin^2\beta\right)}(1-\sin\beta) \\ &= \frac{v_0^2}{g(1+\sin\beta)} \end{aligned} \tag{7.4.48}$$

Similarly maximum range down the inclined plane is

$$x_{Rd-max} = \frac{v_0^2}{g(1-\sin\beta)} \tag{7.4.49}$$

7.5 Projectile Motion with Horizontal Relative Motion

An object is moving horizontally with velocity $\vec{V}$ and a particle of mass m is projected from it with a velocity $\vec{v}_0$, making an angle α with the horizontal. This motion is different only in horizontal motion from the projectile motion discussed in section 7.2.1. Here we will discuss only horizontal motion in the following two cases.

a) Projected in backward direction.

b) Projected in forward direction.

First consider the particle is projected in the backward direction, then both velocities have opposite directions. Consider a cartesian plane as the vertical plane with x axis along horizontal and y axis along vertical. The point O is the point of projection. Since $\vec{v}_0$ is considered positive in section 7.2.1 so $\vec{V}$ will be considered negative. The initial velocity $\vec{v}_0$ can be written as

$$\vec{v}_0 = \vec{v}(0) - \vec{V} = v_0\cos\alpha\hat{\imath} + v_0\sin\alpha\hat{\jmath} - V\hat{\imath} \tag{7.5.1}$$

After time t, the particle is at position P, as shown in Fig. 7.15. Let

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$$

be its position vector. Clearly at $t = 0$

$$\begin{aligned} \vec{r}(0) &= \vec{0} \\ x(0)\hat{\imath} + y(0)\hat{\jmath} &= 0\hat{\imath} + 0\hat{\jmath} \end{aligned}$$

implies that

$$x(0) = 0 \tag{7.5.2}$$

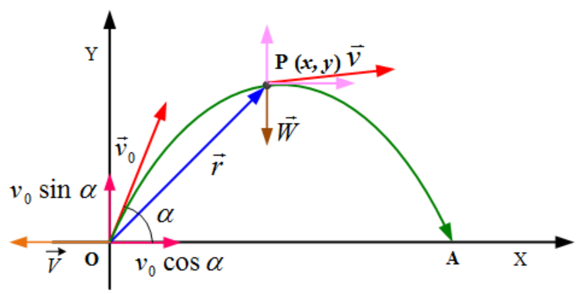


Figure 7.15: Projectile motion with relative horizontal motion.

and

$$y(0) = 0 \tag{7.5.3}$$

Next its velocity is

$$\vec{v}(t) = \dot{\vec{r}}(t) = \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j}$$

at $t = 0$

$$\vec{v}(0) = \dot{x}(0)\hat{i} + \dot{y}(0)\hat{j} \tag{7.5.4}$$

From (7.5.1) and (7.5.4), we can write

$$\dot{x}(0)\hat{i} + \dot{y}(0)\hat{j} = (v_0 \cos\alpha - V)\,\hat{i} + v_0 \sin\alpha\hat{j}$$

Then the initial horizontal scalar component of velocity is

$$\dot{x}(0) = v_0 \cos\alpha - V \tag{7.5.5}$$

And the initial vertical scalar component of velocity is

$$\dot{y}(0) = v_0 \sin\alpha \tag{7.5.6}$$

Finally its acceleration is

$$\vec{a}(t) = \ddot{\vec{r}}(t) = \ddot{x}(t)\hat{i} + \ddot{y}(t)\hat{j}$$

At P, the only force acting on the particle is force of gravity acting in the downward direction. Then by Newton's second law of motion its equation of motion is

$$\begin{aligned} \vec{F} &= \vec{W} \\ m\vec{a} &= -m\vec{g} \\ \ddot{x}\hat{i} + \ddot{y}\hat{j} &= -g\hat{j} \end{aligned} \tag{7.5.7}$$

From (7.5.7), the horizontal component of equation of motion is

$$\ddot{x}(t) = 0 \tag{7.5.8}$$

and the vertical component is

$$\ddot{y}(t) = -g \tag{7.5.9}$$

Integrating (7.5.8) with respect to t

$$\dot{x}(t) = A_1 \tag{7.5.10}$$

At $t = 0$, (7.5.10) becomes

$$\dot{x}(0) = A_1 \tag{7.5.11}$$

Using (7.5.5), (**??**) implies that

$$A_1 = v_0 \cos\alpha - V \tag{7.5.12}$$

Using (7.5.12), (7.5.10) becomes

$$\dot{x}(t) = v_0 \cos\alpha - V \tag{7.5.13}$$

(7.5.13) gives the horizontal scalar component of velocity of the particle at any time t. Integrating it with respect to t

$$x(t) = (v_0 \cos\alpha - V)\, t + B_1 \tag{7.5.14}$$

At $t = 0$, (7.5.14) becomes

$$x(0) = (v_0 \cos\alpha)(0) + B_1 \tag{7.5.15}$$

Using (7.5.2), (7.5.15) becomes

$$B_1 = 0 \tag{7.5.16}$$

Using (7.5.16), (7.5.14) becomes

$$x(t) = (v_0 \cos\alpha - V)\, t \tag{7.5.17}$$

(7.5.17) gives the horizontal component of position of the particle at any time t.
From section 7.2.1 the vertical component of position of the particle at any time t is

$$y(t) = -\frac{1}{2} g t^2 + (v_0 \sin\alpha) t$$

From (7.5.17), we can find the time required to reach the particle at P as

$$t = \frac{x(t)}{v_0 \cos\alpha - V} \tag{7.5.18}$$

Using (7.5.18), (7.5.18) becomes

$$y(t) = -\frac{1}{2}g\left(\frac{x(t)}{v_0\cos\alpha - V}\right)^2 + x(t)\tan\alpha \tag{7.5.19}$$

(7.5.19) gives the path of the projectile of the particle at any time t.
Time of Flight: It's time of flight is the same as discussed in section 7.2.1 and is given by (7.2.31).
Horizontal Range Let the particle hits the x axis at A after projected from O. Then the distance $|\overline{OA}|$ is known as horizontal range and is calculated by using (7.2.31) in (7.5.17)

$$x_R = (v_0\cos\alpha - V)\frac{2v_0\sin\alpha}{g} \tag{7.5.20}$$

Angle of Projection for Maximum Horizontal Range The maximum horizontal range is the maximum of (7.5.20) and can be calculated by using $2nd$ derivative test.
Differentiate (7.5.20) with respect to α

$$\begin{aligned}\frac{dx_R}{d\alpha} &= \frac{dx}{d\alpha}\left(\frac{1}{g}v_0^2\sin 2\alpha - 2Vv_0\sin\alpha\right)\\ &= \frac{v_0}{g}\left(2v_0\cos 2\alpha - 2V\cos\alpha\right)\end{aligned}$$

For critical point we must have

$$\frac{dx_R}{d\alpha} = 0$$

or

$$\frac{2v_0}{g}\left(v_0\cos 2\alpha - V\cos\alpha\right) = 0$$

Since $v_0 \neq 0$ and $g \neq 0$, then

$$\left(2v_0\cos^2\alpha - V\cos\alpha - v_0\right) = 0 \tag{7.5.21}$$

(7.5.21) is quadratic equation in $\cos\alpha$ and its roots are

$$\cos\alpha = \frac{V \pm \sqrt{V^2 + 8v_0^2}}{4v_0}$$

Next second derivative of (7.5.20) is

$$\frac{d^2x_R}{d\alpha^2} = \frac{2v_0}{g}\sin\alpha\left(V - v_0\cos\alpha\right)$$

For maximum horizontal range, we must have

$$\frac{d^2x_R}{d\alpha^2} < 0$$

which is possible only if

$$\cos\alpha = \frac{V + \sqrt{V^2 + 8v_0^2}}{4v_0}$$

or

$$\alpha = \arccos\left(\frac{V + \sqrt{V^2 + 8v_0^2}}{4v_0}\right) \tag{7.5.22}$$

(7.5.22) gives the angle of elevation for maximum range.

7.6 Projectile Motion of a Projectile Projected with a Relative speed from Origin

Consider a cartesian plane as the vertical plane with x axis along horizontal and y axis along vertical. A particle of mass m is projected from a moving cart making an angle α with the horizontal with velocity $\vec{v}_0$. The motion of the cart is along horizontal with velocity V and both motions are in the same directions. Let the point of projection considered as origin O. In the absence of air resistance, the initial velocity $\vec{v}_0$ of the projectile can be written as

$$\begin{aligned}\vec{v}_0 = \vec{v}(0) &= v_0\cos\alpha\hat{i} + v_0\sin\alpha\hat{j} + V\hat{i} \\ &= (V + v_0\cos\alpha)\,\hat{i} + v_0\sin\alpha\hat{j}\end{aligned} \tag{7.6.1}$$

After time t, the particle is at position P, as shown in Fig. 7.16. Let

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

be its position vector. Clearly at $t = 0$

$$\begin{aligned}\vec{r}(0) &= \vec{0} \\ x(0)\hat{i} + y(0)\hat{j} &= 0\hat{i} + 0\hat{j}\end{aligned}$$

implies that

$$x(0) = 0 \tag{7.6.2}$$

and

$$y(0) = 0 \tag{7.6.3}$$

Next its velocity is

$$\vec{v}(t) = \dot{\vec{r}}(t) = \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j}$$

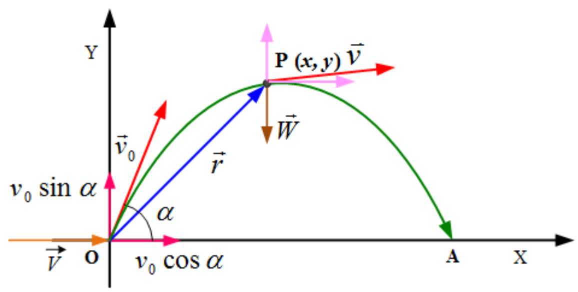


Figure 7.16: Projectile motion.

at $t = 0$

$$\vec{v}(0) \quad = \quad \dot{x}(0)\hat{\imath} + \dot{y}(0)\hat{\jmath} \tag{7.6.4}$$

From (7.6.1) and (7.6.4), we can write

$$\dot{x}(0)\hat{\imath} + \dot{y}(0)\hat{\jmath} \quad = \quad (V + v_0 \cos\alpha)\,\hat{\imath} + v_0 \sin\alpha\hat{\jmath}$$

Then the initial horizontal scalar component of velocity is

$$\dot{x}(0) \quad = \quad V + v_0 \cos\alpha \tag{7.6.5}$$

And the initial vertical scalar component of velocity is

$$\dot{y}(0) \quad = \quad v_0 \sin\alpha \tag{7.6.6}$$

Finally its acceleration is

$$\vec{a}(t) = \ddot{\vec{r}}(t) \quad = \quad \ddot{x}(t)\hat{\imath} + \ddot{y}(t)\hat{\jmath}$$

At P, the only force acting on the particle is force of gravity acting in the downward direction. Then by Newton's second law of motion its equation of motion is

$$\begin{aligned} \vec{F} &= \vec{W} \\ m\vec{a} &= -m\vec{g} \\ \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} &= -g\hat{\jmath} \end{aligned} \tag{7.6.7}$$

From (7.7.1), the horizontal component of equation of motion is

$$\ddot{x}(t) \quad = \quad 0 \tag{7.6.8}$$

Integrating (7.7.2) with respect to t

$$\dot{x}(t) \quad = \quad A_1 \tag{7.6.9}$$

At $t = 0$, (7.7.4) becomes

$$\dot{x}(0) = A_1 \tag{7.6.10}$$

Using (7.6.5), (7.7.5) implies that

$$A_1 = V + v_0 \cos\alpha \tag{7.6.11}$$

Using (7.6.11), (7.7.4) becomes

$$\dot{x}(t) = V + v_0 \cos\alpha \tag{7.6.12}$$

(7.7.6) gives the horizontal scalar component of velocity of the particle at any time t. Integrating it with respect to t

$$x(t) = (v_0 \cos\alpha + V)\, t + B_1 \tag{7.6.13}$$

At $t = 0$, (7.7.7) becomes

$$x(0) = (v_0 \cos\alpha + V)\,(0) + B_1 \tag{7.6.14}$$

Using (7.6.2), (7.7.8) becomes

$$B_1 = 0 \tag{7.6.15}$$

Using (7.7.9), (7.7.7) becomes

$$x(t) = (V + v_0 \cos\alpha)\, t \tag{7.6.16}$$

(7.7.10) gives the horizontal component of position of the particle at any time t.
Since the motion of the cart is along horizontal only, so at ant time t, vertical component speed is given by (7.2.21) and vertical component of position is given by (7.2.25). From (7.7.10), we can find the time required to reach the particle at P as

$$t = \frac{x(t)}{V + v_0 \cos\alpha} \tag{7.6.17}$$

Time of Flight As the particle is moving under gravity, so it will strike horizontal axis (or plane) after time t. This time is obtained by taking vertical component of speed equal to zero. Since there is no change in it so same relation is for time of flight that is given by (7.2.31)

$$t = t_r = \frac{2v_0 \sin\alpha}{g}$$

Horizontal Range Let the particle hits the x axis at A after projected from O. Then the distance $|\overline{OA}|$ is known as horizontal range and is calculated by using (7.2.31) in (7.7.10)

$$x(t) = (V + v_0 \cos\alpha)\,\frac{2v_0 \sin\alpha}{g}$$

$$x_r = \frac{2Vv_0 \sin\alpha}{g} + \frac{v_0^2}{g}\sin 2\alpha \tag{7.6.18}$$

If the projectile has relative horizontal velocity, then increase in range is

$$\begin{aligned}\Delta x &= x_r - x_R \\ &= \frac{2V v_0 \sin\alpha}{g}\end{aligned} \qquad (7.6.19)$$

7.7 Projectile Motion with Air Resistance

In first section we neglect air-resistance (drag force), but it has a major effect on the motion of many objects, including tennis balls, bicycle riders, and airplanes.
Air resistance is a force that retards the motion of the projectile. The force of air resistance always acts in a direction opposite to the direction of motion of the projectile (opposite to the velocity v). The magnitude of force of air resistance depends on the size of the body, its shape and its speed. In many cases, it is proportional to the velocity of the body. Mathematically, it is written as

$$\vec{F}_r = -k\vec{v} = -k\dot{x}\hat{i} - k\dot{y}\hat{j}$$

7.7.1 Path of a Projectile Moving Under Air Resistance

Consider a cartesian plane as the vertical plane with x axis along horizontal and y axis along vertical. A particle of mass m is projected from origin O with a velocity $\vec{v}_0$, making an angle α with the horizontal. The point O is named as point of projection, the velocity $\vec{v}_0$ is the velocity of projection and the angle α is called angle of projection. If the air resistance is proportional to the velocity of the body, The initial conditions are

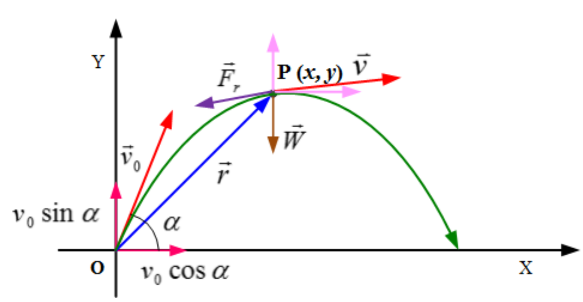


Figure 7.17: Resisted Projectile motion.

$$\begin{aligned}\dot{x}(0) &= v_0\cos\alpha \\ \dot{y}(0) &= v_0\sin\alpha\end{aligned}$$

and

$$\begin{aligned} x(0) &= 0 \\ y(0) &= 0 \end{aligned}$$

then at P, two force are acting on the particle, one is force of gravity, acting in the downward direction and the other is air resistance, opposing the velocity. Then by Newton's second law of motion its equation of motion is

$$\begin{aligned} \vec{F} &= \vec{W} + \vec{F}_r \\ m\vec{a} &= -m\vec{g} - k\vec{v} \\ \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} &= -g\hat{\jmath} - k\dot{x}\hat{\imath} - k\dot{y}\hat{\jmath} \\ &= -k\dot{x}\hat{\imath} - (g + k\dot{y})\hat{\jmath} \qquad (7.7.1) \end{aligned}$$

From (7.7.1), the horizontal component of equation of motion is

$$\begin{aligned} \ddot{x}(t) &= -k\dot{x} \\ \ddot{x}(t) + k\dot{x} &= 0 \qquad (7.7.2) \end{aligned}$$

and the vertical component is

$$\ddot{y}(t) = -(g + k\dot{y}) \qquad (7.7.3)$$

Replace $\dot{x}$ by w and $\ddot{x}$ by $\dot{w}$, (7.7.2) becomes first order differential equation

$$\dot{w}(t) + kw = 0$$

and has solution

$$\begin{aligned} w &= A_1 e^{-kt} \\ \dot{x}(t) &= A_1 e^{-kt} \qquad (7.7.4) \end{aligned}$$

At $t = 0$, (7.7.4) becomes

$$\dot{x}(0) = A_1 \qquad (7.7.5)$$

Using initial condition $\dot{x}(0) = v_0 \cos\alpha$, (7.7.5) implies that

$$A_1 = v_0 \cos\alpha$$

Then, (7.7.4) becomes

$$\dot{x}(t) = v_0 \cos\alpha e^{-kt} \qquad (7.7.6)$$

(7.7.6) gives the horizontal velocity of the particle at any time t. Integrating it with respect to t

$$x(t) = \frac{1}{-k}(v_0 \cos\alpha)e^{-kt} + B_1 \qquad (7.7.7)$$

At $t = 0$, (7.7.7) becomes

$$x(0) = -\frac{1}{k}(v_0 \cos\alpha)(1) + B_1 \tag{7.7.8}$$

Using initial condition $x(0) = 0$, (7.7.8) implies that

$$B_1 = \frac{1}{k}(v_0 \cos\alpha) \tag{7.7.9}$$

Using (7.7.9), (7.7.7) becomes

$$\begin{aligned} x(t) &= -\frac{1}{k}(v_0 \cos\alpha)e^{-kt} + \frac{1}{k}(v_0 \cos\alpha) \\ &= \frac{v_0 \cos\alpha}{k}\left(1 - e^{-kt}\right) \end{aligned} \tag{7.7.10}$$

(7.7.10) gives the horizontal scalar component of position of the particle at any time t. This time can be calculated as

$$\left(1 - e^{-kt}\right) = \frac{kx}{v_0 \cos\alpha} \tag{7.7.11}$$

$$\begin{aligned} e^{-kt} &= \left(1 - \frac{kx}{v_0 \cos\alpha}\right) \\ -kt &= \ln\left(1 - \frac{kx}{v_0 \cos\alpha}\right) \\ -t &= \frac{1}{k}\ln\left(1 - \frac{kx}{v_0 \cos\alpha}\right) \end{aligned} \tag{7.7.12}$$

Next for vertical component consider (7.7.3), replace $\dot{y}$ by z and $\ddot{y}$ by $\dot{z}$, (7.7.2) becomes linear first order differential equation

$$\dot{z}(t) + kz = -g$$

and has solution

$$\begin{aligned} z &= A_2 e^{-kt} - \frac{g}{k} \\ \dot{y}(t) &= A_2 e^{-kt} - \frac{g}{k} \end{aligned} \tag{7.7.13}$$

At $t = 0$, (7.7.13) becomes

$$\dot{y}(0) = A_2 - \frac{g}{k} \tag{7.7.14}$$

Using initial condition $\dot{y}(0) = v_0 \sin\alpha$, (7.7.14) implies that

$$A_2 = v_0 \sin\alpha + \frac{g}{k} \tag{7.7.15}$$

Using (7.7.15), (7.7.13) becomes

$$\dot{y}(t) = \left(v_0 \sin\alpha + \frac{g}{k}\right) e^{-kt} - \frac{g}{k} \tag{7.7.16}$$

(7.7.16) gives the vertical velocity of the particle at any time t.
Integrating (7.7.16), with respect to t

$$y(t) = -\frac{1}{k}\left(v_0 \sin\alpha + \frac{g}{k}\right) e^{-kt} - \frac{g}{k}t + B_2 \tag{7.7.17}$$

At $t = 0$, (7.7.17) becomes

$$y(0) = -\frac{1}{k}\left(v_0 \sin\alpha + \frac{g}{k}\right)(1) - \frac{g}{k}t(0) + B_2 \tag{7.7.18}$$

Using initial condition $y(0) = 0$, (7.7.18) becomes

$$B_2 = \frac{1}{k}\left(v_0 \sin\alpha + \frac{g}{k}\right) \tag{7.7.19}$$

Using (7.7.19), (7.7.17) becomes

$$\begin{aligned} y(t) &= -\frac{1}{k}\left(v_0 \sin\alpha + \frac{g}{k}\right) e^{-kt} - \frac{g}{k}t + \frac{1}{k}\left(v_0 \sin\alpha + \frac{g}{k}\right) \\ &= \frac{1}{k}\left(v_0 \sin\alpha + \frac{g}{k}\right)\left(1 - e^{-kt}\right) - \frac{g}{k}t \end{aligned} \tag{7.7.20}$$

(7.7.20) gives the vertical component of position of the particle at any time t. It can be expressed in terms of x by using (7.7.11) and (7.7.12).

$$\begin{aligned} y &= \frac{1}{k}\left(v_0 \sin\alpha + \frac{g}{k}\right)\frac{kx}{v_0\cos\alpha} + \frac{g}{k}\left[\frac{1}{k}\ln\left(1 - \frac{kx}{v_0\cos\alpha}\right)\right] \\ &= x\tan\alpha + \frac{gx}{kv_0\cos\alpha} + \frac{g}{k^2}\ln\left(1 - \frac{kx}{v_0\cos\alpha}\right) \end{aligned} \tag{7.7.21}$$

Next using the result

$$log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ \ldots$$

(7.7.21) becomes

$$\begin{aligned} y &= x\tan\alpha + \frac{gx}{kv_0\cos\alpha} \\ &+ \frac{g}{k^2}\left(-\frac{kx}{v_0\cos\alpha} - \frac{k^2x^2}{2v_0^2\cos^2\alpha} - \frac{k^3x^3}{3v_0^3\cos^3\alpha} - \ \ldots\right) \\ &= x\tan\alpha + \frac{gx}{kv_0\cos\alpha} - \frac{gx}{kv_0\cos\alpha} - \frac{gx^2}{2v_0^2\cos^2\alpha} - \frac{gkx^3}{3v_0^3\cos^3\alpha} - \ \ldots \\ &= x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha} - \frac{gkx^3}{3v_0^3\cos^3\alpha} - \ \ldots \end{aligned}$$

Ignoring higher powers, the path of the projectile is

$$y = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha} - \frac{gkx^3}{3v_0^3\cos^3\alpha}$$

Exercises

1. A footballer running on ground level kicks a football and the ball is projected with a speed of 10 m/s at an inclination 35° with the ground.
   (a) Write an expression for its horizontal component of position.
   (b) Write an expression for its vertical component of position.
   (c) Find its horizontal range.
   (d) What will be its maximum range.
   (e) Calculate its time of flight.
   (f) Calculate height attained by it.
   (g) What will be its height in case of maximum range.
2. Babur missile is projected with a speed of 880 km/h at an inclination $\frac{\pi}{3}$ with the ground.
   (a) Write an expression for its horizontal component of position.
   (b) Write an expression for its vertical component of position.
   (c) Find its horizontal range.
   (d) What will be its maximum range.
   (e) Calculate its time of flight.
   (f) Calculate height attained by it.
3. The maximum range of Shaheen missile from ground to ground mark is 1500 km.
   (a) What will be its muzzle velocity.
   (b) Calculate its time of flight.
   (c) Calculate height attained by it.

   If it is fired making an angle of 67° with the ground
   (d) Find its horizontal range.
   (e) Write an expression for its horizontal component of position.
   (f) Write an expression for its vertical component of position.
   (g) Calculate height attained by its.
4. A shell bursts on contact with the ground and pieces from it, fly in all directions. If a piece is flying making an angle $\frac{\pi}{6}$ with the ground with a speed of 500 m/s.
   (a) Write an expression for its horizontal component of position.
   (b) Write an expression for its vertical component of position.
   (c) Find its horizontal range.

(d) What will be its maximum range.

(e) Calculate its time of flight.

(f) Calculate height attained by the shell.

5. A particle of mass m is projected from the ground with a velocity $\vec{v}_0$, making an angle α with the ground. If the air resistance is proportional to the square of the velocity of the body.

   (a) Write an expression for its horizontal component of position.

   (b) Write an expression for its vertical component of position.

   (c) Calculate its time of flight.

Chapter 8

Motion Under Central Force

If a particle is moving under the action of a force which is always directed towards or away from a fixed point and depends on its distance from that point, then the force is called a central force, the motion is known as central force motion and the fixed point is called the center of the force. The fixed point is usually taken as the origin. The orbital movement of planets and satellites are such motions. The laws which govern this motion were first postulated by Kepler (1571-1630). His interest was in describing the motion of planets around the sun. He postulated the following laws:

K1 The orbits of the planets are ellipses with the Sun at one focus

K2 The line joining a planet to the Sun sweeps out equal areas in equal intervals of time

K3 The square of the period of a planet is proportional to the cube of the major axis of its elliptical orbit

These three laws can indeed be derived from Newtonian mechanics.

8.1 Keplers Problem

Consider the motion of a particle of mass m, in an inertial reference frame, under the influence of a central force, F, as shown in Fig. 8.1. Let the particle be at P having position vector $\vec{r}$. Then the force at P is

$$\vec{F}(r) = F\hat{r}$$

where $\hat{r}$ is a unit vector in the radial direction. Particularly, if the force is inversely proportional to the square of the distance between the particle and the origin, such as the gravitational force given as,

$$\vec{F}(r) = -\frac{\mu m}{r^2}\hat{r} \tag{8.1.1}$$

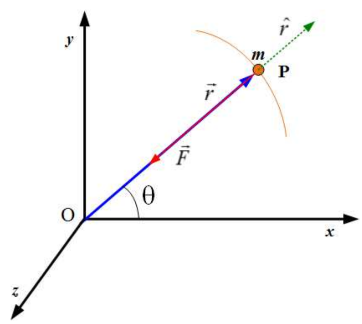


Figure 8.1: Central motion.

where μ is the gravitational parameter.
In general, Keplers problem is equivalent to the two-body problem, in which two masses, m_1 and m_2, can move solely due to their mutual gravitational attraction. This equivalence is obvious when $m_1 >> m_2$, since, in this case, the center of mass of the system can be taken to be at m_1. Even when the two masses are of similar size, the problem can be reduced to a Kepler problem. In this case, the force is directed along the line connecting the (centres of mass of) the two bodies. The most fundamental forces like the gravitational, electrostatic and certain nuclear forces, are of this kind.

8.1.1 Equivalence Between the Two-body Problem and Keplers Problem

Consider a system of two bodies of masses m_1 and m_2 which interact though gravitational attraction. Let $\vec{r}_1$ and $\vec{r}_2$ be their position vectors relative to a fixed origin O, as shown in Fig. 8.2. Thus, the position of the center of gravity, G, of the system will be

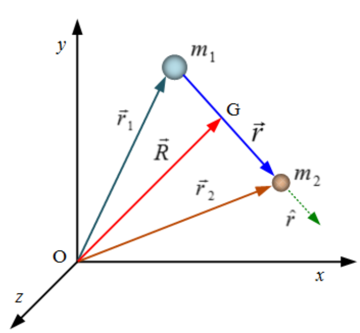


Figure 8.2: Central motion.

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \tag{8.1.2}$$

Also

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \tag{8.1.3}$$

Then

$$\vec{r}_1 = \vec{r}_2 - \vec{r} \tag{8.1.4}$$

and

$$\vec{r}_2 = \vec{r} + \vec{r}_1 \tag{8.1.5}$$

Using (8.1.4) in (8.1.2), we have

$$\begin{aligned}\vec{R} &= \frac{m_1(\vec{r}_2 - \vec{r}) + m_2\vec{r}_2}{m_1 + m_2} \\ &= -\frac{\vec{r}}{m_1 + m_2} + \vec{r}_2\end{aligned}$$

or

$$\vec{r}_2 = \vec{R} + \frac{\vec{r}}{m_1 + m_2} \tag{8.1.6}$$

Similarly

$$\vec{r}_1 = \vec{R} - \frac{\vec{r}}{m_1 + m_2} \tag{8.1.7}$$

The attractive force on m_1 due to m_2 is

$$\vec{F} = G\frac{m_1 m_2}{r^2}\hat{r}$$

where G is the gravitational constant And by Newton's second law of motion, this force is

$$\vec{F} = m_1\ddot{\vec{r}}_1 \tag{8.1.8}$$

Then we can write

$$\begin{aligned}m_1\ddot{\vec{r}}_1 &= G\frac{m_1 m_2}{r^2}\hat{r} \\ \ddot{\vec{r}}_1 &= G\frac{m_2}{r^2}\hat{r} \qquad (8.1.9)\end{aligned}$$

Similarly the attractive force on m_2 due to m_1 is

$$\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{r}$$

And by Newton's second law of motion, this force is

$$\vec{F} = m_2\ddot{\vec{r}}_2 \tag{8.1.10}$$

Then we can write

$$\begin{aligned} m_2\ddot{\vec{r}}_2 &= -G\frac{m_1 m_2}{r^2}\hat{r} \\ \ddot{\vec{r}}_2 &= -G\frac{m_1}{r^2}\hat{r} \end{aligned} \tag{8.1.11}$$

From (8.1.3), we can write

$$\ddot{\vec{r}} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1 \tag{8.1.12}$$

Using (8.1.9) and (8.1.11), (8.1.12) can becomes

$$\begin{aligned} \ddot{\vec{r}} &= -G\frac{m_1}{r^2}\hat{r} - G\frac{m_2}{r^2}\hat{r} \\ &= -G\frac{m_1 + m_2}{r^2}\hat{r} \\ \frac{1}{m_1 + m_2}\ddot{\vec{r}} &= -G\frac{1}{r^2}\hat{r} \end{aligned} \tag{8.1.13}$$

Taking product (8.1.13) with $m_1 m_2$

$$\begin{aligned} \frac{m_1 m_2}{m_1 + m_2}\ddot{\vec{r}} &= -G\frac{m_1 m_2}{r^2}\hat{r} \qquad (8.1.14) \\ \mu\ddot{\vec{r}} &= -G\frac{m_1 m_2}{r^2}\hat{r} \end{aligned}$$

The quantity

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

is known as reduced mass. Multiplying (8.1.8) by m_2

$$m_2\vec{F} = -m_1 m_2\ddot{\vec{r}}_1 \tag{8.1.15}$$

Multiplying (8.1.10) by m_1

$$m_1\vec{F} = m_1 m_2\ddot{\vec{r}}_2 \tag{8.1.16}$$

Adding (8.1.15) and (8.1.16)

$$(m_1 + m_2)\vec{F} = m_1 m_2\left(\ddot{\vec{r}}_2 - \ddot{\vec{r}}_1\right) \tag{8.1.17}$$

Using (8.1.12), (8.1.17) can be written as

$$\vec{F} = \frac{m_1 m_2}{m_1 + m_2}\ddot{\vec{r}} \tag{8.1.18}$$

From (8.1.14) and (8.1.18), we can be written as

$$\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{r} \tag{8.1.19}$$

The above expression shows that the motion of m_2 relative to m_1 is in fact a Kepler problem.

8.2 Motion under Central Force

When a particle moves under the action of a force directed toward a fixed point (center of attraction), the motion is called central-force motion as discussed in the beginning of this chapter.

8.2.1 Angular Momentum under Central Force is Constant or Central Motion is Planar Motion

Consider a particle of mass m is moving under the influence of central force. At time t the particle is at P, having position vector $\vec{r}$. The cental force acts along radial component and the velocity $\vec{v}$ of the particle is tangent to $\vec{r}$ at P. The angular momentum L of the particle is

$$\begin{aligned}\vec{L} &= \vec{r}\times\vec{p} \\ &= \vec{r}\times m\vec{v}\end{aligned}$$

where p is the linear momentum of the particle. Therefore, the angular momentum $\vec{L}$ is always perpendicular to the plane defined by the particle's position vector $\vec{r}$ and velocity $\vec{v}$. The rate of change of the angular momentum $\vec{L}$ equals the net torque

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r}\times\vec{F}$$

Using (8.2.1)

$$\begin{aligned}\frac{d\vec{L}}{dt} &= \dot{\vec{r}}\times m\vec{v} + \vec{r}\times m\dot{\vec{v}} \\ &= \dot{\vec{r}}\times m\dot{\vec{r}} + \vec{r}\times m\ddot{\vec{r}} \\ &= 0 + \vec{r}\times\vec{F}\end{aligned}$$

Since $\vec{r}$ and $\vec{F}$ points in the same or opposite direction so

$$\vec{r}\times\vec{F} = r\hat{r}\times F(r)\hat{r} = 0$$

Hence, the angular momentum $\vec{L}$ is constant. Consequently, the particle's position $\vec{r}$ and velocity $\vec{v}$ always lie in a single plane perpendicular to $\vec{L}$, as shown in Fig. 8.3.

8.2.2 Equation of Motion in Polar Coordinates

Since the motion is confined to a plane, we can choose our coordinate system with $\hat{z}$ being parallel to $\vec{L}$, so that the motion is in the xy plane with origin at the centre of force. The equation of motion is of a particle of mass m moving under cental force is

$$\vec{F}(r) = m\ddot{\vec{r}}$$

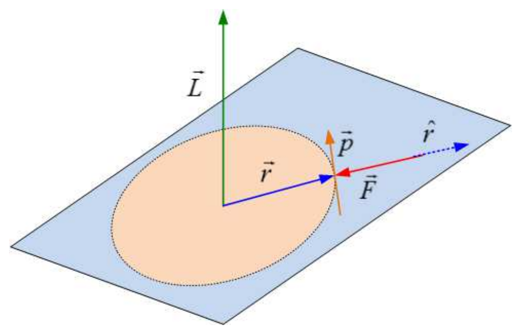


Figure 8.3: Central force motion is planer motion.

Introducing polar coordinates, the equation of motion is

$$F(r)\hat{r} = m\left[\left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(2\dot{r}\dot{\theta} + \ddot{\theta}\right)\hat{\theta}\right] \tag{8.2.1}$$

Using (8.1.1), (8.2.2) becomes

$$-\frac{\mu m}{r^2}\hat{r} = m\left[\left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(2\dot{r}\dot{\theta} + \ddot{\theta}\right)\hat{\theta}\right] \tag{8.2.2}$$

Since $m \neq 0$, the radial component is

$$-\frac{\mu}{r^2} = \left(\ddot{r} - r\dot{\theta}^2\right) \tag{8.2.3}$$

and the transverse component is

$$\left(2\dot{r}\dot{\theta} + \ddot{\theta}\right) = 0 \tag{8.2.4}$$

(8.2.4) can be written as

$$\frac{1}{r}\left[\frac{d}{dt}\left(r^2\dot{\theta}\right)\right] = \left(2\dot{r}\dot{\theta} + \ddot{\theta}\right)$$

or

$$r^2\dot{\theta} = h\ (constant) \tag{8.2.5}$$

$$r^2 = \frac{h}{\dot{\theta}} \tag{8.2.6}$$

and

$$\dot{\theta} = \frac{h}{r^2} \tag{8.2.7}$$

8.2.3 Motion in a Central-Force Field (Gravitational Field)

In a uniform gravitational field, the gravitational acceleration is everywhere constant and depends only on its distance r from the center of force. Here, the magnitude $F(r)$ (which is positive for a repulsive force and negative for an attractive force) is defined in terms of the central potential $U(r)$ as

$$F(r) = -U^{'}(r)$$

K2 The line joining a planet to the Sun sweeps out equal areas in equal intervals of time. That is

$$\frac{dA}{dt} = \text{constant} \tag{8.2.8}$$

Proof Consider a planet P moves around the sun S. The trajectory is in the xy plane. At any time t, P has coordinates $P(r, \theta)$ relative to S. At time $t + dt$, the planet is at Q, having position vector $r + dr$ as shown in Fig. 8.4. Since dr is very very small, then

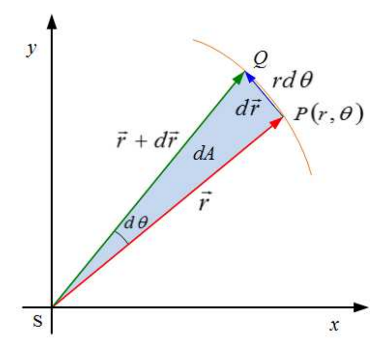


Figure 8.4: Planet motion around sun.

$$dr \cong ds = rd\theta$$

Let dA be the area swept by $\vec{r}$, which is the area of the small region SPQ, that is

$$dA = \frac{1}{2}r^2 d\theta$$

The areal velocity of the planet is

$$\begin{aligned}\frac{dA}{dt} &= \frac{1}{2}r^2\frac{d\theta}{dt} \\ &= \frac{1}{2}r^2\dot{\theta} \qquad (8.2.9)\end{aligned}$$

Using (8.2.5), (8.2.21) becomes

$$\frac{dA}{dt} = \frac{h}{2} \qquad \text{(constant)} \tag{8.2.10}$$

which proves Keplers second law.

8.2.4 Radial Component

Since

$$\frac{d}{dt}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\dot{r}$$

or

$$\dot{r} = -r^2\frac{d}{dt}\left(\frac{1}{r}\right) \tag{8.2.11}$$

Using (8.2.6), (8.2.11) becomes

$$\begin{aligned} \dot{r} &= -\frac{h}{\dot{\theta}}\frac{d}{dt}\left(\frac{1}{r}\right) \\ &= -\frac{h}{\dot{\theta}}\frac{d}{d\theta}\left(\frac{1}{r}\right)\frac{d\theta}{dt} \\ &= -h\frac{d}{d\theta}\left(\frac{1}{r}\right) \end{aligned} \tag{8.2.12}$$

Differentiate (8.2.12) with respect to time

$$\begin{aligned} \ddot{r} &= -h\frac{d}{dt}\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right) \\ &= -h\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\frac{d\theta}{dt} \\ &= -h\dot{\theta}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) \end{aligned} \tag{8.2.13}$$

Using (8.2.7), (8.2.13) becomes

$$\ddot{r} = -\frac{h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) \tag{8.2.14}$$

Using (8.2.14), (8.2.3), can be written as

$$\begin{aligned} -\frac{\mu}{r^2} &= \left(-\frac{h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) - r\dot{\theta}^2\right) \\ \mu &= \left(h^2\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}r^4\dot{\theta}^2\right) \end{aligned} \tag{8.2.15}$$

Since $h = r^2\dot{\theta}$, *i.e.* $h^2 = r^4\dot{\theta}^2$, then (8.2.15) can be written as

$$\mu = \left(h^2 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{h^2}{r} \right)$$

or

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{\mu}{h^2} \tag{8.2.16}$$

(8.2.16) is a linear second order nonhomogeneous differential equation for $\frac{1}{r}$ as a function of θ. Its general solution is

$$\frac{1}{r} = \left(\frac{1}{r}\right)_c + \left(\frac{1}{r}\right)_p$$

Where $\left(\frac{1}{r}\right)_c$ is complementary solution and $\left(\frac{1}{r}\right)_p$ is particular integral. For $\left(\frac{1}{r}\right)_c$, the characteristic equation is

$$m^2 + 1 = 0$$

and the solution is

$$\left(\frac{1}{r}\right)_c = A_1 \cos\theta + A_2 \sin\theta$$

Where A_1 and A_2 are constants of integration, let $A_1 = e\frac{\mu}{h^2}\cos\psi$ and $A_2 = -e\frac{\mu}{h^2}\sin\psi$, with e and ψ are constants. Then

$$\left(\frac{1}{r}\right)_c = e\frac{\mu}{h^2}\cos(\theta + \psi)$$

The particular integral is

$$\left(\frac{1}{r}\right)_p = \frac{\mu}{h^2}$$

Hence the general solution is

$$\frac{1}{r} = \frac{\mu}{h^2}\left(1 + e\cos(\theta + \psi)\right)$$

The constant $\psi = 0$ and $e > 0$ if we rotate the base line $\theta = 0$, then the equation describing the trajectory will be

$$r = \frac{h^2}{\mu\left(1 + e\cos\theta\right)} \tag{8.2.17}$$

The standard polar form of a conic section is

$$r = \frac{l}{(1 + e\cos\theta)} \tag{8.2.18}$$

Hence (8.2.17) represents the equation of a conic in polar coordinates with one focus at the center of the force. The length of the semi latus rectum is

$$l = \frac{h^2}{\mu} \tag{8.2.19}$$

Hence the reduced mass is

$$\mu = \frac{h^2}{l} \tag{8.2.20}$$

The constant $e \geq 0$ is called the eccentricity, defining the conic surface as follows

$$\begin{aligned} e &= 0 \quad \text{the curve is a circle} \\ e &< 1 \quad \text{the curve is an ellipse} \\ e &= 1 \quad \text{the curve is a parabola} \\ e &> 1 \quad \text{the curve is a hyperbola} \end{aligned}$$

When $e < 1$, the trajectory given by (8.2.17) is an ellipse, thus proving Keplers first law.

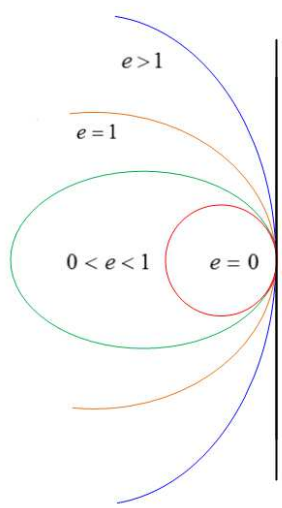


Figure 8.5: conic surfaces

The point in the trajectory which is closest to the focus is called the *periapsis* and is denoted

by π. For elliptical orbits, the point in the trajectory which is farthest away from the focus is called the *apoapsis* and is denoted by α. When considering orbits around the earth, these points are called the *perigee* and *apogee*, whereas for orbits around the sun, these points are called the *perihelion* and aphelion, respectively.
At perihelion and aphelion, the radial velocity is zero, so $\vec{v}$ is orthogonal to the radius vector $\vec{r}$. At these points the areal velocity of the planet is

$$\frac{dA}{dt} = \frac{1}{2}rv$$

8.2.5 Elliptical Trajectories

When $e < 1$, (8.2.17) represents an ellipse with one focus is $S(0,0)$, as shown in Fig. 8.7. Its trajectory is

$$r = \frac{l}{(1 + e\cos\theta)} \tag{8.2.21}$$

The point π is closest to the focus S, so is the *perihelion* and the point α is farthest away

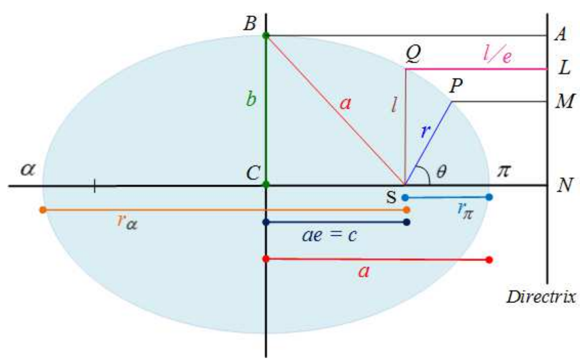


Figure 8.6: Elliptic trajectories

from the focus S, so is the *aphelion.* If a is the semi major axis and b is the semi minor axis of the ellipse, then

$$a^2 = b^2 + c^2 \tag{8.2.22}$$

Also the distance between the focus S and the center of the ellipse is

$$c = ae = a - r_\pi \tag{8.2.23}$$

The eccentricity is

$$e = \frac{c}{a} \tag{8.2.24}$$

Then (8.2.22) becomes

$$a^2 = b^2 + (ae)^2 \tag{8.2.25}$$

or

$$b^2 = a^2(1 - e^2) \tag{8.2.26}$$

And the semi-minor axis of the ellipse is

$$b = a\sqrt{(1 - e^2)} \tag{8.2.27}$$

Also l is its semi latus rectum, given as

$$l = \frac{b^2}{a} \tag{8.2.28}$$

Using (8.2.28), (8.2.26) becomes

$$l = a(1 - e^2) \tag{8.2.29}$$

or

$$a = \frac{l}{(1 - e^2)} \tag{8.2.30}$$

From Fig. 8.7, the major axis can be written as

$$2a = r_\pi + r_\alpha \tag{8.2.31}$$

Using (8.2.30), (8.2.31) can be written as

$$2a = r_\pi + r_\alpha = \frac{2l}{(1 - e^2)} \tag{8.2.32}$$

Apply partial fraction technique on right hand side of (8.2.32), then we have

$$2a = r_\pi + r_\alpha = \frac{l}{(1 + e)} + \frac{l}{(1 - e)} \tag{8.2.33}$$

so from (8.2.33), we can suppose that

$$r_\pi = \frac{l}{(1 + e)} = a(1 - e) \tag{8.2.34}$$

and

$$r_\alpha = \frac{l}{(1 - e)} = a(1 + e) \tag{8.2.35}$$

The area of the ellipse is

$$\begin{aligned} A &= \pi ab \\ &= \pi a^2 \sqrt{(1-e^2)} \end{aligned} \tag{8.2.36}$$

Consider (8.2.10)

$$\frac{dA}{dt} = \frac{h}{2}$$

which is first order differential equation and has solution

$$A = \frac{h}{2}T \tag{8.2.37}$$

where T is constant of integration, known as period of the orbit. Equating (8.2.38) and (8.2.39)

$$\begin{aligned} \frac{h}{2}T &= \pi a^2 \sqrt{(1-e^2)} \\ h &= \frac{2\pi}{T} a^2 \sqrt{(1-e^2)} \end{aligned} \tag{8.2.38}$$

Using (8.2.29), (8.2.38) can be written as

$$h^2 = \left(\frac{2\pi}{T}\right)^2 a^3 l \tag{8.2.39}$$

Using (8.2.39), the reduced mass given by (8.2.20) can be written as

$$\begin{aligned} \mu &= \frac{h^2}{l} \\ &= \left(\frac{2\pi}{T}\right)^2 a^3 \end{aligned}$$

or

$$T^2 = \left(\frac{4\pi^2}{\mu}\right) a^3 \tag{8.2.40}$$

(8.2.40) is the Keplers third law.

Example 8.2.1. *The planet Mercury orbits the Sun in* 87.97 *days, in an elliptic orbit with semi major axis is* 57.91×10^6 *km and semi minor axis is* 56.67×10^6 *km. Find*

1. *the coordinates of the center of the elliptic path.*

2. *the eccentricity.*

3. *the length of the semi latusrectum.*

4. *the area of the ellipse*

5. *the areal velocity.*

6. *its speed*

   a) *at its perihelion which is* 46.00×10^6 *km away from the sun;*

   b) *at its aphelion which is* 69.82×10^6 *km away from the sun.*

Solution The given data is

$$\begin{aligned} T &= 87.97 \ \ days = 7.6 \times 10^5 \ s \\ a &= 57.91 \times 10^6 \ km \\ b &= 56.67 \times 10^6 \ km \\ r_\pi &= 46.00 \times 10^6 \ km \\ r_\alpha &= 69.82 \times 10^6 \ km \end{aligned}$$

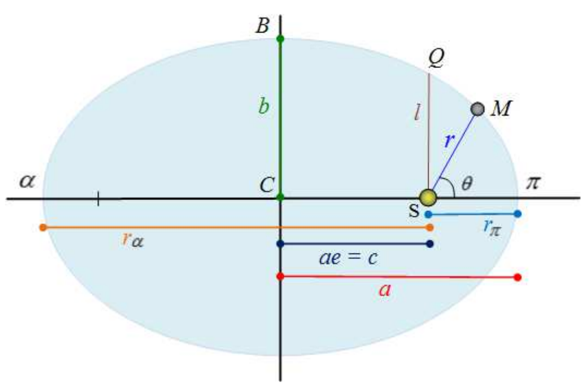


Figure 8.7: The planet Mercury moves in elliptic orbit around the Sun

1. To find the center of the elliptic path, we use (8.2.23), to calculate the distance of the focus $S(0,0)$ from the center $C(-c,0)$ as

$$\begin{aligned} c &= a - r_\pi \\ &= 57.91 \times 10^6 - 46.00 \times 10^6 \\ &= 11.91 \times 10^6 \quad km \end{aligned} \tag{8.2.41}$$

Hence the coordinate of the center are $C(-11.91 \times 10^6, 0)$.

2. The eccentricity is given by (8.2.24)

$$\begin{aligned} e &= \frac{c}{a} \\ &= \frac{11.91 \times 10^6}{57.91 \times 10^6} \\ &\approx 0.2057 \end{aligned} \tag{8.2.42}$$

3. The length of semi latusrectum is given by (8.2.28)

$$\begin{aligned} l &= \frac{b^2}{a} \\ &= \frac{\left(56.67 \times 10^6\right)^2}{57.91 \times 10^6} \\ &= 36.54 \times 10^6 \quad km \end{aligned} \tag{8.2.43}$$

4. the area of the ellipse is

$$A = \pi ab$$
$$= \pi \left(56.67 \times 10^6\right)\left(57.91 \times 10^6\right) \tag{8.2.44}$$
$$= 8.37 \times 10^{15} \quad km^2 \tag{8.2.45}$$

5. The areal velocity is

Consider (8.2.10)

$$\begin{aligned} \frac{dA}{dt} &= \frac{A}{T} \\ &= \frac{8.37 \times 10^{15}}{7.6 \times 10^5} \\ &= 1.1 \times 10^9 \quad km^2/s \end{aligned} \tag{8.2.46}$$

At perihelion and aphelion, the areal velocity of the planet is

$$\frac{dA}{dt} = \frac{1}{2} rv$$

6. So its speed

   a) at its perihelion is

$$v = \frac{2}{r}\frac{dA}{dt}$$

At perihelion, $r_\pi = 46.00 \times 10^6 \quad km$, then the speed is

$$\begin{aligned} v &= \frac{2\left(1.1 \times 10^9\right)}{46.00 \times 10^6} \\ &= 47.8 \quad km/s \end{aligned} \tag{8.2.47}$$

   b) At aphelion, $r_\alpha = 69.82 \times 10^6$, then the speed is

$$\begin{aligned} v &= \frac{2\left(1.1 \times 10^9\right)}{69.82 \times 10^6} \\ &= 31.5 \quad km/s \end{aligned} \tag{8.2.48}$$

Exercise A satellite is launched in a direction parallel to the surface of the earth with a speed $36,900$ km/h in an elliptic orbit from *perigee*, 500 km away from the earth. Let 63700 be the radius of the earth, then find

1. the coordinates of the center of the elliptic path.
2. the eccentricity.
3. the length of the semi latusrectum.
4. the semi minor axis
5. semi major axis
6. the time period.
7. the area of the ellipse
8. the areal velocity.
9. its speed at *apogee*
10. *apogee* distance from earth.

Chapter 9

Small Oscillation

9.1 Small Oscillation

Consider one dimensional motion of a particle of mass m under a conservative force. If $U(x)$ is the potential function then the force is

$$F = -\frac{dU}{dx} \tag{9.1.1}$$

Let the particle initially be at rest at a local minimum of $U(x)$. Let this minimum is zero and it exists at $x = x_0$. Then

$$U(x_0) = 0 \tag{9.1.2}$$

and

$$\frac{dU}{dx_0} = 0 \tag{9.1.3}$$

Let it be given a small kick, from its equilibrium position, so that it moves back and forth around it, executing simple harmonic motion of small amplitude. To see this, expand $U(x)$ in a Taylor series around the equilibrium point, x_0.

$$\begin{aligned} U(x) &= U(x_0) + U'(x_0)(x - x_0) + \frac{1}{2!}U''(x_0)(x - x_0)^2 \\ &+ \frac{1}{3!}U'''(x_0)(x - x_0)^3 + \cdot\ \cdot\ \cdot \end{aligned} \tag{9.1.4}$$

Using (9.1.2) and (9.1.3) in (9.1.4), we are left with the $U''(x_0)$ and higher-order terms. But for sufficiently small displacements, these higher-order terms are negligible compared to the $U''(x_0)$ term, and we are left with

$$U(x) \cong \frac{1}{2}U''(x_0)(x - x_0)^2 \tag{9.1.5}$$

Also the Hookes-law potential is

$$U(x) = \frac{1}{2}k(x_0)(x-x_0)^2 \tag{9.1.6}$$

From (9.1.5) and (9.1.6) the spring constant is

$$k = U''(x_0) \tag{9.1.7}$$

And the frequency of small oscillations is

$$\begin{aligned} \omega &= \sqrt{\frac{k}{m}} \\ &= \sqrt{\frac{U''(x_0)}{m}} \end{aligned} \tag{9.1.8}$$

Example 9.1.1. *A particle of mass m is moving under the influence of the potential*

$$U(x) = \frac{A}{x^2} - \frac{B}{x} \tag{9.1.9}$$

Find

a) Equilibrium position.

b) Angular frequency.

Solution Using (9.1.3), the equilibrium point, x_0 is

$$U'(x) = 0 \tag{9.1.10}$$

$$-\frac{A}{x^3} + \frac{B}{x^2} = 0 \tag{9.1.11}$$

$$x = 2\frac{A}{B} \tag{9.1.12}$$

or

$$x_0 = 2\frac{A}{B} \tag{9.1.13}$$

The spring constant is given by using Using (9.1.7). First the second derivative of $U(x)$ is

$$U''(x) = \frac{A}{x^4} - \frac{B}{x^3} \tag{9.1.14}$$

Using (9.1.13),(9.1.14) becomes,

$$\begin{aligned} \omega &= \sqrt{\frac{U''(x_0)}{m}} \\ &= \sqrt{\frac{B^4}{8mA^3}} \end{aligned} \tag{9.1.15}$$

(9.1.15) gives the frequency of oscillation.

9.2 Equilibrium and Stability

A body is said to be in equilibrium if the vector sum of all the external forces acting on it is zero. If F given by (9.1.1), is the only acting force, then for equilibrium, we can write

$$\begin{aligned} F &= 0 \\ -U'(x) &= 0 \end{aligned}$$

or

$$U'(x) = 0$$

Next we consider this equilibrium is stable or unstable.

9.2.1 Stable Equilibrium

The equilibrium of a particle is stable if

$$\begin{aligned} U'(x) &= 0 \\ U''(x) &> 0 \end{aligned}$$

9.2.2 Unstable Equilibrium

The equilibrium of a particle is unstable if

$$\begin{aligned} U'(x) &= 0 \\ U''(x) &< 0 \end{aligned}$$

Example 9.2.1. *In spring mass system with $k > 0$, the potential energy function is*

$$U(x) = \frac{1}{2}kx^2$$

and

$$U'(x) = kx$$

then

$$U'(x) = 0$$

gives

$$x = 0$$

Next

$$\begin{aligned} U''(x) &= k \\ U''(0) &= k > 0 \end{aligned}$$

Hence the system has stable equilibrium at $x = 0$.

9.3 Motion in a Rapidly Oscillating Field under Harmonic Force

In this section, the application of small but fast oscillation is presented accompanied with the stability of the system. This stabilization was initialized by Stephenson in 1908. Later, in 1950 Kapitza renewed it. In 1961 Landau introduced oscillation under the harmonic force. For stabilization, the potential energy function is obtained by using averaging procedure. For this consider one dimensional motion of a particle of mass m under the action of a time independent potential field force

$$F_1 = \frac{-dU}{dx} \tag{9.3.1}$$

and a small but fast oscillating force

$$F_2 = f(x,t) = f_1 \cos \omega t + f_2 \sin \omega t. \tag{9.3.2}$$

The system has time period T_U due to force F_1, so may oscillate due to this force with frequency $\omega_U = \frac{1}{T_U}$ and T is the time period due the force F_2, and oscillation due to this force is $\omega = \frac{1}{T}$. The fast oscillation means the frequency $\omega \gg \omega_U$. Also the coefficients f_1, f_2 are functions of co-ordinates only. The magnitude of F_2 is not assumed small in comparison with the force F_1 but we shall assume that the oscillation of the particle (denoted below by ξ) due to this force is small.
Since the system has one dimensional motion it means it depends only on the space co-ordinate x. Then by Newton'second law of motion, equation of motion of the particle is

$$m\ddot{x} = F_1 + F_2 \tag{9.3.3}$$

Since the particle transverse a slow path and at the same time execute fast but small oscillations of frequency ω about the path, the function $x(t)$ (path) is a sum of both parts as shown in Fig. 9.5.17. If we represent $X(t)$ the slow path and $\xi(t)$ the fast but small oscillations, the path is

$$x(t) = X(t) + \xi(t) \tag{9.3.4}$$

By (9.3.4) the functions f and U will also be transformed. Using Taylor's expansion in

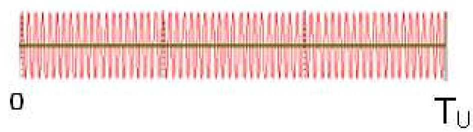


Figure 9.1: Path of the particle

powers of ξ up to first order terms we have

$$f(x) = f(X+\xi) = f(X) + \xi \frac{\partial f}{\partial X}$$

$$U(x) = U(X+\xi) = U(X) + \xi \frac{\partial U}{\partial X}$$

Since U is a function of coordinates only so $x \to X$ and will treated independent of ξ. Hence for (9.3.1) we can write

$$\frac{dU}{dx} = \frac{d}{dX}[U(x) = U(X+\xi)] = \frac{d}{dX}[U(X) + \xi \frac{\partial U}{\partial X}] = \frac{dU}{dX} + \xi \frac{d^2U}{dX^2} \qquad (9.3.5)$$

substituting Eq.(9.3.5) in Eq.(9.3.3) we have

$$m\ddot{X} + m\ddot{\xi} = -\frac{dU}{dX} - \xi \frac{d^2U}{dX^2} + f(X,t) + \xi \frac{df}{dX} \qquad (9.3.6)$$

This equation involves both fast and slow motions, which must be separately equal. For the fast oscillating term we can put simply

$$m\ddot{\xi} = f(X,t) \qquad (9.3.7)$$

and the slow term is

$$m\ddot{X} = -\frac{dU}{dX} - \xi \frac{d^2U}{dX^2} + \xi \frac{df}{dX} \qquad (9.3.8)$$

Which contain the small factor ξ and are therefore of a higher order of smallness (but the derivative $\ddot{\xi}$ is proportional to the large quantity ω^2 and so is not small). Integrating equation Eq. (9.3.7) with the function f given by Eq. (9.3.2). Since the coefficients f_1 and f_2 are functions of coordinates only, so will be treated as a constant.

$$\begin{aligned} m\dot{\xi} &= \int f dt \\ &= \int (f_1 \cos\omega t + f_2 \sin\omega t)\, dt \\ &= \frac{1}{\omega}(-f_1 \sin\omega t + f_2 \cos\omega t) + C_1 \end{aligned}$$

The term $C_1 = \dot{\xi}_0$ is the initial speed of the small oscillation. Let at $t = 0$, $\dot{\xi}_0 = 0$, then above relation is

$$m\dot{\xi} = \frac{1}{\omega}\left(-f_1 \sin\omega t + f_2 \cos\omega t\right)$$

Again integrating, we have

$$m\xi = -\frac{1}{\omega^2}\left(f_1 \cos\omega t + f_2 \sin\omega t\right) + C_2$$

The term $C_2 = \xi_0$ is the initial position of the small oscillation. Let at $t = 0$, $\xi_0 = 0$, then above relation is

$$m\xi = -\frac{f}{\omega^2}$$

or

$$\xi = -\frac{f}{m\omega^2} \tag{9.3.9}$$

(9.3.9) gives the small but fast oscillation. Next the mean value of a T periodic function is denoted by bar and is given by

$$\bar{f} = \frac{1}{T}\int_0^T f(t)dt \tag{9.3.10}$$

Using Eq. (9.3.10), the mean value of $f(t)$ over its period $T = \frac{2\pi}{\omega}$ is

$$\begin{aligned}
\bar{f} &= \frac{1}{T}\int_0^T \left(f_1 \cos\omega t + f_2 \sin\omega t\right) dt \\
&= \frac{1}{T}\left(-\frac{f_1}{\omega}\sin\omega t + \frac{f_2}{\omega}\cos\omega t\right)_0^T \\
&= \frac{1}{2\pi}\Big[- f_1\left[\sin(2\pi) - \sin(0)\right] + f_2\left[\cos(2\pi) - \cos(0)\right]\Big] \\
&= 0
\end{aligned}$$

Hence the mean value of $f(t)$ over its period T is zero. Since $\xi(t)$ contains f, so its mean value over same period is also zero. Also

$$\bar{\ddot{\xi}} = 0$$

During this time averaging, the function $X(t)$ remains invariant. We therefore have $\overline{x} = X(t)$, i.e. $X(t)$ describes the slow motion of the particle averaged over the rapid oscillations. Using this time averaging technique, we shall derive a relation which will be the function $X(t)$ only.

Next time averaging (9.3.6) over its period ($0 \to T$). As calculated above, the mean values of the first powers of f and ξ over this period are zero, so the mean value of (9.3.6) over this period is

$$\begin{aligned}
m\ddot{X} &= -\frac{dU}{dX} - \overline{\xi\frac{df}{dX}} \\
&= -\frac{dU}{dX} - \frac{1}{m\omega^2}\overline{f\frac{df}{dX}}
\end{aligned} \tag{9.3.11}$$

This relation represent smooth motion of the particle averaged over small oscillation. Consider

$$\frac{1}{2}\frac{df^2}{dX} = f\frac{df}{dX} \tag{9.3.12}$$

Using Eq. (9.3.12), Eq. (9.3.11) becomes

$$m\ddot{X} = -\frac{dU}{dX} - \frac{1}{m\omega^2}\frac{1}{2}\overline{\frac{df^2}{dX}} \tag{9.3.13}$$

(9.3.13) is a function of X only so can be written as

$$m\ddot{X} = -\frac{d}{dX}\left(U + \frac{\overline{f}^2}{2m\omega^2}\right) \tag{9.3.14}$$

In (9.3.14) the quantity $\left(U + \frac{\overline{f}^2}{2m\omega^2}\right)$ can be regarded as potential energy function. We call it effective potential energy function and is given by

$$U_{eff} = U + \frac{\overline{f}^2}{2m\omega^2} \tag{9.3.15}$$

Using (9.3.15), (9.3.14) can be written as

$$m\ddot{X} = -\frac{dU_{eff}}{dX} \tag{9.3.16}$$

(9.3.16) represents that after averaging the net force is conservative. To calculate $\overline{f}^2$ we first calculate the quantity f^2 from (9.3.2) as

$$\begin{aligned} f^2 &= (f_1\cos\omega t + f_2\sin\omega t)^2 \\ &= f_1^2\cos^2\omega t + f_2^2\sin^2\omega t + 2f_1f_2\cos\omega t\sin\omega t \end{aligned} \tag{9.3.17}$$

Taking time average of (9.3.17), we have

$$\overline{f^2} = f_1^2\overline{\cos^2\omega t} + f_2^2\overline{\sin^2\omega t} + 2f_1f_2\overline{\cos\omega t\sin\omega t} \tag{9.3.18}$$

To simplify this calculation we make use of the following trigonometric orthogonal relations.

$$\begin{aligned} \overline{\sin\omega t\cos\omega t} &= \frac{1}{T}\int_0^T \sin\omega t\cos\omega t dt = 0 \\ \overline{\cos^2\omega t} &= \frac{1}{T}\int_0^T \cos^2\omega t dt = \frac{1}{T}\int_0^T \frac{1}{2}(1+\cos 2\omega t)dt = \frac{1}{2} \\ \overline{\sin k\omega^2 t} &= \frac{1}{T}\int_0^T \sin^2\omega t dt = \frac{1}{T}\int_0^T \frac{1}{2}(1-\cos 2\omega t)dt = \frac{1}{2} \end{aligned}$$

Hence (9.3.18) can be written as

$$\overline{f^2} \;=\; \frac{1}{2}(f_1^2 + f_2^2) \tag{9.3.19}$$

Using (9.3.19), (9.3.15) can be written as

$$U_{eff} = U + \frac{f_1^2 + f_2^2}{4m\omega^2} \tag{9.3.20}$$

Thus the motion of the particle averaged over the small but fast oscillations is the same as if the constant potential U were augmented by a constant quantity proportional to the squared amplitude of the variable field.
The system moving under the action small but fast oscillation will be stabilized by minimizing (9.3.20).

9.4 Stabilization of Modulated Pendulum with Harmonic Force

The modulated pendulum consists of a bob of mass m attached a massless string of length l, whose point of support oscillates horizontally or vertically with a high frequency. This pendulum is stabilized by minimizing its effective potential energy function given by Eq. (9.3.20). Here we will consider two cases:

- Modulation with a high frequency horizontal small oscillations.
- Modulation with a high frequency vertical small oscillations.

9.4.1 Horizontal Modulation with Sin External Force

Consider a bob of mass m is attached with a massless string of length l and is free to oscillate. Let it make an angle ϕ with the vertical. If g is the gravitational acceleration, then its natural frequency is $\omega_0 = \sqrt{\frac{g}{l}}$. Its potential energy function is

$$U \;=\; -mgy \;= -mgl\cos\phi \tag{9.4.1}$$

An external force $\sin\omega t$ of high frequency with amplitude a is applied at the pivot point to have a horizontal modulation, as shown in Fig. (9.5). This high frequency means $\omega \gg \omega_0$. Then at any time the position of the particle is

$$\begin{aligned} x &= a\sin\omega t + l\sin\phi \\ y &= l\cos\phi \end{aligned}$$

We can write down the equation of motion of the above system by finding Euler-Lagrange equations. The time independent potential energy function is given by (9.4.1), so we need

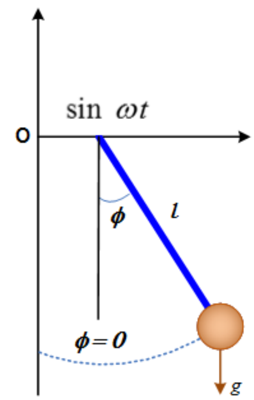


Figure 9.2: Horizontally modulated Pendulum with sin force

to find its kinetic energy only. For this the velocity components are

$$\begin{aligned}\dot{x} &= a\omega\cos\omega t + \dot{\phi} l\cos\phi \\ \dot{y} &= -\dot{\phi} l\sin\phi\end{aligned}$$

Then the kinetic energy is

$$\begin{aligned}K &= \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) \\ &= \frac{m}{2}\left(a^2\omega^2\cos^2\omega t + \dot{\phi}^2 l^2\cos^2\phi + 2a\omega\dot{\phi} l\cos\omega t\cos\phi + \dot{\phi}^2 l^2\sin^2\phi\right) \\ &= \frac{m}{2}\left(a^2\omega^2\cos^2\omega t + \dot{\phi}^2 l^2 + 2a\omega\dot{\phi} l\cos\omega t\cos\phi\right)\end{aligned}$$

And the Lagrangian is

$$\begin{aligned}L &= K-U \\ &= \frac{m\dot{\phi}^2 l^2}{2} + \frac{ma^2\omega^2}{2}\cos^2\omega t + ma\omega\dot{\phi} l\cos\omega t\cos\phi + mgl\cos\phi\end{aligned}$$

Now $\cos^2\omega t$ can be written as

$$\begin{aligned}\cos^2\omega t &= \frac{1}{2}\left(1+cos2\omega t\right) \\ &= \frac{d}{dt}\left(\frac{t}{2}+\frac{sin2\omega t}{4\omega}\right)\end{aligned}$$

Also consider

$$\frac{d}{dt}(\sin\phi\cos\omega t) = -\omega\sin\omega t\sin\phi + \dot{\phi}\cos\omega t\cos\phi$$

then

$$\dot{\phi}\cos\omega t cos\phi = \frac{d}{dt}(\sin\phi\cos\omega t) + \omega\sin\omega t\sin\phi$$

then Lagrangian becomes

$$\begin{aligned} L &= \frac{m\dot{\phi}^2 l^2}{2} + \frac{ma^2\omega^2}{2}\frac{d}{dt}\left(\frac{t}{2} + \frac{sin2\omega t}{4\omega}\right) + mgl\cos\phi \\ &+ mla\omega\left(\frac{d}{dt}(\sin\phi\cos\omega t) + \omega\sin\omega t\sin\phi\right) \\ &= \frac{m\dot{\phi}^2 l^2}{2} + mgl\cos\phi + mla\omega^2\sin\omega t\sin\phi \\ &+ \frac{ma^2\omega^2}{2}\frac{d}{dt}\left(\frac{t}{2} + \frac{\sin 2\omega t}{4\omega}\right) + mla\omega\frac{d}{dt}(\sin\phi\cos\omega t) \end{aligned}$$

Which is of the form

$$L' = L + \frac{d}{dt}f(\phi, t)$$

Since Lagrangian remains invariant if we add or subtract $\frac{d}{dt}f(q,t)$. So omitting total derivatives we have

$$L = \frac{m\dot{\phi}^2 l^2}{2} + mgl\cos\phi + mla\omega^2\sin\omega t\sin\phi$$

Here $q = l\phi$ is the only generalized coordinates. So equation of motion is

$$\frac{\partial L}{\partial(l\phi)} - \frac{d}{dt}\left(\frac{\partial L}{\partial(l\dot{\phi})}\right) = 0 \tag{9.4.2}$$

$$\begin{aligned} \frac{\partial L}{\partial(l\phi)} &= \frac{1}{l}\frac{\partial L}{\partial(\phi)} \\ &= \frac{1}{l}[mgl(-\sin\phi) + mla\omega^2\sin\omega t\cos\phi] \\ &= m[g(-\sin\phi) + a\omega^2\sin\omega t\cos\phi] \\ \left(\frac{\partial L}{\partial(l\dot{\phi})}\right) &= \frac{1}{l}\left(\frac{\partial L}{\partial(\dot{\phi})}\right) \\ &= \frac{1}{l}[ml^2\dot{\phi}] \\ \frac{d}{dt}\left(\frac{\partial L}{\partial(l\dot{\phi})}\right) &= ml\ddot{\phi} \end{aligned}$$

Using above results in (9.4.2), the equation of motion is

$$ml\ddot{\phi} = ma\omega^2 \sin\omega t \cos\phi - mg(\sin\phi) \tag{9.4.3}$$

comparing (9.4.3) with (9.3.3) we have

$$f = ma\omega^2 \sin\omega t \cos\phi \tag{9.4.4}$$

Next comparing (9.4.4) with (9.3.2), the coefficients are

$$\begin{aligned} f_1 &= 0 \quad and \; f_2 = ma\omega^2 \cos\phi \\ f_2^2 &= m^2 a^2 \omega^4 \cos^2\phi \\ \overline{f^2} &= \frac{1}{2} f_1^2 = \frac{1}{2} m^2 a^2 \omega^4 \cos^2\phi \end{aligned}$$

Using (9.3.20) the effective potential energy is

$$\begin{aligned} U_{eff} &= -mgl\cos\phi + \frac{1}{4m\omega^2} m^2 a^2 \omega^4 \cos^2\phi \\ U_{eff} &= mgl\left(-\cos\phi + \frac{a^2\omega^2}{4gl}\cos^2\phi\right) \end{aligned} \tag{9.4.5}$$

The dimensionless potential energy function denoted by $\widetilde{U}_{eff}$ is

$$\widetilde{U}_{eff} = \frac{U_{eff}}{mgl} = \left(-\cos\phi + \frac{a^2\omega^2}{4gl}\cos^2\phi\right) \tag{9.4.6}$$

Stability

The positions of stable equilibrium correspond to the minima of U_{eff} in $[-\pi, \pi]$. The first step is to differentiate (9.4.5) with respect to ϕ.

$$\begin{aligned} \frac{dU_{eff}}{d\phi} &= mgl\left(\sin\phi - \frac{2a^2\omega^2}{4gl}\sin\phi\cos\phi\right) \\ &= mgl\sin\phi\left(1 - \frac{a^2\omega^2}{2gl}\cos\phi\right) \end{aligned} \tag{9.4.7}$$

To find the critical points we have $\frac{dU_{eff}}{d\phi} = 0$, which means

$$mgl\sin\phi\left(1 - \frac{a^2\omega^2}{2gl}\cos\phi\right) = 0$$

Since $m \neq 0$, $g \neq 0$ and $l \neq 0$. That means

$$\sin\phi = 0$$

or

$$\left(1 - \frac{a^2\omega^2}{2gl}\cos\phi\right) = 0$$

First consider

$$\begin{aligned} \sin\phi &= 0 \\ \Rightarrow \phi &= 0\,,\ \pi \end{aligned}$$

Next consider

$$\begin{aligned} 1 - \frac{a^2\omega^2}{2gl}\cos\phi &= 0 \\ \frac{a^2\omega^2}{2gl}\cos\phi &= 1 \\ \cos\phi &= \frac{2gl}{a^2\omega^2} \\ \phi &= \pm\arccos\left(\frac{2gl}{a^2\omega^2}\right) \end{aligned}$$

For minimization we must have

$$\frac{d^2U_{eff}}{d\phi^2} > 0 \text{ at } 0, \pi \text{ and } \pm\arccos\left(\frac{2gl}{a^2\omega^2}\right)$$

Differentiate (9.4.7) with respect to ϕ

$$\frac{d^2U_{eff}}{d\phi^2} = mgl(cos\phi - \frac{a^2\omega^2}{2gl}cos2\phi) \tag{9.4.8}$$

(1) **Stability at** $\phi = 0$

The system at $\phi = 0$ is stable if U_{eff} is minimum at $\phi = 0$. This minimum exist if

$$\frac{d^2U_{eff}}{d\phi^2}\,|_{\phi=0} > 0$$

Since $\cos(0) = 1$

$$\Rightarrow mgl\left(1 - \frac{a^2\omega^2}{2gl}\right) > 0$$

or

$$\begin{aligned} \left(1 - \frac{a^2\omega^2}{2gl}\right) &> 0 \\ \frac{a^2\omega^2}{2gl} &< 1 \end{aligned}$$

It means

$$a^2\omega^2 < 2gl$$

If $a^2\omega^2 < 2gl$ the position $\phi = 0$ is stable. It means the the position $\phi = 0$ is conditionally stable.

(2) **Stability at** $\phi = \pi$

The system at $\phi = \pi$ is stable if U_{eff} is minimum at $\phi = \pi$. This minimum exist if

$$\frac{d^2U_{eff}}{d\phi^2} |_{\phi=\pi} \quad > \quad 0$$

Since $cos(\pi) = -1$

$$\Rightarrow mgl\left(-1 - \frac{a^2\omega^2}{2gl}\right) \quad > \quad 0$$
$$-\left(1 + \frac{a^2\omega^2}{2gl}\right) \quad > \quad 0$$

Which is impossible, hence the system is unstable $\phi = \pi$.

(3) **Stability at** $\phi = \pm \arccos\left(\frac{2gl}{a^2\omega^2}\right)$

we can write (9.4.13) as

$$\frac{d^2U_{eff}}{d\phi^2} \quad = \quad mgl\left(cos\phi - \frac{a^2\omega^2}{2gl}(2cos^2\phi - 1)\right) \tag{9.4.9}$$

Since $cos\phi = \frac{2gl}{a^2\omega^2}$ then (9.4.9) can be written as

$$= \quad mgl\left(\frac{2gl}{a^2\omega^2} - \frac{a^2\omega^2}{2gl}(2\frac{4g^2l^2}{a^4\omega^4} - 1)\right)$$
$$= \quad mgl\left(\frac{2gl}{a^2\omega^2} - \frac{4gl}{a^2\omega^2} + 2\frac{a^2\omega^2}{2gl}\right)$$

for minimization of effective potential energy function, we must have

$$mgl\left(\frac{-2gl}{a^2\omega^2} + \frac{a^2\omega^2}{2gl}\right) > 0$$

or

$$\left(\frac{-2gl}{a^2\omega^2} + \frac{a^2\omega^2}{2gl}\right) > 0$$

$$\frac{a^2\omega^2}{2gl} > \frac{2gl}{a^2\omega^2}$$

It means that

$$a^2\omega^2 > 2gl$$

Hence if $a^2\omega^2 > 2gl$ the position given by $\cos\phi = \frac{2gl}{a^2\omega^2}$ is stable. It is also conditional stable position.
All these stable points are shown in Fig. 9.3.
We can write the summery as:

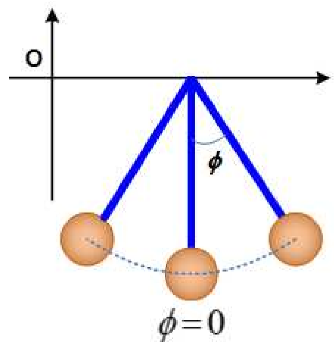


Figure 9.3: Stable points with horizontal oscillation.

(1) If $a^2\omega^2 < 2gl$, the position $\phi = 0$ is stable. $\Big($Graphically the minimization of U_{eff} at $\phi = 0$ is shown in Fig. 9.4 (a)$\Big)$.

(2) The position $\phi = \pi$ is unstable.

(3) If $a^2\omega^2 > 2gl$ the position given by $\cos\phi = \frac{2gl}{a^2\omega^2}$ is stable. $\Big($Graphically the minimization of U_{eff} at $\phi = \pm\arccos\left(\frac{2gl}{a^2\omega^2}\right)$ shown in Fig. 9.4 (b)$\Big)$.

9.4.2 Vertical Modulation with Harmonic Force

Consider a bob of mass m is attached with a massless string of length l and is free to oscillate. Let it make an angle ϕ with the vertical. If g is the gravitational acceleration, then its natural frequency is $\omega_0 = \sqrt{\frac{g}{l}}$. Its potential energy function is same as above, hence given by (9.4.1). An external force $\sin\omega t$ of high frequency with amplitude a is applied at the pivot point to have vertical modulation, as shown in Fig. 9.5. Then at any time the position of the particle is

$$\begin{aligned} x &= l\sin\phi \\ y &= l\cos\phi + a\sin\omega t \end{aligned}$$

We can write down the equation of motion of the above system by finding Euler-Lagrange equations. The time independent potential energy function is given by (9.4.1), so we need to find its kinetic energy only. For this the velocity components are

$$\begin{aligned} \dot{x} &= \dot{\phi}l\cos\phi \\ \dot{y} &= -\dot{\phi}l\sin\phi + a\omega\cos\omega t \end{aligned}$$

Then the kinetic energy is

$$\begin{aligned} K &= \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) \\ &= \frac{m}{2}\left(\dot{\phi}^2 l^2\cos^2\phi+\dot{\phi}^2 l^2\sin^2\phi+a^2\omega^2\cos^2\omega t-2\dot{\phi}la\omega\sin\phi\cos\omega t\right) \\ &= \frac{m}{2}\left(\dot{\phi}^2 l^2+a^2\omega^2\cos^2\omega t-2\dot{\phi}la\omega\sin\phi\cos\omega t\right) \end{aligned}$$

And the Lagrangian is

$$\begin{aligned} L &= K-U \\ &= \frac{m}{2}\dot{\phi}^2 l^2+\frac{m}{2}a^2\omega^2\frac{d}{dt}\left(\frac{t}{2}+\frac{sin2\omega t}{4\omega}\right)+mla\omega\left(\omega\cos\phi\sin\omega t+\frac{d}{dt}(\cos\omega t\cos\phi)\right) \\ &+ mglcos\phi-\frac{mga}{\omega}\frac{d}{dt}(sin\omega t) \end{aligned}$$

Omitting total derivative

$$L = \frac{m}{2}\dot{\phi}^2 l^2+mla\omega^2\cos\phi\sin\omega t+mgl\cos\phi$$

Here $l\phi$ is the only generalized coordinate

$$\begin{aligned} \frac{\partial L}{\partial l\phi} &= \frac{1}{l}\frac{\partial L}{\partial\phi} \\ &= \frac{1}{l}\left(-mla\omega^2\sin\phi\sin\omega t-mgl\sin\phi\right) \\ &= \left(-ma\omega^2\sin\phi\sin\omega t-mg\sin\phi\right) \\ \frac{\partial L}{\partial l\dot{\phi}} &= \frac{\partial L}{l\partial\dot{\phi}} \\ &= \frac{1}{l}m\dot{\phi}l^2=ml\dot{\phi} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial l\dot{\phi}}\right) &= ml\ddot{\phi} \end{aligned}$$

Hence the equation of motion is

$$ml\ddot{\phi} = -ma\omega^2\sin\phi\sin\omega t-mg\sin\phi \qquad (9.4.10)$$

comparing it with Eq. (9.3.3) the external force acting on the bob is

$$f = -ma\omega^2\sin\phi\sin\omega t$$

Now comparing it with Eq. (9.3.2)

$$\begin{aligned} f_1 &= 0 \;\; and \;\; f_2=-ma\omega^2\sin\phi \\ f_2^2 &= m^2a^2\omega^4\sin^2\phi \end{aligned}$$

By using Eq. (9.3.20) the "effective potential energy" is

$$\begin{aligned} U_{eff} &= mgl\left(-\cos\phi + \frac{a^2\omega^2}{4gl}\sin^2\phi\right) \\ \widetilde{U}_{eff} &= \left(-\cos\phi + \frac{a^2\omega^2}{4gl}\sin^2\phi\right) \end{aligned} \tag{9.4.11}$$

Stability

The positions of stable equilibrium correspond to the minima of U_{eff} in $[-\pi, \pi]$. The first step is to differentiate (9.4.5) with respect to ϕ.

$$\begin{aligned} \frac{dU_{eff}}{d\phi} &= mgl\left(\sin\phi + \frac{2a^2\omega^2}{4gl}\sin\phi cos\phi\right) \\ &= mgl\sin\phi\left(1 + \frac{a^2\omega^2}{2gl}\cos\phi\right) \end{aligned} \tag{9.4.12}$$

To find the critical points we have $\frac{dU_{eff}}{d\phi} = 0$, which means

$$mglsin\phi\left(1 + \frac{a^2\omega^2}{2gl}cos\phi\right) = 0$$

Since $m \neq 0$, $g \neq 0$ and $l \neq 0$. That means

$$sin\phi\left(1 + \frac{a^2\omega^2}{2gl}cos\phi\right) = 0$$

so we have

$$sin\phi = 0$$

or

$$\left(1 + \frac{a^2\omega^2}{2gl}cos\phi\right) = 0$$

First consider

$$\begin{aligned} sin\phi &= 0 \\ \Rightarrow \phi &= 0\,,\ \pi \end{aligned}$$

Next consider

$$\begin{aligned} 1 + \frac{a^2\omega^2}{2gl}cos\phi &= 0 \\ \frac{a^2\omega^2}{2gl}cos\phi &= -1 \\ cos\phi &= -\frac{2gl}{a^2\omega^2} \end{aligned}$$

and

$$\phi = \pm \arccos\left(-\frac{2gl}{a^2\omega^2}\right)$$

For minimization we must have

$$\frac{d^2U_{eff}}{d\phi^2} > 0 \text{ at } 0, \pi \text{ and } \pm \arccos\left(-\frac{2gl}{a^2\omega^2}\right)$$

Differentiate (9.4.12) with respect to ϕ

$$\frac{d^2U_{eff}}{d\phi^2} = mgl\left(\cos\phi + \frac{a^2\omega^2}{2gl}\cos 2\phi\right) \tag{9.4.13}$$

(1) **Stability at** $\phi = 0$

The system at $\phi = 0$ is stable if U_{eff} is minimum at $\phi = 0$. This minimum exist if

$$\frac{d^2U_{eff}}{d\phi^2}\,|_{\phi=0} > 0$$

Since $\cos(0) = 1$

$$\Rightarrow mgl\left(1 + \frac{a^2\omega^2}{2gl}\right) > 0 \text{ always} > 0$$

It means the position $\phi = 0$ (vertically downward position) is always stable.

(2) **Stability at** $\phi = \pi$

The system at $\phi = \pi$ is stable if U_{eff} is minimum at $\phi = \pi$. This minimum exist if

$$\frac{d^2U_{eff}}{d\phi^2}\,|_{\phi=\pi} > 0$$

Since $cos(\pi) = -1$

$$\Rightarrow mgl\left(-1 + \frac{a^2\omega^2}{2gl}\right) > 0$$
$$\left(-1 + \frac{a^2\omega^2}{2gl}\right) > 0$$

It means that

$$a^2\omega^2 > 2gl$$

Hence if $a^2\omega^2 > 2gl$ the position given by $\phi = \pi$ is stable. It is conditional stable position.

(3) **Stability at** $\phi = \pm \arccos\left(\frac{2gl}{a^2\omega^2}\right)$

For $cos\phi = -\frac{2gl}{a^2\omega^2}$

$$\begin{aligned}\frac{d^2U_{eff}}{d\phi^2} &= mgl\left(\frac{-2gl}{a^2\omega^2} + \frac{a^2\omega^2}{2gl}\left(2\cdot\frac{4g^2l^2}{a^4\omega^4} - 1\right)\right)\\ &= mgl\left(\frac{2gl}{a^2\omega^2} - 1\right)\end{aligned}$$

Now $\mid cos\phi \mid = \frac{2gl}{a^2\omega^2} < 1$
then we have $\frac{d^2U_{eff}}{d\phi^2} < 0$
so the position given by $cos\phi = \frac{-2gl}{a^2\omega^2}$ is not stable.
All these stable points are shown in Fig. 9.6.
We can write the summery as:

(1) the position $\phi = 0$(vertically downward position) is always stable. $\Big($Graphically minimization of U_{eff} at $\phi = 0$ is shown in Fig. 9.7 (a)$\Big)$.

(2) If $a^2\omega^2 > 2gl$ then $\phi = \pi$ (vertically upward position) is also stable. $\Big($Graphically minimization of U_{eff} at $\phi = \pi$ is shown in Fig. 9.7 (b)$\Big)$

(3) the position given by $\cos\phi = \frac{-2gl}{a^2\omega^2}$ is unstable.

9.5 Motion in a Rapidly Oscillating Field under Arbitrary Periodic Force with Zero Mean

In 2009, I and Borisenok (My PhD advisor) studied the motion of a particle in a rapidly oscillating field under an arbitrary periodic force with zero mean. In this motion external harmonic force is replaced with an arbitrary force extended in Fourier Series. It means a nonlinear force may be replaced by a linear one. Let's discuss the one-dimensional motion of a classical particle of mass m under a force due to time-independent potential $U(x)$

$$F_1 = \frac{-dU}{dx} \tag{9.5.1}$$

and a fast oscillating periodic force with zero mean. The Fourier expansion of this periodic force is

$$f(x,t) = \sum_{k=1}^{\infty} \Big[a_k(x)\cos(k\omega t) + b_k(x)\sin(k\omega t) \Big] \ , \tag{9.5.2}$$

The system has time period T_U due to force F_1, so may oscillate due to this force with frequency $\omega_U = \frac{1}{T_U}$ and T is the time period due the force F_2, and oscillation due to this force is $\omega = \frac{1}{T}$. The fast oscillation means the frequency $\omega \gg \omega_U$. The coefficients a_k and b_k are functions of co-ordinates only given by (9.5.3) and (9.5.4) respectively.

$$a_k(x) = \frac{2}{T}\int_0^T f(x,t)\cos k\omega t \ dt \ ; \tag{9.5.3}$$

$$b_k(x) = \frac{2}{T}\int_0^T f(x,t)\sin k\omega t \ dt \ . \tag{9.5.4}$$

The equation of the particle motion is:

$$m\ddot{x} = -\frac{dU(x)}{dx} + f(x,t) \ . \tag{9.5.5}$$

We present the movement as a slow path and at the same time execute fast but small oscillations of frequency ω about the path: $x(t) = X(t) + \xi(t)$, where $\xi(t)$ corresponds to these small oscillations. The mean value of the function $\xi(t)$ over its period T is zero, and the function $X(t)$ changes only slightly in that time. Denoting this average by a bar, we therefore have $\bar{x} = X(t)$. Now Taylor's expansion in powers of ξ up to the first order term provides us:

$$\frac{dU}{dx} = \frac{dU}{dX} + \xi\frac{d^2U}{dX^2} \ . \tag{9.5.6}$$

Substituting (9.5.6) in (9.5.5) we have:

$$m\ddot{X} + m\ddot{\xi} = -\frac{dU}{dX} - \xi\frac{d^2U}{dX^2} + f(X,t) + \xi\frac{df}{dX} \ . \tag{9.5.7}$$

This equation involves both fast and slow terms, which must evidently be separately equal. For the fast term we can put simply

$$m\ddot{\xi} = f(X, t) \tag{9.5.8}$$

and the slow term with small oscillations is

$$m\ddot{X} = -\frac{dU}{dX} - \xi\frac{d^2U}{dX^2} + \xi\frac{df}{dX} \; .$$

Integrating (9.5.8) with the function f given by (9.5.2) and regarding X as a constant, we get

$$\xi = -\frac{1}{m\omega^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\Big(a_k\cos k\omega t + b_k\sin k\omega t\Big)$$

Next we average equation (9.5.7) with respect to the time interval $[0, T]$: Since $\bar{\xi} = 0$ and $\bar{f} = 0$,

$$m\ddot{X} = -\frac{dU}{dX} + \overline{\xi\frac{df}{dX}} \tag{9.5.9}$$

and

$$\frac{df}{dX} = \sum_{k=1}^{\infty}\Big(a_k'\cos k\omega t + b_k'\sin k\omega t\Big) \; ,$$

where $a_k' = da_k/dX$ and $b_k' = db_k/dX$. Then we apply the time averaging:

$$\overline{\xi\cdot\frac{df}{dX}} = -\frac{1}{m\omega^2}\sum_{k,j=1}^{\infty}\Bigg[\frac{a_k a_j'}{k^2}\cdot\overline{\cos k\omega t\cos j\omega t} + \frac{b_k a_j'}{k^2}\cdot\overline{\sin k\omega t\cos j\omega t} + \frac{a_k b_j'}{k^2}\cdot\overline{\cos k\omega t\sin j\omega t} + \frac{b_k b_j'}{k^2}\cdot\overline{\sin k\omega t\sin j\omega t}\Bigg] \; .$$

Since

$$\overline{\sin k\omega t\cos j\omega t} = \overline{\cos k\omega t\sin j\omega t} \quad = \quad 0 \; ; \tag{9.5.10}$$

and

$$\overline{\cos k\omega t\cos j\omega t} = \overline{\sin k\omega t\sin j\omega t} \quad = \quad \begin{cases} 0 & k \neq j \\ \frac{1}{2} & k = j \end{cases} \tag{9.5.11}$$

then we have

$$\overline{\xi \cdot \frac{df}{dX}} = -\frac{1}{4m\omega^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{da_k^2}{dX}+\frac{db_k^2}{dX}\right) . \tag{9.5.12}$$

Substituting (9.5.12) in (9.5.9),

$$m\ddot{X} \quad = \quad -\frac{dU}{dX} - \frac{1}{4m\omega^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{d(a_k^2+b_k^2)}{dX} . \tag{9.5.13}$$

Eq.(9.5.13) involves only the function $X(t)$. It can be written as

$$m\ddot{X} = -\frac{dU_{eff}}{dX} ,$$

where the effective potential energy is defined as

$$U_{eff} = U + \frac{1}{4m\omega^2}\sum_{k=1}^{\infty}\frac{\left(a_k^2+b_k^2\right)}{k^2} . \tag{9.5.14}$$

If $a_k, b_k = 0$ for any $k \geq 2$, (9.5.14) coincides with (9.3.20). It means that (9.3.20) is a particular case of (9.5.14). So we can call it generalized averaging method in a rapidly oscillating field under arbitrary periodic force with zero mean. If a system is modulated by a periodic force with zero mean with high frequency, then it is stabilized by minimizing (9.5.14).

9.5.1 Stabilization of Modulated Pendulum with Harmonic Force

Consider a bob of mass m is attached with a massless string of length l and is free to oscillate. Let it make an angle ϕ with the vertical. If g is the gravitational acceleration, then its natural frequency is $\omega_0 = \sqrt{\frac{g}{l}}$. Its potential energy function is

$$U \quad = \quad -mgy \quad = -mgl\cos\phi$$

An external harmonic force of high frequency given by (9.5.15)

$$f(t) \quad = \quad \sin\omega t \quad \text{if} \quad 0 \leq t < T ; \tag{9.5.15}$$

and shown in Fig. 9.8. The Fourier expansion of (9.5.15) is given by (9.5.16)

$$f \quad = \quad m\omega^2\cos\phi\sum_{k=1}^{\infty}\Big[a_k(x)\cos(k\omega t)+b_k(x)\sin(k\omega t)\Big] \tag{9.5.16}$$

This force is applied at the pivot point to have a horizontal modulation, as shown in Fig. (9.5.1). Since the first Fourier coefficient is

$$a_0(x) = \frac{2}{T}\int_0^T f(x,t)\ dt \tag{9.5.17}$$

Hence the zero meaning $\overline{f} = 0$ is equivalent to show $a_0 = 0$. The first Fourier coefficient of (9.5.15) is

$$a_0 = \frac{2}{T}\int_0^T \sin\omega t\ dt$$

Using (9.5.10) we have

$$a_0 = 0 \tag{9.5.18}$$

Hence the mean value of f is zero. The other Fourier coefficients are given by (9.3.3)

$$a_k = \frac{2}{T}\int_0^T \sin\omega t\cos k\omega t\ dt$$

Using (9.5.11) we have

$$a_k = 0$$

The other fourier coefficient is

$$b_k = \frac{2}{T}\int_0^T \sin\omega t\sin k\omega t\ dt$$

Again using (9.5.11) we have the result for $k = 1$ only

$$\begin{aligned} b_1 &= \frac{2}{T}\int_0^T \sin^2\omega t\ dt \\ &= 1 \end{aligned} \tag{9.5.19}$$

Hence with Fourier coefficient

$$b_k = m\omega^2\cos\phi \tag{9.5.20}$$

the force acting on the particle is

$$f = m\omega^2\cos\phi\sin(\omega t) \tag{9.5.21}$$

Using (9.5.14) the effective potential energy is

$$U_{eff} = mgl\left(-\cos\phi + \frac{\omega^2}{4gl}\cos^2\phi\right)\ .$$

Which is same as given by (9.4.5) with $a = 1$. Hence the stability of extremum positions $\phi = 0\,,\ \pi\,,\ \pm\arccos 2gl/\omega^2$ is as following.

(1) If $\omega^2 < 2gl$, the downward position $\phi = 0$ is stable.

(2) The Vertically upward position $\phi = \pi$ is unstable.

(3) If $\omega^2 > 2gl$ the position given by $\cos\phi = \frac{2gl}{\omega^2}$ is stable.

The stable points with horizontal modulation are shown in Fig. 9.10. The same we will observe for vertical modulation.

9.5.2 Stabilization of Modulated Pendulum with Linear Force

The potential energy function of a periodic force with zero mean can be calculated using (9.5.14). Here the nonlinear force (harmonic force) is replaced with some linear force. For horizontal modulation, then the net acting force is

$$f(t) = m\omega^2 \cos\phi \cdot R(t,n) \quad ,$$

where the function R is T-periodic $R(t+T,n) \equiv R(t,n)$. Let's introduce the triangular shape linear force:$R_s(t)$ given by (10.3.2)

$$R_S(t) = \begin{cases} \frac{4t}{T} & \text{if } \ 0 \le t < \frac{T}{4} \ ; \\ \frac{4}{T}\left(\frac{T}{2} - t\right) & \text{if } \ \frac{T}{4} \le t < \frac{3T}{4} \ ; \\ \frac{4(t-T)}{T} & \text{if } \ \frac{3T}{4} \le t < T \ . \end{cases} \tag{9.5.22}$$

and is shown in Fig. (9.11)

9.5.3 Horizontal Modulation

For horizontal modulation, the force acting on the particle is

$$f(t) = m\omega^2 \cos\phi \cdot R_S(t,n)$$

Then by (10.33)Fourier coefficient a_0 of (10.3.2) is:

$$\begin{aligned} a_0 &= \frac{2}{T}\left[\int_0^{\frac{T}{4}} \frac{4t}{T} dt + \int_{\frac{T}{4}}^{\frac{3T}{4}} \frac{4}{T}\left(\frac{T}{2} - t\right) dt + \int_{\frac{3T}{4}}^{T} \frac{4(t-T)}{T} dt\right] \\ &= \frac{2}{T}\left[\frac{4}{T}\frac{t^2}{2}\Big|_0^{\frac{T}{4}} + 2t\Big|_{\frac{T}{4}}^{\frac{3T}{4}} - \frac{4}{T}\frac{t^2}{2}\Big|_{\frac{T}{4}}^{\frac{3T}{4}} + \frac{4}{T}\frac{t^2}{2}\Big|_{\frac{3T}{4}}^{T} - 4t\Big|_{\frac{3T}{4}}^{T}\right] \\ &= \frac{2}{T}\left[\frac{T}{8} + \frac{3T}{2} - \frac{T}{2} - \frac{9T}{8} + \frac{T}{8} + 2T - \frac{9T}{8} - 4T + 3T\right] \\ &= 0 \end{aligned} \tag{9.5.23}$$

Whic is equivalent to show that $\bar{R}_s = 0$, and we can proceed further. The other Fourier coefficients are given by Eq.(9.3.3)

$$a_k = \frac{2}{T}\int_0^T R_S(t) \cos k\omega t \ dt$$

by using Eq. (9.5.11) we have

$$\begin{aligned}
a_k &= \frac{2}{T}\left[\int_0^{\frac{T}{4}} \frac{4t}{T}\cos k\omega t dt + \int_{\frac{T}{4}}^{\frac{3T}{4}} \frac{4}{T}\left(\frac{T}{2}-t\right)\cos k\omega t dt + \int_{\frac{3T}{4}}^{T} \frac{4(t-T)}{T}\cos k\omega t dt\right] \\
&= \frac{2}{T}\left[\frac{4}{T}\left(\frac{t\sin k\omega t}{k\omega}+\frac{\cos k\omega t}{k^2\omega^2}\right)\Big|_0^{\frac{T}{4}} + 2\frac{\sin k\omega t}{k\omega}\Big|_{\frac{T}{4}}^{\frac{3T}{4}} - \frac{4}{T}\left(\frac{t\sin k\omega t}{k\omega}+\frac{\cos k\omega t}{k^2\omega^2}\right)\Big|_{\frac{T}{4}}^{\frac{3T}{4}}\right] \\
&+ \frac{2}{T}\left[\frac{4}{T}\left(\frac{t\sin k\omega t}{k\omega}+\frac{\cos k\omega t}{k^2\omega^2}\right)\Big|_{\frac{3T}{4}}^{T} - 4\frac{\sin k\omega t}{k\omega}\Big|_{\frac{3T}{4}}^{T}\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{2}{T}\left[\frac{4}{T}\left(\frac{T}{4}\frac{\sin k\omega\frac{T}{4}}{k\omega}+\frac{\cos k\omega\frac{T}{4}}{k^2\omega^2}\right) + \frac{2}{k\omega}\left(\sin k\omega\frac{3T}{4} - \sin k\omega\frac{T}{4}\right)\right] \\
&- \frac{2}{T}\frac{4}{T}\left[\left(\frac{3T}{4}\frac{\sin k\omega\frac{3T}{4}}{k\omega} - \frac{T}{4}\frac{\sin k\omega\frac{T}{4}}{k\omega}\right) + \left(\frac{\cos k\omega\frac{3T}{4} - \cos k\omega\frac{T}{4}}{k^2\omega^2}\right)\right] \\
&+ \frac{2}{T}\frac{4}{T}\left[\left(T\frac{\sin k\omega T}{k\omega} - \frac{3T}{4}\frac{\sin k\omega\frac{3T}{4}}{k\omega}\right) + \left(\frac{\cos k\omega T - \cos k\omega\frac{3T}{4}}{k^2\omega^2}\right)\right] \\
&- \frac{2}{T}4\left[\frac{\sin k\omega T - \sin k\omega\frac{3T}{4}}{k\omega}\right] \\
&= \frac{2}{T}\left[\frac{8}{T}\frac{\cos k\omega\frac{T}{4}}{k^2\omega^2} - \frac{4}{Tk^2\omega^2} - \frac{8}{T}\frac{\cos k\omega\frac{3T}{4}}{k^2\omega^2} + \frac{4}{T}\frac{\cos k\omega T}{k^2\omega^2}\right]
\end{aligned}$$

using $T = \frac{2\pi}{\omega}$, we have

$$a_k = 0 \tag{9.5.24}$$

The other fourier coefficient is

$$b_k = \frac{2}{T}\int_0^T R_S(t)\sin k\omega t\ dt$$

by using Eq. (9.5.11) we have

$$\begin{aligned}
b_k &= \frac{2}{T}\left[\int_0^{\frac{T}{4}} \frac{4t}{T}\sin k\omega t dt + \int_{\frac{T}{4}}^{\frac{3T}{4}} \frac{4}{T}\left(\frac{T}{2}-t\right)\sin k\omega t dt + \int_{\frac{3T}{4}}^{T} \frac{4(t-T)}{T}\sin k\omega t dt\right] \\
&= \frac{2}{T}\left[\frac{4}{T}\left(\frac{-t\cos k\omega t}{k\omega}+\frac{\sin k\omega t}{k^2\omega^2}\right)\Big|_0^{\frac{T}{4}} + 2\frac{-\cos k\omega t}{k\omega}\Big|_{\frac{T}{4}}^{\frac{3T}{4}} - \frac{4}{T}\left(\frac{-t\cos k\omega t}{k\omega}+\frac{\sin k\omega t}{k^2\omega^2}\right)\Big|_{\frac{T}{4}}^{\frac{3T}{4}}\right] \\
&+ \frac{2}{T}\left[\frac{4}{T}\left(\frac{-t\cos k\omega t}{k\omega}+\frac{\sin k\omega t}{k^2\omega^2}\right)\Big|_{\frac{3T}{4}}^{T} + 4\frac{\cos k\omega t}{k\omega}\Big|_{\frac{3T}{4}}^{T}\right]
\end{aligned}$$

$$
\begin{aligned}
= & \frac{2}{T}\left[\frac{4}{T}\left(-\frac{T}{4}\frac{\cos k\omega\frac{T}{4}}{k\omega}+\frac{\sin k\omega\frac{T}{4}}{k^2\omega^2}\right)+\frac{2}{k\omega}\left(-\cos k\omega\frac{3T}{4}+\cos k\omega\frac{T}{4}\right)\right] \\
+ & \frac{2}{T}\frac{4}{T}\left[\left(\frac{3T}{4}\frac{\cos k\omega\frac{3T}{4}}{k\omega}-\frac{T}{4}\frac{\cos k\omega\frac{T}{4}}{k\omega}\right)-\left(\frac{\sin k\omega\frac{3T}{4}-\sin k\omega\frac{T}{4}}{k^2\omega^2}\right)\right] \\
+ & \frac{2}{T}\frac{4}{T}\left[\left(-T\frac{\cos k\omega T}{k\omega}-\frac{3T}{4}\frac{\cos k\omega\frac{3T}{4}}{k\omega}\right)+\left(\frac{\sin k\omega T-\sin k\omega\frac{3T}{4}}{k^2\omega^2}\right)\right] \\
- & \frac{8}{T}\left[\frac{\cos k\omega T-\cos k\omega\frac{3T}{4}}{k\omega}\right]
\end{aligned}
$$

using $T=\frac{2\pi}{\omega}$, we have

$$
b_k = \frac{8}{k^2\pi^2}\left[\sin k\frac{\pi}{2}\right]
$$

when k is even, $b_k=0$
for odd b_k can be written as;

$$
b_{2j+1} = \frac{8}{\pi^2}\frac{(-1)^j}{(2j+1)^2}
$$

$$
b_k = m\omega^2\cos\phi\frac{8}{\pi^2}\frac{(-1)^j}{(2j+1)^2} \tag{9.5.25}
$$

Then the force acting on the particle is

$$
R_S(t)=m\omega^2\cos\phi\frac{8}{\pi^2}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^2}\sin\left(\frac{2\pi(2j+1)t}{T}\right) .
$$

For horizontal modulation, the effective potential energy is

$$
\begin{aligned}
U_{eff} &= U+m\omega^2\cos^2\phi\cdot\frac{1}{4}\left(\frac{8}{\pi^2}\right)^2\sum_{j=0}^{\infty}\frac{1}{(2j+1)^6} \\
&= U+\frac{\pi^2}{60}m\omega^2\cos^2\phi
\end{aligned}
$$

which has extremum at $\phi=0\,,\ \pi\,,\ \pm\arccos 30gl/\omega^2\pi^2$.
Stability We see that

(1) The downward point $\phi=0$ is stable if $\omega^2<3.0396gl$.

(2) Vertically upward point $\phi=\pi$ is not a stable.

(3) The point given by $\cos\phi=3.0396gl/\omega^2$ is stable if $\omega^2>3.0396gl$.

9.5.4 Rectangular Shape Force

Here we introduce rectangular shape periodic force:$R_l(t)$ given by

$$R_l(t) = \begin{cases} 1 & 0 \le t \le \frac{T}{2} \\ -1 & \frac{T}{2} \le t \le T \end{cases} \tag{9.5.26}$$

This force also satisfy the property $\bar{R}_l = 0$ as the Fourier coefficient $a_0 = 0$. For horizontal modulation, the force acting on the particle is

$$f(t) = m\omega^2 \cos\phi \cdot R_l(t, n) \quad ,$$

Then by Fourier expansion in the place of (9.5.26) we get:

$$a_k = 0 \tag{9.5.27}$$

$$b_k = m\omega^2 \cos\phi \frac{4}{(2k-1)\pi} \tag{9.5.28}$$

with the force acting on the particle is

$$R_L(t) = m\omega^2 \cos\phi \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k-1)} \sin(2k-1)\omega t$$

For horizontal modulation, the effective potential energy is

$$\begin{aligned} U_{eff} &= U + m\omega^2 \cos^2\phi \cdot \frac{1}{4} \left(\frac{16}{\pi^2}\right)^2 \sum_{k=0}^{\infty} \frac{1}{(2k-1)^4} \\ &= U + 0.4112 m\omega^2 \cos^2\phi \end{aligned}$$

which has extremum at $\phi = 0\,,\ \pi\,,\ \pm\arccos 1.2159 gl/\omega^2$.
Stability We see that

(1) The downward point $\phi = 0$ is stable if $\omega^2 < 1.2159 gl$.

(2) Vertically upward point $\phi = \pi$ is not a stable.

(3) The point given by $\cos\phi = 1.2159 gl/\omega^2$ is stable if $\omega^2 > 1.2159 gl$.

For horizontal modulation the the stability results at nontrivial positions are summarized in table 9.1. At nontrivial positions, we see that when we apply external triangular type force, whose area under the curve is less as compared to area under sine curve, we stabilize the pendulum with relatively high frequency, as compared to external sine force. And when we apply external rectangular linear force, whose area under the curve is more as compared to area under sine curve. Hence we can stabilize the pendulum with relatively low frequency as compared to external sine force.

Table 9.1: Stability Conditions for Horizontal Modulation.

Force Type	Trivial position	Stability condition	Non-trivial position	Stability condition
sin	0	$\omega^2 < 2gl$	$\pm \arccos 2gl/\omega^2$	$\omega^2 > 2gl$
triangular	0	$\omega^2 < 3.0396gl$	$\pm \arccos 3.0396gl/\omega^2$	$\omega^2 > 3.0396gl$
rectangular	0	$\omega^2 < 1.2159gl$	$\pm \arccos 1.2159gl/\omega^2$	$\omega^2 > 1.2159gl$

9.5.5 Stabilization of Vertical Oscillations

The same effect we observe for ***vertical modulation***:
The force acting on the particle for external harmonic force is

$$f = m\omega^2 \sin\phi \cdot \sin\omega t \ . \tag{9.5.29}$$

Here the vertical downward position $\phi = 0$ is always stable, the inverse point $\phi = \pi$ is stable under the condition $\omega^2 > 2gl$.
In place of external harmonic force, we use an arbitrary periodic force. Then the force acting on the particle is

$$f = m\omega^2 \sin\phi \cdot R(t,n) \ ,$$

where $R(t,n)$ represents an arbitrary periodic force.

For vertical modulation the point $\phi = 0$ is always stable while for upper point $\phi = \pi$ we reproduce conditions as:

Stability Conditions for Vertical Oscillation

Force Type	Position	Stability condition	Position	Stability condition
sin	0	always	π	$\omega^2 > 2gl$
triangular	0	always	π	$\omega^2 > 3.0396gl$
rectangular	0	always	π	$\omega^2 > 1.2159gl$

These stable positions are illustrated in the Fig (9.13) as:

Exercise

1. Discuss the stability of a pendulum in the following cases using 9.4
   (a) Modulation with a high frequency horizontal small oscillating $\cos\omega t$ external force.
   (b) Modulation with a high frequency vertical small oscillating $\cos\omega t$ external force.
2. Discuss the stability of a modulated pendulum with a high frequency horizontal small oscillating following external force using 9.5.14
   (a) $f(t) = \cos\omega t$
   (b)

$$f(t) = \begin{cases} \frac{4t}{T} & \text{if } 0 \le t < \frac{T}{4}\ ; \\ \frac{4}{T}\left(\frac{T}{2} - t\right) & \text{if } \frac{T}{4} \le t < \frac{3T}{4}\ ; \\ \frac{4(t-T)}{T} & \text{if } \frac{3T}{4} \le t < T \end{cases}$$

   (c)

$$f(t) = \begin{cases} \frac{1}{2} & \text{if } 0 \le t < \frac{1}{6}T; \\ 1 & \text{if } \frac{1}{6}T \le t < \frac{1}{3}T\ ; \\ \frac{1}{2} & \text{if } \frac{1}{3}T \le t < \frac{1}{2}T\ ; \\ -\frac{1}{2} & \text{if } \frac{1}{2}T \le t < \frac{2}{3}T\ ; \\ -1 & \text{if } \frac{2}{3}T \le t < \frac{5}{6}T\ ; \\ -\frac{1}{2} & \text{if } \frac{5}{6}T \le t < T \end{cases}$$

   (d)

$$f(t) = \begin{cases} \frac{8t}{T} & \text{if } 0 \le t < \frac{T}{8}\ ; \\ 1 & \text{if } \frac{T}{8} \le t < \frac{3T}{8}\ ; \\ \frac{8}{T}\left(\frac{T}{2} - t\right) & \text{if } \frac{3T}{8} \le t < \frac{5T}{8}\ ; \\ -1 & \text{if } \frac{5T}{8} \le t < \frac{7T}{8}\ ; \\ \frac{8(t-T)}{T} & \text{if } \frac{7T}{8} \le t < T \end{cases}$$

   (e)

$$f(t) = \begin{cases} \frac{8t}{T} & \text{if } 0 \le t < \frac{3T}{8}\ ; \\ \frac{8}{T}\left(\frac{T}{2} - t\right) & \text{if } \frac{3T}{8} \le t < \frac{5T}{8}\ ; \\ \frac{8(t-T)}{T} & \text{if } \frac{5T}{8} \le t < T \end{cases}$$

(f)

$$f(t,n) = \begin{cases} 1 & 0 \le t < \tau \\ -(n-1) & \tau \le t < T, \qquad n > 1 \in \mathbb{Z}. \end{cases}$$

3. Discuss the stability of a modulated pendulum with a high frequency vertical small oscillating following external force using 9.5.14

   (a) $f(t) = \cos \omega t$

   (b)

$$f(t) = \begin{cases} \frac{4t}{T} & \text{if } 0 \le t < \frac{T}{4}\ ; \\ \frac{4}{T}\left(\frac{T}{2} - t\right) & \text{if } \frac{T}{4} \le t < \frac{3T}{4}\ ; \\ \frac{4(t-T)}{T} & \text{if } \frac{3T}{4} \le t < T \end{cases}$$

   (c)

$$f(t) = \begin{cases} \frac{1}{2} & \text{if } 0 \le t < \frac{1}{6}T; \\ 1 & \text{if } \frac{1}{6}T \le t < \frac{1}{3}T\ ; \\ \frac{1}{2} & \text{if } \frac{1}{3}T \le t < \frac{1}{2}T\ ; \\ -\frac{1}{2} & \text{if } \frac{1}{2}T \le t < \frac{2}{3}T\ ; \\ -1 & \text{if } \frac{2}{3}T \le t < \frac{5}{6}T\ ; \\ -\frac{1}{2} & \text{if } \frac{5}{6}T \le t < T \end{cases}$$

   (d)

$$f(t) = \begin{cases} \frac{8t}{T} & \text{if } 0 \le t < \frac{T}{8}\ ; \\ 1 & \text{if } \frac{T}{8} \le t < \frac{3T}{8}\ ; \\ \frac{8}{T}\left(\frac{T}{2} - t\right) & \text{if } \frac{3T}{8} \le t < \frac{5T}{8}\ ; \\ -1 & \text{if } \frac{5T}{8} \le t < \frac{7T}{8}\ ; \\ \frac{8(t-T)}{T} & \text{if } \frac{7T}{8} \le t < T \end{cases}$$

   (e)

$$f(t) = \begin{cases} \frac{8t}{T} & \text{if } 0 \le t < \frac{3T}{8}\ ; \\ \frac{8}{T}\left(\frac{T}{2} - t\right) & \text{if } \frac{3T}{8} \le t < \frac{5T}{8}\ ; \\ \frac{8(t-T)}{T} & \text{if } \frac{5T}{8} \le t < T \end{cases}$$

(f)

$$f(t,n) \quad = \quad \begin{cases} 1 & 0 \le t < \tau \\ -(n-1) & \tau \le t < T, \qquad n > 1 \in \mathbb{Z}. \end{cases}$$

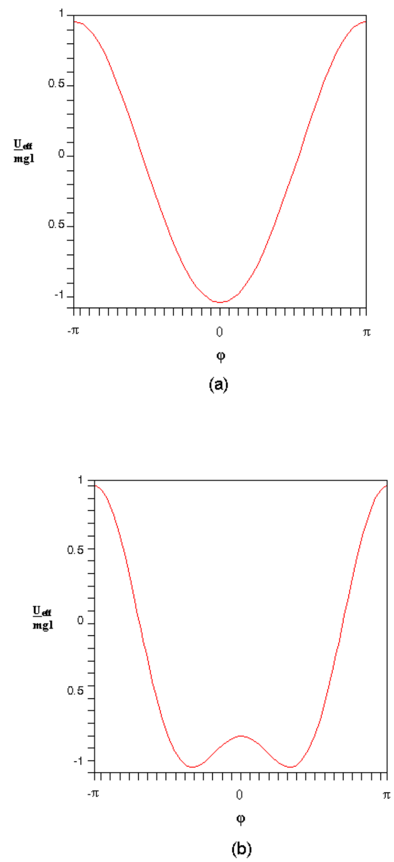


Figure 9.4: (a): Stable point $\phi = 0$ of Kapitza Pendulum with Horizontal Oscillation under the condition $\omega^2 < 2gl$
(b): Stable points given by $cos\phi = \frac{2gl}{\omega^2}$ of Kapitza Pendulum with Horizontal Oscillation under the condition $\omega^2 > 2gl$

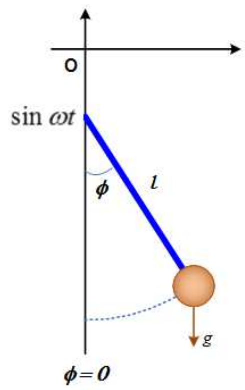


Figure 9.5: Vertically modulated Pendulum with sin force

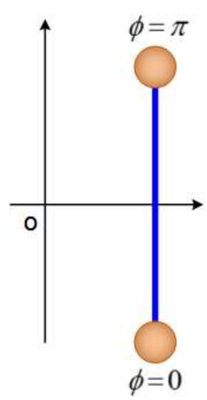


Figure 9.6: Stable positions for vertical oscillation

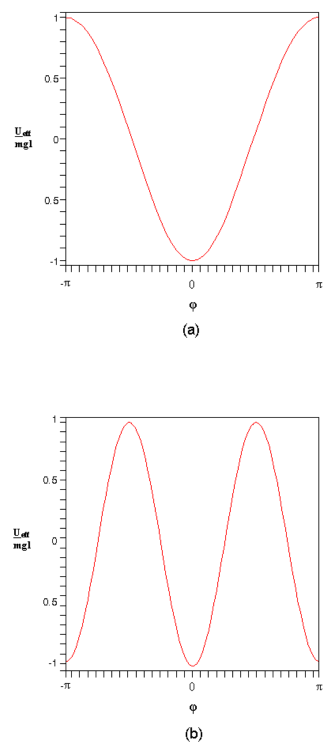


Figure 9.7: (a): Stable point $\phi = 0$ of Kapitza Pendulum with vertical Oscillation
(b): Stable points given by $\phi = \pi$ of Kapitza Pendulum with Vertical Oscillation under the condition $\omega^2 > 2gl$

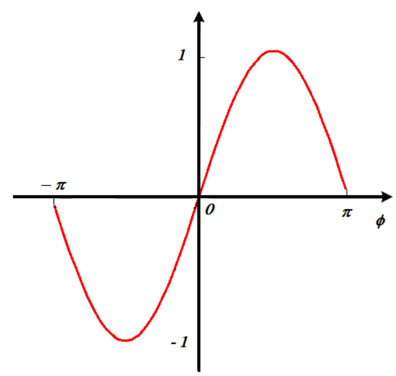


Figure 9.8: Sin type Signal

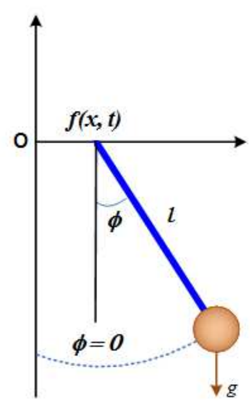


Figure 9.9: Horizontally modulated Pendulum with sin force

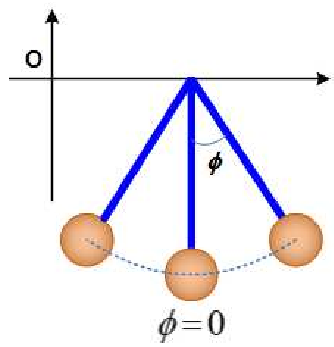


Figure 9.10: Stable points with horizontal oscillation.

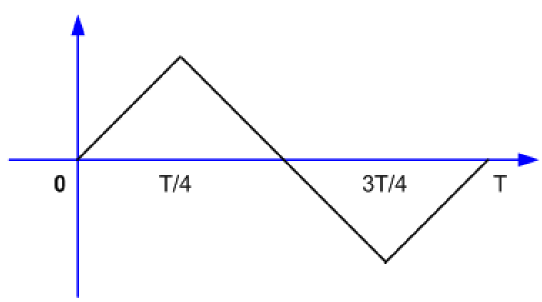
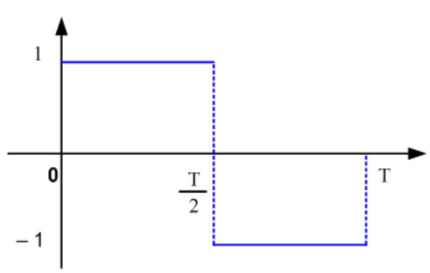


Figure 9.11: Saw type pulses



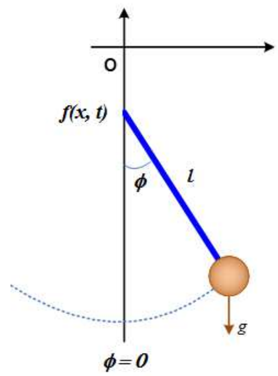


Figure 9.12: Kapitza Pendulum with Vertical Oscillation

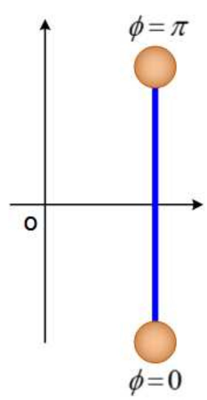
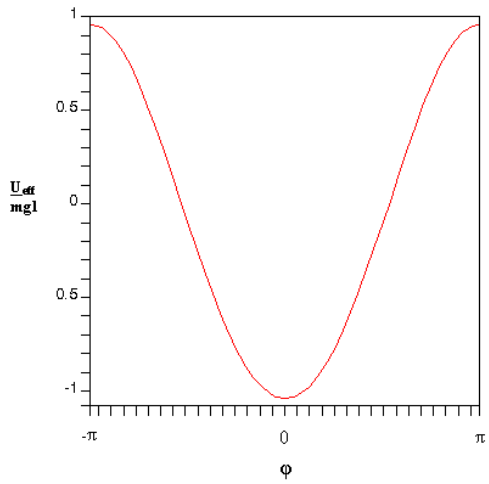
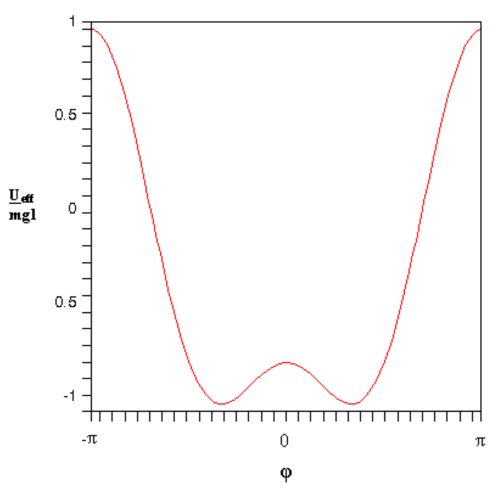


Figure 9.13: Stable positions with Vertical Oscillation



(a)



(b)

Figure 9.14: (a): Stable point $\phi = 0$ of Kapitza Pendulum with Horizontal Oscillation under the condition $\omega^2 < 1.2159gl$
(b): Stable points given by $cos\phi = \frac{1.2159gl}{\omega^2}$ of Kapitza Pendulum with Horizontal Oscillation under the condition $\omega^2 > S_n gl$

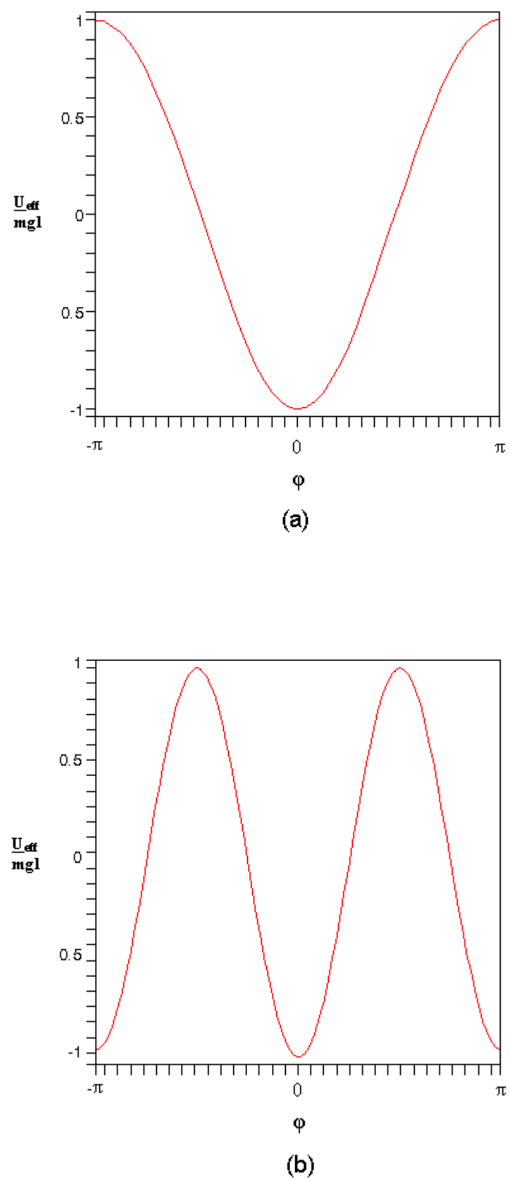


Figure 9.15: (a): Stable point $\phi = 0$ of Kapitza Pendulum with vertical Oscillation
(b): Stable points given by $\phi = \pi$ of Kapitza Pendulum with Vertical Oscillation under the condition $\omega^2 > S_n gl$

Chapter 10

Rotation and Rigid Bodies

Rotation is very common in our daily life. We can see it in fan, clock (with needles), swing, merry go round, speedometer and dvd in use. In rotation the orientation of a point or line is fixed, if a point is fixed, we say rotation is about a point and if a line is fixed, we say rotation is about a line. The line is known as the axis of rotation. In rotation the angular displacement is called angle of rotation. For example, if we move our arm, we have rotation about our shoulder (may be consider as a point) and if we close or open the door, there is rotation about fixed side, can be regarded as the axis of rotation. In chapter 5, we have already discussed angular motion. In this chapter, we will discuss some particular aspects of rotation in $2-space$ (Rotation around origin) and rotation in $3-space$ (Rotation about an axis passing through origin). First consider rotation in $2-space$.

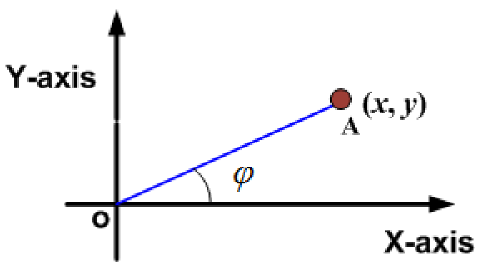
10.1 Rotations in $2-space$ (Rotation about point)

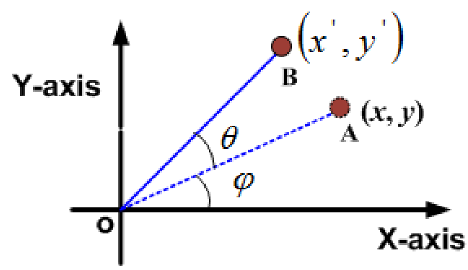
In plane, the mathematics of rotations is fairly trivial. Any rotation takes place around a fixed point, and is uniquely characterized by its direction cosines. Here we will consider xy plane and fixed point as origin.
Note: The rotation of a vector around origin in xy plane is linear transformation but other than origin is not.

10.1.1 Rotation of a vector about origin

Consider $xy-coordinate$ system. Let $A(x,y)$ be a point having radius r and making an angle ϕ with $x-axis$. Then

$$x = r\cos\phi \tag{10.1.1}$$
$$y = r\sin\phi \tag{10.1.2}$$





Let the vector $\vec{OA}$ rotates about origin O making an angle θ, known as angle of rotation. The new position $B(x^{'}, y^{'})$ of A makes an angle $\theta + \phi$ with $x - axis$. Then

$$x^{'} = r\cos(\theta + \phi) \tag{10.1.3}$$
$$y^{'} = r\sin(\theta + \phi) \tag{10.1.4}$$

using trigonometric results we have

$$x^{'} = r(\cos\theta\cos\phi - \sin\theta\sin\phi) \tag{10.1.5}$$
$$y^{'} = r(\sin\theta\cos\phi + \cos\theta\sin\phi) \tag{10.1.6}$$

Using (10.1.1) and (10.1.2), (10.1.5) and (10.1.6) can be written as

$$x^{'} = x\cos\theta - y\sin\theta \tag{10.1.7}$$
$$y^{'} = x\sin\theta + y\cos\theta \tag{10.1.8}$$

(10.1.7) and (10.1.8) give the coordinates of a point after a rotation θ about origin. Above equations in matrix form can be written as

$$\begin{pmatrix} x^{'} \\ y^{'} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{10.1.9}$$

Here $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is known as rotational matrix. Its action on the coordinates is:

$$\begin{pmatrix} x^{'} \\ y^{'} \end{pmatrix} = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{10.1.10}$$

Where l_{ij} are direction cosines which define the orientation of the rigid body.

Let $\vec{x} = \begin{pmatrix} x^{'} \\ y^{'} \end{pmatrix}$ and $\vec{X} = \begin{pmatrix} x \\ y \end{pmatrix}$, then (10.2.1) can be written as

$$\vec{x} = R\vec{X} \tag{10.1.11}$$

(10.1.11) is known as vector equation. It gives the relation how the rotational matrix R connects the vectors $\vec{x}$ and $\vec{X}$. The inverse transformation of (10.1.9) is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \tag{10.1.12}$$

If the coordinates of a point are given before rotation, we will use (10.1.9) to calculate its coordinates after rotation, and if we are given coordinates of a point after rotation, we will use (10.1.12) to calculate its coordinates before rotation.
From above discussion a general result can be stated as:
If the rotation about origin is anticlockwise, then coordinates of a point can be calculated by using (10.1.9) and if the rotation about origin is clockwise, then coordinates of a point can be calculated by using (10.1.12)

Example 10.1.1. *Let a point* $(2,4)$ *is rotated about the origin through an angle of* $\theta = 30^\circ$*.*

Find its new coordinates.

Solution Here the rotation is anticlockwise, so we will use (10.1.7) and (10.1.8) to find new coordinates of A. Given $A = (2,4)$ and $\theta = 30^\circ$ so $x = 2$ and $y = 4$, then

$$\cos\theta = \cos 30 = \frac{\sqrt{3}}{2} \tag{10.1.13}$$

and

$$\sin\theta = \sin 30 = \frac{1}{2} \tag{10.1.14}$$

we obtain

$$\begin{aligned} x' &= 2\left(\frac{\sqrt{3}}{2}\right) - 4\left(\frac{1}{2}\right) \\ y' &= 2\left(\frac{1}{2}\right) + 4\left(\frac{\sqrt{3}}{2}\right) \end{aligned}$$

Thus, the new coordinates are

$$\begin{aligned} x' &= \sqrt{3} - 2 \\ y' &= 1 + 2\sqrt{3} \end{aligned}$$

In other words we can say that the point after rotation is $\left(\sqrt{3} - 2, 1 + 2\sqrt{3}\right) \simeq (-0.3, 4.5)$.

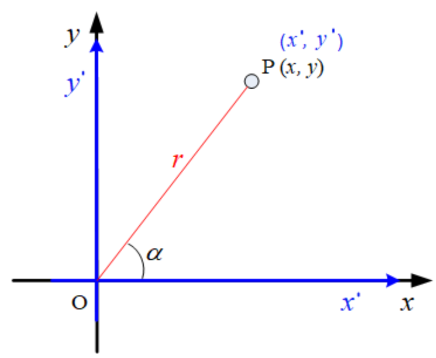


Figure 10.1: Rotation of coordinate axes

10.1.2 Rotation of Coordinate Axes

Consider $xy-$coordinate system. Let it be fixed. Let $P(x, y)$ be a point whose distance from O is r and makes an angle α with $Ox-$axis. This can be viewed as in Fig 10.1. Then polar coordinates of P are

$$x = r\cos\alpha \tag{10.1.15}$$
$$y = r\sin\alpha \tag{10.1.16}$$

Introduce another $x^{'}y^{'}-$coordinate system with same origin O. Let it be rotatable. Initially both system coincides (see Fig. 10.1). Let it be rotated in anticlockwise direction about the origin through an angle ϕ relative to $xy-$coordinate system. This rotation is shown in Fig. 10.2. As r be the distance from the common origin O to the point P, and let r makes an angle θ with $Ox^{'}-$axis as shown in Fig. 10.3. Then coordinates of P relative to $x^{'}y^{'}-$coordinate system can be written as

$$x^{'} = r\cos\theta \tag{10.1.17}$$
$$y^{'} = r\sin\theta \tag{10.1.18}$$

From Fig. 10.3, θ can be written as

$$\theta = \alpha - \phi$$

It follows that

$$x^{'} = r\cos(\alpha - \phi) \tag{10.1.19}$$
$$y^{'} = r\sin(\alpha - \phi) \tag{10.1.20}$$

using trigonometric results we have

$$x^{'} = r(\cos\alpha\cos\phi + \sin\alpha\sin\phi) \tag{10.1.21}$$
$$y^{'} = r(\sin\alpha\cos\phi - \cos\alpha\sin\phi) \tag{10.1.22}$$

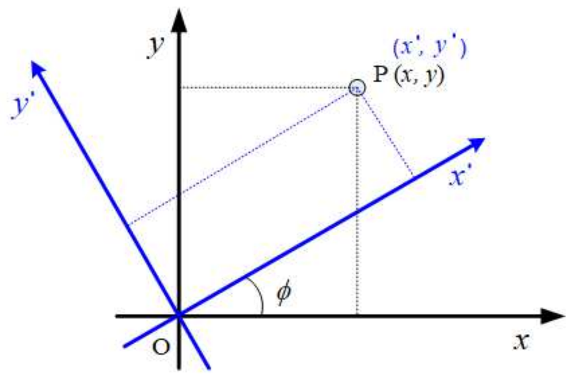


Figure 10.2: Rotation of coordinate axes

Using (10.1.15) and (10.1.16), (10.1.21) and (10.1.22) can be written as

$$x^{'} = x\cos\phi + y\sin\phi \qquad (10.1.23)$$
$$y^{'} = -x\sin\phi + y\cos\phi \qquad (10.1.24)$$

The above system in matrix form can be written as

$$\begin{pmatrix} x^{'} \\ y^{'} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (10.1.25)$$

Its inverse transformation can be given as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x^{'} \\ y^{'} \end{pmatrix} \qquad (10.1.26)$$

Here we observe that there is same coordinate transformation or rotational matrix when a point has anticlockwise rotation about origin and coordinate axes have clockwise rotation about origin.

Example 10.1.2. *If the coordinate axes are rotated about the origin through an angle of* $\theta = 30°$. *Find new coordinates of a point* $A = (2, 4)$.

Solution Here the rotation is anticlockwise, so we will use (10.1.25) to find new coordinates of A. Given $A = (2, 4)$ and $\theta = 30°$ so $x = 2$ and $y = 4$, then

$$\cos\theta = \cos 30 = \frac{\sqrt{3}}{2}$$

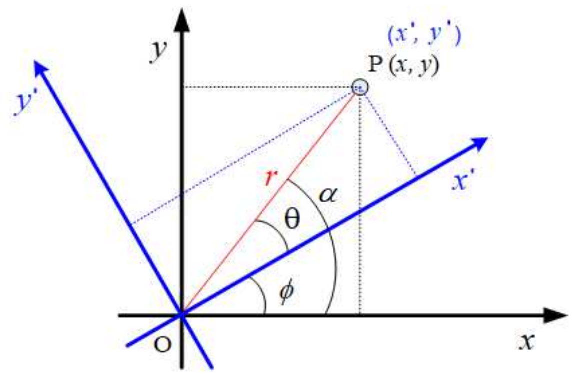


Figure 10.3: Rotation of coordinate axes

and

$$\sin\theta \;=\; \sin 30 = \frac{1}{2}$$

we obtain

$$x' \;=\; 2\left(\frac{\sqrt{3}}{2}\right) + 4\left(\frac{1}{2}\right)$$
$$y' \;=\; -2\left(\frac{1}{2}\right) + 4\left(\frac{\sqrt{3}}{2}\right)$$

Thus, the new coordinates are

$$x' \;=\; \sqrt{3} + 2$$
$$y' \;=\; -1 + 2\sqrt{3}$$

In other words we can say that the point after rotation is $\left(\sqrt{3}+2, -1+2\sqrt{3}\right) \simeq (3.7, 2.5)$. If the equation of a curve is given before rotation, then equation (10.1.25) will give coordinates of fixed system in terms of rotatable system. Using this transformation the curve after rotation can be easily calculated. The equation (10.1.26) is inverse transformation of equation (10.1.25). It gives coordinates of rotatable system in terms of fixed system.

Example 10.1.3. *Suppose that the axes of an $\acute{x}\acute{y}-$coordinate system are rotated through an angle of $\theta = 45$ to obtain an $xy-$coordinate system. Find the equation of the curve*

$$\acute{x}^2 - \acute{x}\acute{y} + \acute{y}^2 - 6 \;=\; 0 \qquad (10.1.27)$$

in xy−coordinates.

Solution Here we have $\theta = 45°$, then

$$\cos\theta = \cos 45 = \frac{1}{\sqrt{2}}$$

and

$$\sin\theta = \sin 45 = \frac{1}{\sqrt{2}}$$

Using the rotation equations in (10.1.7) and (10.1.8), we obtain

$$\begin{aligned} x^{'} &= \left(\frac{x}{\sqrt{2}}\right) - \left(\frac{y}{\sqrt{2}}\right) \\ y^{'} &= \left(\frac{x}{\sqrt{2}}\right) + \left(\frac{y}{\sqrt{2}}\right) \end{aligned}$$

Substituting these into equation (10.1.28) yields

$$\begin{aligned} \left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}\right)^2 - \left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}\right)\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right) + \left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)^2 - 6 &= 0 \\ \frac{x^2}{12} + \frac{y^2}{4} &= 1 \end{aligned} \tag{10.1.28}$$

Example 10.1.4. *Suppose that the axes of an xy−coordinate system are rotated through an angle of θ. to obtain an x́ý−coordinate system. Find the equation of the curve*

$$x^2 + y^2 = a^2 \tag{10.1.29}$$

in x́ý−coordinates.

Solution Since we have given equation of a circle before rotation, so will will use inverse transformation

$$\begin{aligned} x &= \acute{x}\cos\theta + \acute{y}\sin\theta \\ y &= -\acute{x}\sin\theta + \acute{y}\cos\theta \end{aligned}$$

in given equation.

$$\begin{aligned} x^2 + y^2 &= a^2 \\ (\acute{x}\cos\theta + \acute{y}\sin\theta)^2 + (-\acute{x}\sin\theta + \acute{y}\cos\theta)^2 &= a^2 \\ \left(\acute{x}^2\cos^2\theta + \acute{y}^2\sin^2\theta + 2\acute{x}\acute{y}\cos\theta\sin\theta\right) + \left(\acute{x}^2\sin^2\theta + \acute{y}^2\cos^2\theta - 2\acute{x}\acute{y}\cos\theta\sin\theta\right) &= a^2 \\ \acute{x}^2\left(\cos^2\theta + \sin^2\theta\right) + \acute{y}^2\left(\sin^2\theta + \cos^2\theta\right) &= a^2 \\ \acute{x}^2 + \acute{y}^2 &= a^2 \end{aligned}$$

It means that x is transformed into $\acute{x}$ and y into $\acute{y}$ and the equation of a circle remains invariant.

Remark 10.1.1. A circle centred at origin is same after any rotation about the origin.

10.2 Rotational Matrix in $2-space$

The direction cosines which define the orientation of the rigid body can be written as a matrix $R = (l_{ij})_{2\times 2}$ known as rotational matrix.

$$R = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \tag{10.2.1}$$

Rotational matrices are orthogonal (*i.e.* they preserve lengths and angles), hence if R^{-1} is inverse and R^T is transpose of R then the criteria for a matrix to be rotational is

a) $detR = |R| = +1$

b) $RR^T = R^T R = I$ or $R^{-1} = R^T$

If any of above conditions violate, the matrix is not rotational matrix.

10.2.1 Construction of Rotational Matrix in $2-space$

To construct a rotational matrix, consider $xy-$coordinate system. Let it be fixed. Introduce another $x^{'}y^{'}-$coordinate system with same origin O. Let it be rotatable. Initially both system coincides (see Fig. 10.4). Let $x^{'}y^{'}-$coordinate system is rotated in anticlockwise

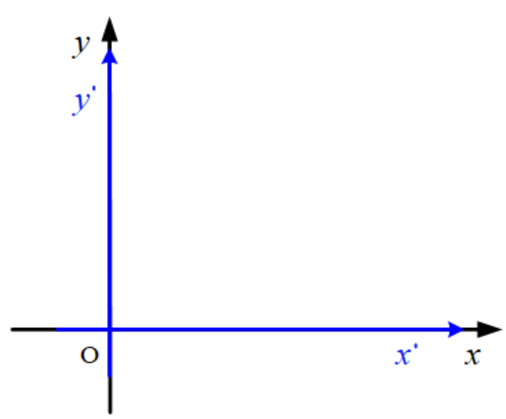


Figure 10.4: Rotation of coordinate axes

direction about the origin through an angle θ relative to $xy-$coordinate system. This

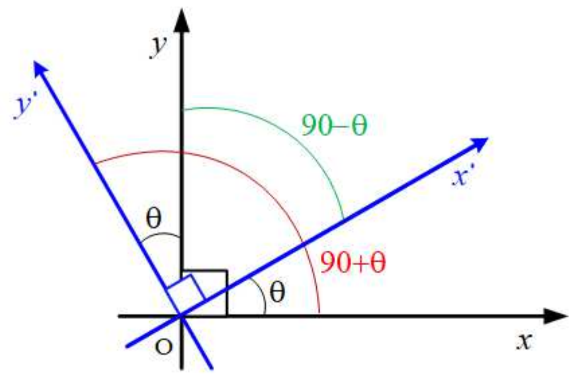


Figure 10.5: Rotation of coordinate axes

rotation is shown in Fig. 10.5. In (10.2.1), l_{ij} are cosines of the angles between the axes. Then rotation matrix can be written as

$$R = \begin{pmatrix} \cos\left(Ox, Ox'\right) & \cos\left(Ox, Oy'\right) \\ \cos\left(Oy, Ox'\right) & \cos\left(Oy, Oy'\right) \end{pmatrix}$$

In Fig. 10.5, the angles between the axes are as following.

$$\angle\left(Ox', Ox\right) = \theta \qquad \angle\left(Ox', Oy\right) = 90 - \theta$$
$$\angle\left(Oy', Ox\right) = 90 + \theta \qquad \angle\left(Oy', Oy\right) = \theta$$

Then the rotational matrix is

$$\begin{aligned} R &= \begin{pmatrix} \cos(\theta) & \cos(90+\theta) \\ \cos(90-\theta) & \cos(\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \end{aligned}$$

Example 10.2.1. *Determine whether the following matrices are rotational matrices.*

1) $A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

2) $B = \begin{pmatrix} 1 & -2 \\ \frac{1}{2} & 1 \end{pmatrix}$

3) $C = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$

Solution For a rotational matrix we have to show

a) $|R| = 1$

b) $RR^T = R^T R = I$ or $R^{-1} = R^T$

1) The determinant of A is

$$|A| = \begin{vmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{vmatrix} = \frac{3}{4} + \frac{1}{4} = 1$$

The transpose of A is $A^T = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ Next AA^T is

$$\begin{aligned} AA^T &= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{4} + \frac{1}{4} & \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} & \frac{3}{4} + \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I \end{aligned}$$

As both conditions are satisfied, the matrix A is rotational matrix.

2) The determinant of B is

$$|B| = \begin{vmatrix} 1 & -2 \\ \frac{1}{2} & 1 \end{vmatrix} = 1 + 1 = 2$$

As $|B| \neq 1$, hence B is not rotational matrix.

3) The determinant of C is

$$|C| = \begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = 6 - 5 = 1$$

The transpose of C is $C^T = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ Next CC^T is

$$\begin{aligned} CC^T &= \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 9+1 & 15+2 \\ 15+2 & 25+4 \end{pmatrix} = \begin{pmatrix} 10 & 17 \\ 17 & 29 \end{pmatrix} \\ &\neq I \end{aligned}$$

As one of the conditions is not satisfied, the matrix C is not rotational matrix.

10.3 Rotations in $3-space$

Most people have played with spinning tops, and know that their dynamics is very rich. The subject is of considerable practical importance, say to spacecraft designers. To understand it we must understand the concept of rotations in three dimensional space, and we will discuss it in this section. The axis of rotation may be coordinate axis or other than coordinate axis.

10.3.1 Position of a Rigid Body in $3-space$

Let us take a regular trihedral $OX_1X_2X_3$, fixed in space. Introduce another regular trihedral $O^{'}x_1x_2x_3$, fixed in the rigid body with $O^{'}$ as origin of the rigid body. Both trihedral are right-handed. The position of the body is completely determined by giving the coordinates $\xi_1\xi_2\xi_3$ with respect to origin $O^{'}$ and by direction cosines of the axes $O^{'}x_1x_2x_3$ relative to the axes $OX_1X_2X_3$. If $l_{ij};\ i,j = 1,2,3$ is the cosines of the angle between $O^{'}x_i$ and OX_j. If $\alpha_{ij};\ i,j = 1,2,3$ are the angle between $O^{'}x_i$ and OX_j Then $l_{ij} = \cos\alpha_{ij};\ i,j = 1,2,3$ From these nine direction cosines only three are independent, as they are connected by six orthogonality conditions which can be written as

$$l_{ki}l_{kj} = l_{ik}l_{jk} = \delta_{ij};\ i,j,k = 1,2,3. \tag{10.3.1}$$

Where δ_{ij} is the Kronecker $\delta-$symbol, defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } \ i = j\ ; \\ 0 & \text{if } \ i \neq j\ ;\ i,j = 1,2,3. \end{cases} \tag{10.3.2}$$

10.3.2 Rotational Matrix in $3-space$

The direction cosines which define the orientation of the rigid body can be written as a matrix $R = (l_{ij})_{3\times3}$.

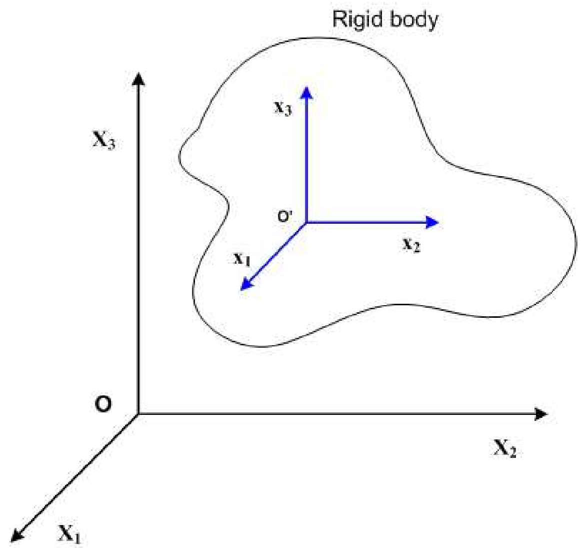


Figure 10.6: Position of a body

$$R = \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$$

in other words

$$R = \begin{pmatrix} \cos\left(OX_1, O^{'}x_1\right) & \cos\left(OX_1, O^{'}x_2\right) & \cos\left(OX_1, O^{'}x_3\right) \\ \cos\left(OX_2, O^{'}x_1\right) & \cos\left(OX_2, O^{'}x_2\right) & \cos\left(OX_2, O^{'}x_3\right) \\ \cos\left(OX_3, O^{'}x_1\right) & \cos\left(OX_3, O^{'}x_2\right) & \cos\left(OX_3, O^{'}x_3\right) \end{pmatrix}$$

Determinant of the matrix R denoted by $|R|$ is the volume of the rectangular parallelepiped whose sides have unit length and lie along the axes of the trihedral $O^{'}x_1x_2x_3$. Since both the trihedral are right-handed, so this volume is $+1$

$$|R| \quad = \quad +1 \tag{10.3.3}$$

The matrix R is called rotational matrix. If R^T denotes the transpose of R, the orthogonality conditions (10.3.1) show that

$$RR^T \quad = \quad R^T R = I \tag{10.3.4}$$

$I_{3\times 3}$ is unit matrix. It follows that

$$R^T = R^{-1} \tag{10.3.5}$$

(10.3.4) and (10.3.5) confirm that the rotational matrix R is a proper orthogonal matrix. For next discussion, we consider O and O' coincide, then rotational matrix is

$$R = \begin{pmatrix} \cos(\angle X_1 O x_1) & \cos(\angle X_1 O x_2) & \cos(\angle X_1 O x_3) \\ \cos(\angle X_2 O x_1) & \cos(\angle X_2 O x_2) & \cos(\angle X_2 O x_3) \\ \cos(\angle X_3 O x_1) & \cos(\angle X_3 O x_2) & \cos(\angle X_3 O x_3) \end{pmatrix}$$

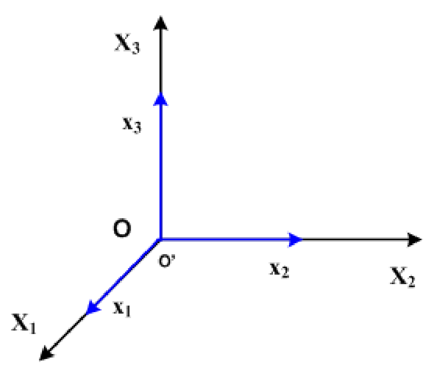


Figure 10.7: Both systems coincides

Example 10.3.1. *Determine whether the following matrix is rotational matrix.*

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{3}{4} & \frac{5}{8} & \frac{\sqrt{3}}{8} \\ -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{8} & \frac{7}{8} \end{pmatrix}$$

Solution For a rotational matrix we have to show

a) $|R| = 1$

b) $RR^T = R^T R = I$ or $R^{-1} = R^T$

a) The determinant of A is

$$
\begin{aligned}
|A| &= \begin{vmatrix} \frac{1}{2} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{3}{4} & \frac{5}{8} & \frac{\sqrt{3}}{8} \\ -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{8} & \frac{7}{8} \end{vmatrix} \\
&= \frac{1}{2}\left(\frac{35}{64} - \frac{3}{64}\right) + \frac{3}{4}\left(\frac{21}{32} + \frac{3}{32}\right) + \frac{\sqrt{3}}{4}\left(\frac{3\sqrt{3}}{32} + \frac{5\sqrt{3}}{32}\right) \\
&= \frac{1}{4} + \frac{9}{16} + \frac{3}{16} \\
&= 1
\end{aligned}
$$

The transpose of A is

$$
A^T = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{3}{4} & \frac{5}{8} & \frac{\sqrt{3}}{8} \\ \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{8} & \frac{7}{8} \end{pmatrix}
$$

Next AA^T is

$$
\begin{aligned}
AA^T &= \begin{pmatrix} \frac{1}{2} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{3}{4} & \frac{5}{8} & \frac{\sqrt{3}}{8} \\ -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{8} & \frac{7}{8} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{3}{4} & \frac{5}{8} & \frac{\sqrt{3}}{8} \\ \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{8} & \frac{7}{8} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{4} + \frac{9}{16} + \frac{3}{16} & \frac{3}{8} - \frac{15}{32} + \frac{3}{32} & -\frac{\sqrt{3}}{8} - \frac{3\sqrt{3}}{32} + \frac{7\sqrt{3}}{32} \\ \frac{3}{8} - \frac{15}{32} + \frac{3}{32} & \frac{9}{16} + \frac{25}{64} + \frac{3}{64} & -\frac{3\sqrt{3}}{16} + \frac{5\sqrt{3}}{64} + \frac{7\sqrt{3}}{64} \\ -\frac{\sqrt{3}}{8} - \frac{3\sqrt{3}}{32} + \frac{7\sqrt{3}}{32} & -\frac{3\sqrt{3}}{16} + \frac{5\sqrt{3}}{64} + \frac{7\sqrt{3}}{64} & \frac{3}{16} + \frac{3}{64} + \frac{49}{64} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= I
\end{aligned}
$$

As both conditions are satisfied, the matrix A is rotational matrix.

10.4 Rotation about Coordinate axis in $3-space$

If axis of rotation is one of the coordinate axes, then we have some special rotational matrix.

10.4.1 Rotational Matrix when Rotation is about x_3 axis

If axis of rotation is x_3 axis, then it coincides with X_3 axis. If the rotation of θ radian is made about x_3 in anticlockwise direction, then the angles between axes are

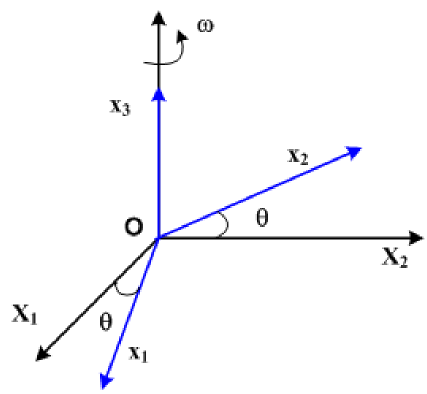


Figure 10.8: Rotation about X_3

$\angle X_1Ox_1 = \theta, \quad \angle X_2Ox_1 = \frac{\pi}{2} - \theta, \quad \angle X_3Ox_1 = \frac{\pi}{2}$
$\angle X_1Ox_2 = \frac{\pi}{2} + \theta, \quad \angle X_2Ox_2 = \theta, \quad \angle X_3Ox_2 = \frac{\pi}{2},$
$\angle X_1Ox_3 = \frac{\pi}{2}, \quad \angle X_2Ox_3 = \frac{\pi}{2}, \quad \angle X_3Ox_3 = 0$
and the rotational matrix is:

$$R = \begin{pmatrix} \cos\theta & \cos\left(\frac{\pi}{2} + \theta\right) & \cos\left(\frac{\pi}{2}\right) \\ \cos\left(\frac{\pi}{2} - \theta\right) & \cos\theta & \cos\left(\frac{\pi}{2}\right) \\ \cos\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) & \cos(0) \end{pmatrix}$$

on simplification we can write

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10.4.1}$$

If the rotation is in clockwise direction, the rotational matrix is

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

10.4.2 Rotation about x_1 axis

Similarly if the rotation is about x_1 axis, then the rotational matrix is

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

10.4.3 Rotation about x_2 axis

Similarly if the rotation is about x_2 axis, then the rotational matrix is

$$R = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

Example 10.4.1. *Find rotational matrix if a rotation of angle $\frac{\pi}{6}$ radian is made about $\langle 0,0,1\rangle$ axis in counter clockwise direction.*

Solution Here the axis of rotation is x_3 axis, to find rotational matrix we can use (10.4.1) with $\theta = \frac{\pi}{6}$

$$\begin{aligned} R &= \begin{pmatrix} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) & 0 \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

10.5 Angle and axis of Rotation

The **angle of rotation** can be measured by the trace of rotational matrix. If the rotation is about x_3 axis, then trace is

$$\begin{aligned} Tr(R) &= \cos\theta + \cos\theta + 1 \\ &= 2\cos\theta + 1 \\ \theta &= \arccos\left(\frac{Tr(R)-1}{2}\right) \end{aligned} \qquad (10.5.1)$$

(10.5.1) gives the angle of rotation for any axis of rotation.

10.5.1 Axis of Rotation

If we consider an arbitrary axis of rotation and let P be a point on the axis of rotation. Then its position relative to $RHRTS\ OX_1X_2X_3$ is $P(X_1, X_2, X_3)$ and its position relative to $RHRTS\ O'x_1x_2x_3$ is $P(x_1, x_2, x_3)$. Since on rotational axis, both axes have the same coordinates. So we have

$$P(X_1, X_2, X_3) = P(x_1, x_2, x_3)$$

Let the position vector of P is

$$\vec{X} = (x_1, x_2, x_3)^T \tag{10.5.2}$$

Then the vector equation (10.1.11) can be written as

$$\vec{X} = R\vec{X} \tag{10.5.3}$$

with $R_{3\times 3}$ rotational matrix. Consequently we have:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{10.5.4}$$

Solving (10.5.4), we have the vector (10.5.2), known as the axis of rotation.

Example 10.5.1. *For rotational matrix*

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{3}{4} & \frac{5}{8} & \frac{\sqrt{3}}{8} \\ -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{8} & \frac{7}{8} \end{pmatrix}$$

determine angle and axis of rotation.

Solution Angle of rotation can be determined by using equation (10.5.1). Trace of R is

$$\begin{aligned} Tr(R) &= \frac{1}{2} + \frac{5}{8} + \frac{7}{8} = \frac{16}{8} \\ &= 2 \end{aligned}$$

Let θ be the angle of rotation then

$$\begin{aligned} \theta &= \arccos\left(\frac{2-1}{2}\right) = \frac{1}{2} \\ &= \frac{\pi}{3} \text{ or } \frac{\pi}{3} + 2n\pi \end{aligned}$$

Let $P = (x, y, z)$, be a point on the axis of rotation, then its position vector relative to origin is: $\vec{x} = (x, y, z)^T$ then the vector equation (10.5.4), becomes

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{3}{4} & \frac{5}{8} & \frac{\sqrt{3}}{8} \\ -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{8} & \frac{7}{8} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and the system of equations is

$$\begin{aligned} x &= \frac{1}{2}x - \frac{3}{4}y + \frac{3}{4}z \\ y &= \frac{3}{4}x + \frac{5}{8}y - \frac{3}{8}z \\ z &= -\frac{3}{4}x + \frac{3}{8}y + \frac{7}{8}z \end{aligned}$$

This system can be rearranged as:

$$\begin{aligned} 2x + 3y - \sqrt{3}z &= 0 \\ 6x - 3y + \sqrt{3}z &= 0 \\ 2\sqrt{3}x - \sqrt{3}y + z &= 0 \end{aligned}$$

This system reduces to

$$\begin{aligned} x &= 0 \\ 3y - \sqrt{3}z &= 0 \end{aligned}$$

The above system is in reduced Echelon form with $x = 0$ and z as free variable. Set $z = 1$, we have $y = \frac{1}{\sqrt{3}}$.

Hence the axis of rotation is $\overline{x} = \left(0, \frac{1}{\sqrt{3}}, 1\right)^T$ and the unit vector along the axis of rotation is $\hat{x} = \left\langle 0, \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$.

Normal vector indicates that axis of rotation lies in yz plane. More information about its direction can be calculated as

$$\tan\theta = \frac{z}{y} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

and

$$\theta = \frac{\pi}{3}$$

Hence axis of rotation passing through origin lies in the $yz-$ plane and is inclined at an angle of $\theta = \frac{\pi}{3}$ to the positive $y - axis$.

10.6 Euler's Angles (ϕ, θ and ψ)

These angles are generated in the above rotations, considering one after the other. The first rotation is counterclockwise through an angle ϕ radian about x_3 axis which initially coincides with X_3 axis. This rotation takes place in x_1x_2 plane, then the rotational matrix is

$$R_\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

After first rotation the coordinate system $Ox_1x_2x_3$ transform to $Ox_1'x_2'x_3'$ and the vector

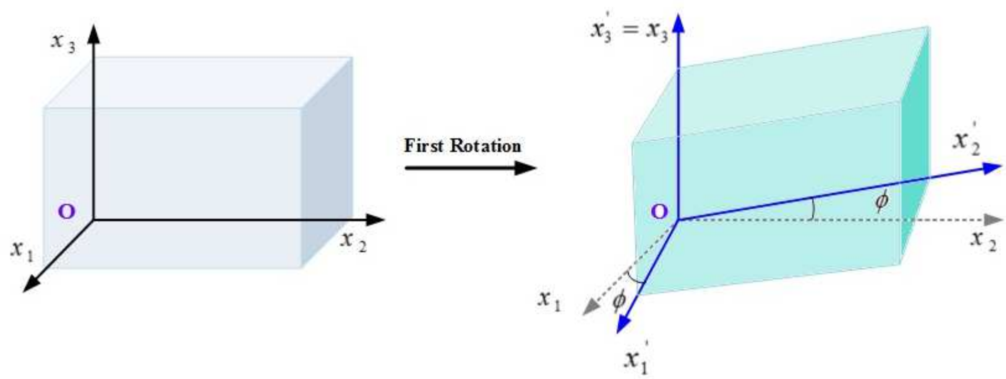


Figure 10.9: First Rotation

equation (10.1.11) can be written as

$$\vec{x'} = R_\phi \vec{x} \tag{10.6.1}$$

The next rotation is counterclockwise through an angle θ radian about x_1' axis. This rotation takes place in $x_2'x_3'$ plane,then the rotational matrix is

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

After second rotation the coordinate system $Ox_1'x_2'x_3'$ transform to $Ox_1''x_2''x_3''$ and the vector

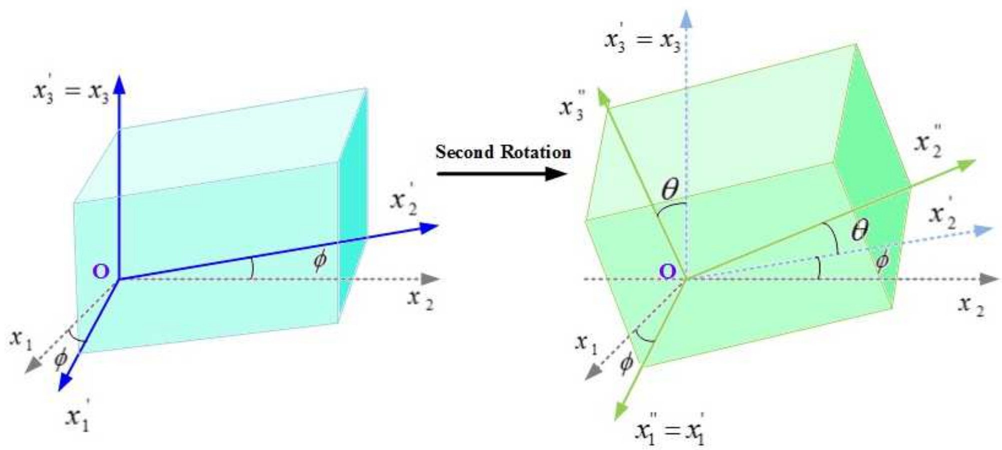


Figure 10.10: Second Rotation

equation is

$$\vec{x''} = R_\theta \vec{x'} \tag{10.6.2}$$

The next rotation is counterclockwise through an angle ψ radian about $x_3^{''}$ axis. This rotation takes place in $x_1^{''}x_2^{''}$ plane, then the rotational matrix is

$$R_\psi = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

After third rotation the coordinate system $Ox_1^{''}x_2^{''}x_3^{''}$ transform to $Ox_1^{'''}x_2^{'''}x_3^{'''}$ and the vector equation is

$$\vec{x'''} = R_\psi \vec{x''} \tag{10.6.3}$$

After these three rotations, the coordinate system $Ox_1x_2x_3$ transform to $Ox_1^{'''}x_2^{'''}x_3^{'''}$ and the complete transformation is given by backward substitution (10.6.3 → 10.6.2 → 10.6.1)

$$\begin{aligned} \vec{x'''} &= R_\psi \vec{x''} = R_\psi R_\theta \vec{x'} \\ &= R_\psi R_\theta R_\phi \vec{x} \end{aligned} \tag{10.6.4}$$

And the rotational matrix is

$$R = R_\psi R_\theta R_\phi \tag{10.6.5}$$

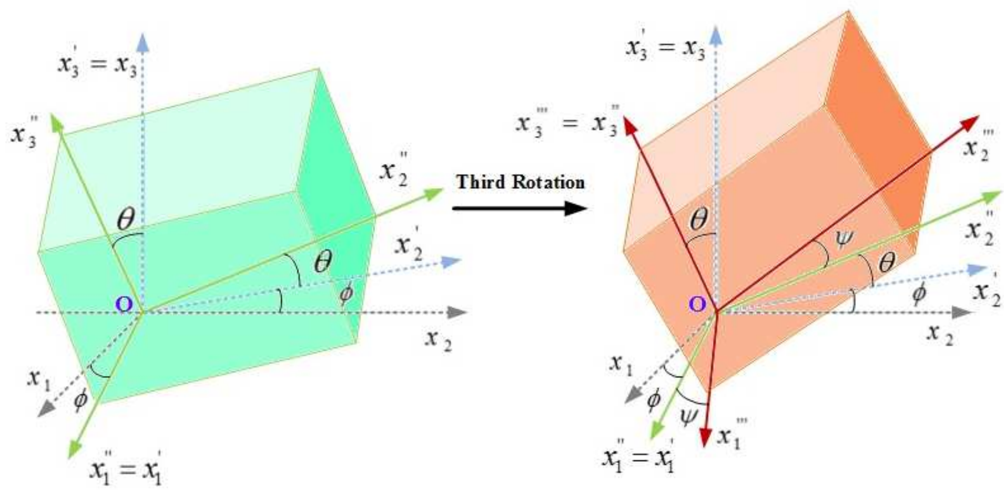


Figure 10.11: Third Rotation

$$R = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\cos\psi\sin\phi - \cos\theta\cos\phi\sin\psi & \sin\theta\sin\psi \\ \sin\psi\cos\phi - \cos\theta\cos\psi\sin\phi & -\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi & -\sin\theta\cos\psi \\ \sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

The lines common to the planes x_1x_2 plane and $x_1^{'''}x_2^{'''}$ planes is called the **line of nodes**.

10.7 Rotation about arbitrary axis in $3 - space$

Let a rigid body rotates about an arbitrary axis OC passing through origin O. Let $\hat{n}$ be the unit vector in this direction. Let P (initial position) be a fixed point in the body having position vector $\vec{r}$. Let it makes angle ϕ with OC axis.

$$\vec{OP} = \vec{r} \tag{10.7.1}$$

Let the point P has anticlockwise rotation of angle θ, then Q be its new (final) position with position vector $\vec{r}_1$ and is shown in Fig. 10.12 (a)

$$\vec{OQ} = \vec{r}_1 \tag{10.7.2}$$

By vector addition, the position vector of this new position of the particle is

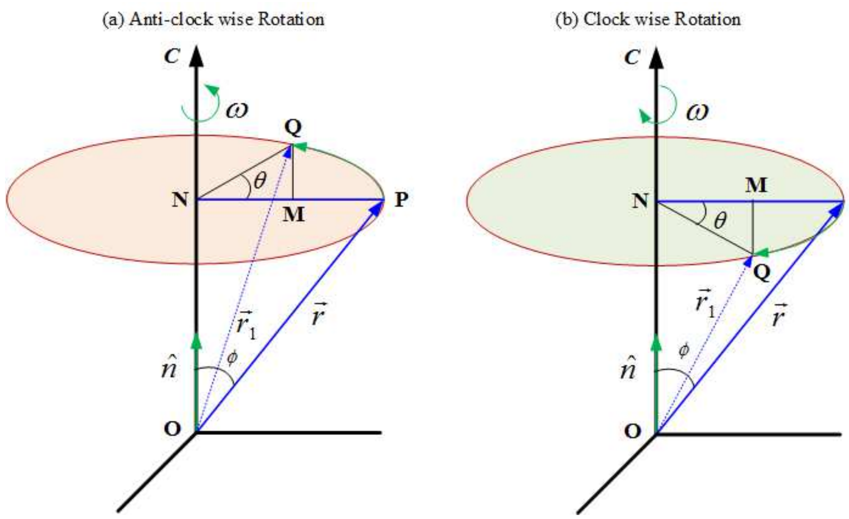


Figure 10.12: Addition of vectors

$$\vec{OQ} = \vec{r}_1 = \vec{ON} + \vec{NQ} \tag{10.7.3}$$

$\vec{ON}$ is along the axis of rotation. To calculate $\vec{NQ}$, consider right angle triangle NMQ as shown in Fig. 10.13 (a)

$$\vec{NQ} = \vec{MN} + \vec{MQ}$$

Then (10.7.3) becomes

$$\vec{r}_1 = \vec{ON} + \vec{NM} + \vec{MQ} \tag{10.7.4}$$

(10.7.4) gives the position of the particle after rotation. To calculate $\vec{ON}$, consider right angle triangle ONP

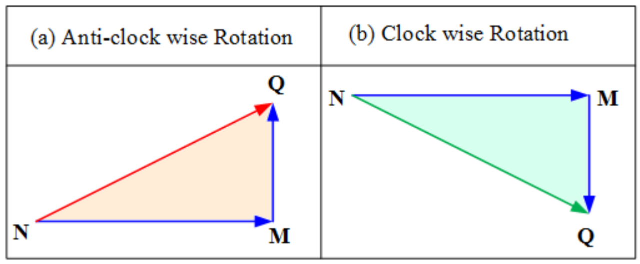


Figure 10.13: Orientation of $\vec{NQ}$

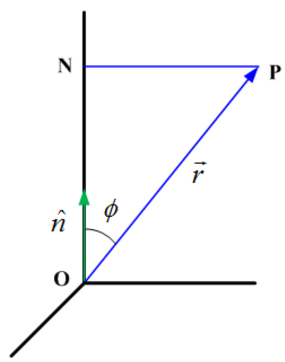


Figure 10.14: Rotation of a Vector

$$\begin{aligned} ON = r\cos\phi &= 1\ r\cos\phi \\ &= \parallel \hat{n} \parallel \parallel \vec{r} \parallel \cos\phi \\ &= \hat{n}\cdot\vec{r} \end{aligned}$$

$\vec{ON}$ can be written as

$$\vec{ON} = (\hat{n}\cdot\vec{r})\,\hat{n} \tag{10.7.5}$$

For $\vec{NM}$ and $\vec{MQ}$, consider right angle triangle NMQ in the circle, we have

$$\begin{aligned} NM &= NQ\cos\theta \\ MQ &= NQ\sin\theta \end{aligned}$$

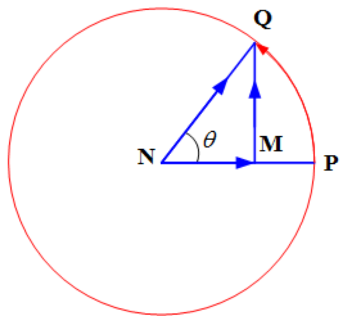


Figure 10.15: triangle NMQ

Next $NQ = NP$ (radius of the circle with centre at N) hence above equations can be written as

$$NM = NP\cos\theta \quad (10.7.6)$$

$$MQ = NP\sin\theta \quad (10.7.7)$$

Since $\vec{NM}$ is in the direction of $\vec{NP}$, so we can write

$$\vec{NM} = \vec{NP}\cos\theta \quad (10.7.8)$$

Consider again right angle triangle ONP

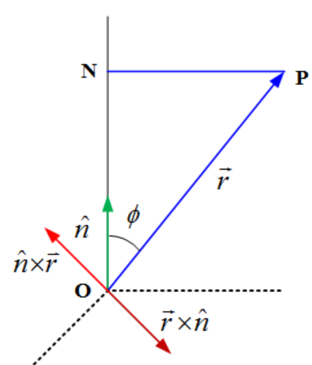


Figure 10.16: right angle triangle ONP

$$\begin{aligned} NP = OP \sin\phi &= r \sin\phi \\ &= 1\ r \sin\phi \\ &= \| \hat{n} \| \| \vec{r} \| \sin\phi \\ &= \| \hat{n} \times \vec{r} \| \end{aligned} \tag{10.7.9}$$

Using (10.7.9), (10.7.7) can be written as

$$MQ = \| \hat{n} \times \vec{r} \| \sin\theta \tag{10.7.10}$$

It can be viewed from the Fig. (10.17), MQ is in the direction of $\hat{n} \times \vec{r}$. Let $\hat{e}$ be the unit vector in this direction.

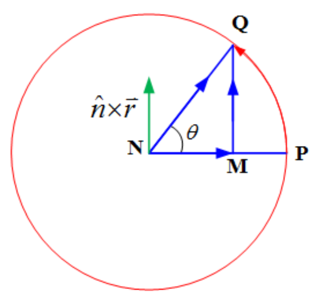


Figure 10.17: Direction of $\vec{MQ}$

$$\begin{aligned} \vec{MQ} &= \| \hat{n} \times \vec{r} \| \hat{e} \sin\theta \\ &= (\hat{n} \times \vec{r}) \sin\theta \end{aligned} \tag{10.7.11}$$

For $\vec{NP}$, again consider right angle triangle ONP. Since

$$\vec{OP} = \vec{ON} + \vec{NP}$$

then $\vec{NP}$ is

$$\vec{NP} = \vec{OP} - \vec{ON} \tag{10.7.12}$$

Using (10.7.1) and (10.7.5), (10.7.12) can be written as

$$\vec{NP} = \vec{r} - (\hat{n} \cdot \vec{r})\, \hat{n} \tag{10.7.13}$$

Hence (10.7.8) can be written as

$$\vec{NM} = [\vec{r} - (\hat{n}\cdot\vec{r})\,\hat{n}]\cos\theta \tag{10.7.14}$$

Hence after rotation, the new position of the particle is

$$\begin{aligned}\vec{OQ} = \vec{r}_1 &= (\hat{n}\cdot\vec{r})\,\hat{n} + [\vec{r} - (\hat{n}\cdot\vec{r})\,\hat{n}]\cos\theta + (\hat{n}\times\vec{r})\sin\theta \\ &= \vec{r}\cos\theta + (1-\cos\theta)\,(\hat{n}\cdot\vec{r})\,\hat{n} + (\hat{n}\times\vec{r})\sin\theta \end{aligned} \tag{10.7.15}$$

Which is also known as rotational formula for finite rotations.

If the rotation is clockwise then the rotational formula is

$$\vec{r}_1 = \vec{r}\cos\theta + (1-\cos\theta)\,(\hat{n}\cdot\vec{r})\,\hat{n} + (\vec{r}\times\hat{n})\sin\theta \tag{10.7.16}$$

This formula can also be written as

$$\vec{r}_1 = \vec{r}\cos\theta + (1-\cos\theta)\,(\hat{n}\cdot\vec{r})\,\hat{n} - (\hat{n}\times\vec{r})\sin\theta$$

10.7.1 Rotational Matrix when rotation is about arbitrary axis

In this case rotational matrix can be calculated by (10.7.15) into vector equation

$$\vec{r_1} = R\vec{r}$$

where $\vec{r_1}$ is vector after rotation, $\vec{r}$ is vector before rotation and R is rotational matrix. Let

$$\begin{aligned}\vec{r} &= \langle x, y, z\rangle \\ \hat{n} &= \langle a, b, c\rangle, \quad \text{then } \; a^2+b^2+c^2 = 1\end{aligned}$$

We will convert all terms on right side of (10.7.15) in matrix form. First consider the term $\vec{r}\cos\theta$

$$\begin{aligned}\vec{r}\cos\theta &= I\vec{r}\cos\theta, \quad \text{where } I \text{ is identity matrix of order } 3\times 3 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\vec{r}\cos\theta \\ &= \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{pmatrix}\vec{r}\end{aligned}$$

Next consider the term $(1-\cos\theta)\,(\hat{n}\cdot\vec{r})\,\hat{n}$

$$\begin{aligned}
(1-\cos\theta)\,(\hat{n}\cdot\vec{r})\,\hat{n} &= (1-\cos\theta)\,(ax+by+cz)\begin{pmatrix} a \\ b \\ c \end{pmatrix} \\
&= (1-\cos\theta)\begin{pmatrix} a^2x+aby+caz \\ abx+b^2y+bcz \\ acx+bcy+c^2z \end{pmatrix} \\
&= (1-\cos\theta)\begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}\begin{pmatrix} x \\ y \\ x \end{pmatrix} \\
&= \begin{pmatrix} a^2(1-\cos\theta) & ab(1-\cos\theta) & ac(1-\cos\theta) \\ ab(1-\cos\theta) & b^2(1-\cos\theta) & bc(1-\cos\theta) \\ ac(1-\cos\theta) & bc(1-\cos\theta) & c^2(1-\cos\theta) \end{pmatrix}\vec{r}
\end{aligned}$$

Finally consider the term $(\hat{n}\times\vec{r})\sin\theta$

$$\begin{aligned}
(\hat{n}\times\vec{r})\sin\theta &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ x & y & z \end{vmatrix}\sin\theta \\
&= \langle bz-cy, cx-az, ay-bx\rangle\sin\theta \\
&= \sin\theta\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ x \end{pmatrix} \\
&= \begin{pmatrix} 0 & -c\sin\theta & b\sin\theta \\ c\sin\theta & 0 & -a\sin\theta \\ -b\sin\theta & a\sin\theta & 0 \end{pmatrix}\vec{r}
\end{aligned}$$

substitution all these results in (10.7.15), we have

$$\begin{aligned}
\vec{r}_1 &= \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{pmatrix}\vec{r} + \begin{pmatrix} a^2(1-\cos\theta) & ab(1-\cos\theta) & ac(1-\cos\theta) \\ ab(1-\cos\theta) & b^2(1-\cos\theta) & bc(1-\cos\theta) \\ ac(1-\cos\theta) & bc(1-\cos\theta) & c^2(1-\cos\theta) \end{pmatrix}\vec{r} \\
&+ \begin{pmatrix} 0 & -c\sin\theta & b\sin\theta \\ c\sin\theta & 0 & -a\sin\theta \\ -b\sin\theta & a\sin\theta & 0 \end{pmatrix}\vec{r} \\
&= \begin{pmatrix} \cos\theta + a^2(1-\cos\theta) & ab(1-\cos\theta) - c\sin\theta & ac(1-\cos\theta) + b\sin\theta \\ ab(1-\cos\theta) + c\sin\theta & \cos\theta + b^2(1-\cos\theta) & bc(1-\cos\theta) - a\sin\theta \\ ac(1-\cos\theta) - b\sin\theta & bc(1-\cos\theta) + a\sin\theta & \cos\theta + c^2(1-\cos\theta) \end{pmatrix}\vec{r}
\end{aligned} \tag{10.7.17}$$

Here

$$R = \begin{pmatrix} a^2(1-\cos\theta) + \cos\theta & ab(1-\cos\theta) - c\sin\theta & ac(1-\cos\theta) + b\sin\theta \\ ab(1-\cos\theta) + c\sin\theta & b^2(1-\cos\theta) + \cos\theta & bc(1-\cos\theta) - a\sin\theta \\ ac(1-\cos\theta) - b\sin\theta & bc(1-\cos\theta) + a\sin\theta & c^2(1-\cos\theta) + \cos\theta \end{pmatrix} \tag{10.7.18}$$

is the rotational matrix when rotation is about arbitrary axis.

Example 10.7.1. *Find rotational matrix if a rotation of angle θ radian is made about $\langle 0,0,1\rangle$ axis in counter clockwise direction.*

Solution Here the axis of rotation is x_3 axis with $a = 0, b = 0$ and $c = 1$. To find rotational matrix we can use (10.7.18)

$$R = \begin{pmatrix} (0)^2(1-\cos\theta) + \cos\theta & (0)(0)(1-\cos\theta) - (1)\sin\theta & (0)(1)(1-\cos\theta) + (0)\sin\theta \\ (0)(0)(1-\cos\theta) + (1)\sin\theta & (0)^2(1-\cos\theta) + \cos\theta & (0)(1)(1-\cos\theta) - (0)\sin\theta \\ (0)(1)(1-\cos\theta) - (0)\sin\theta & (0)(1)(1-\cos\theta) + (0)\sin\theta & (1)^2(1-\cos\theta) + \cos\theta \end{pmatrix}$$

then we get

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 10.7.2. *A point* $(2,1,1)$ *has rotation of angle* $\theta = \frac{\pi}{3}$ *about an axis passing through origin lying in the* $yz-$ *plane and is inclined at an angle of* $\frac{\pi}{3}$ *to the positive* $y-axis$*. Determine:*

a) Its new position after rotation.

b) Its rotational matrix.

Solution

a) The new position of a point after rotation is given by (10.7.15)

$$\vec{r}_1 = \vec{r}\cos\theta + (1-\cos\theta)(\hat{n}\cdot\vec{r})\,\hat{n} + (\hat{n}\times\vec{r})\sin\theta$$

Consider cartesian coordinate system with O as origin. The position vector of point $P(2,1,1)$ is

$$\vec{r} = \langle 2,1,1\rangle$$

Let OC be the axis of rotation making an angle $\frac{\pi}{3}$ to the positive $y-axis$. Accordingly the axis of rotation lies in yz plane with inclination $\frac{\pi}{3}$ with $y-axis$. All this information is shown in Fig. 10.18 The unit vector along axis of rotation is

$$\begin{aligned} \hat{n} &= \left\langle 0, \cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right)\right\rangle \\ &= \left\langle 0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle \end{aligned}$$

The angle of rotation is

$$\theta = \frac{\pi}{3}$$

The trigonometric relations are

$$\begin{aligned} \cos\left(\frac{\pi}{3}\right) &= \frac{1}{2} \\ \sin\left(\frac{\pi}{3}\right) &= \frac{\sqrt{3}}{2} \\ 1-\cos\left(\frac{\pi}{3}\right) &= 1-\frac{1}{2} = \frac{1}{2} \end{aligned}$$

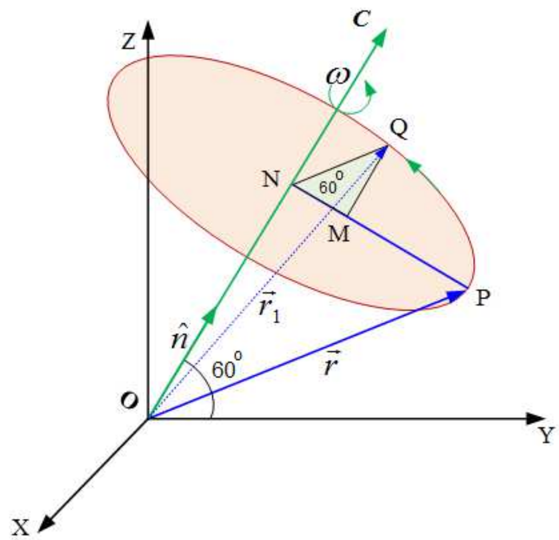


Figure 10.18: Rotation of a Vector

From given data the term $\vec{r}\cos\theta$ can be calculated as

$$\begin{aligned}\vec{r}\cos\theta &= \langle 2,1,1\rangle \cos\left(\frac{\pi}{3}\right)\\ &= \langle 2,1,1\rangle\left(\frac{1}{2}\right) = \left\langle 1,\frac{1}{2},\frac{1}{2}\right\rangle\end{aligned}$$

The dot product of $\hat{n}$ and $\vec{r}$ is

$$\begin{aligned}\hat{n}\cdot\vec{r} &= \left\langle 0,\frac{1}{2},\frac{\sqrt{3}}{2}\right\rangle\cdot\langle 2,1,1\rangle\\ &= 0+\frac{1}{2}+\frac{\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2}\end{aligned}$$

Then the term $(1-\cos\theta)\,(\hat{n}\cdot\vec{r})\,\hat{n}$ is

$$\begin{aligned}(1-\cos\theta)\,(\hat{n}\cdot\vec{r})\,\hat{n} &= \left(\frac{1}{2}\right)\left(\frac{1+\sqrt{3}}{2}\right)\left\langle 0,\frac{1}{2},\frac{\sqrt{3}}{2}\right\rangle\\ &= \left\langle 0,\frac{1+\sqrt{3}}{8},\frac{3+\sqrt{3}}{8}\right\rangle\end{aligned}$$

The cross product of $\hat{n}$ and $\vec{r}$ is

$$
\begin{aligned}
(\hat{n} \times \vec{r}) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 2 & 1 & 1 \end{vmatrix} \\
&= \left\langle \frac{1-\sqrt{3}}{2}, \sqrt{3}, -1 \right\rangle
\end{aligned}
$$

Then the term $(\hat{n} \times \vec{r}) \sin\theta$ is

$$
\begin{aligned}
(\hat{n} \times \vec{r}) \sin\theta &= \left\langle \frac{1-\sqrt{3}}{2}, \sqrt{3}, -1 \right\rangle \left(\frac{\sqrt{3}}{2} \right) \\
&= \left\langle \frac{\sqrt{3}-3}{4}, \frac{3}{2}, -\frac{\sqrt{3}}{2} \right\rangle
\end{aligned}
$$

After rotation the the position vector of Q is

$$
\begin{aligned}
\vec{r}_1 &= \left\langle 1, \frac{1}{2}, \frac{1}{2} \right\rangle + \left\langle 0, \frac{1+\sqrt{3}}{8}, \frac{3+\sqrt{3}}{8} \right\rangle + \left\langle \frac{\sqrt{3}-3}{4}, \frac{3}{2}, -\frac{\sqrt{3}}{2} \right\rangle \\
&= \left\langle \frac{1+\sqrt{3}}{4}, \frac{17+\sqrt{3}}{8}, \frac{7-3\sqrt{3}}{8} \right\rangle
\end{aligned}
$$

Hence new position of P after rotation is $\left(\frac{1+\sqrt{3}}{4}, \frac{17+\sqrt{3}}{8}, \frac{7-3\sqrt{3}}{8} \right)$

b) Its rotational matrix is given by (10.7.18)

$$
R = \begin{pmatrix} a^2(1-\cos\theta)+\cos\theta & ab(1-\cos\theta)-c\sin\theta & ac(1-\cos\theta)+b\sin\theta \\ ab(1-\cos\theta)+c\sin\theta & b^2(1-\cos\theta)+\cos\theta & bc(1-\cos\theta)-a\sin\theta \\ ac(1-\cos\theta)-b\sin\theta & bc(1-\cos\theta)+a\sin\theta & c^2(1-\cos\theta)+\cos\theta \end{pmatrix}
$$

Here $a = 0, b = \frac{1}{2}$, and $c = \frac{\sqrt{3}}{2}$, then the rotational matrix is

$$
\begin{aligned}
R &= \begin{pmatrix} (0)^2\left(\frac{1}{2}\right)+\frac{1}{2} & (0)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) & (0)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\ (0)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) & \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right) & \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)-(0)\left(\frac{\sqrt{3}}{2}\right) \\ (0)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)-\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) & \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)+(0)\left(\frac{\sqrt{3}}{2}\right) & \left(\frac{\sqrt{3}}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{3}{4} & \frac{5}{8} & \frac{\sqrt{3}}{8} \\ -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{8} & \frac{7}{8} \end{pmatrix}
\end{aligned}
$$

As

a) $RR^T = R^T R = I$ or $R^{-1} = R^T$

b) $detR = |R| = 1$

Hence R is rotational matrix.

10.8 Euler's Theorem

Next we study Euler's theorem, was given by Euler in 1776. It has a fundamental importance in the theory of rotations of a rigid body.

Theorem 10.8.1. *The general displacement of a rigid body with one point fixed is a rotation about some axis.*

Proof: Let us take a regular trihedral $OX_1X_2X_3$, fixed in space. Introduce another regular trihedral $Ox_1x_2x_3$, fixed in the rigid body. Initially both systems coincides with O as same origin. The position of the body is completely determined by giving the coordinates $P\,(X_1, X_2, X_3)$ with respect to origin O relative to the axes $OX_1X_2X_3$, while $P\,(x_1, x_2, x_3)$ with respect to origin O relative to the axes $Ox_1x_2x_3$. If the body undergoes some displacement, then the representation of

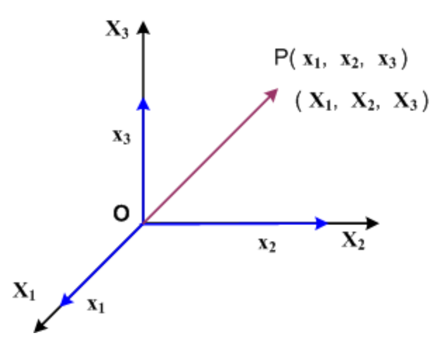


Figure 10.19: Position of a body

any point is given by vector equation:

$$\vec{x} = R\vec{X} \tag{10.8.1}$$

with $\vec{x}$ is the representation with respect to $Ox_1x_2x_3$ and $\vec{X}$ is the representation with respect to $OX_1X_2X_3$. Here R is a rotational matrix which connects the vectors $\vec{x}$ and $\vec{X}$.

If the theorem is true, the points on the axis of rotation are unaltered in position. If we consider $\vec{OP} = \vec{x}$ as the axis of rotation, then by the displacement, on the axis of rotation we have:

$$\vec{x} = \vec{X} \qquad (10.8.2)$$

and consequently Eq. (10.8.1) becomes:

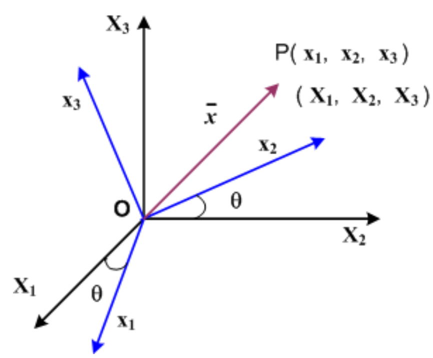


Figure 10.20: Rotation about OP

$$\vec{x} = R\vec{x} \qquad (10.8.3)$$

Hence Euler's theorem will be true if it can be shown that thee exist a vector $\vec{x}$ having the same components in both the systems. Consider the rotational matrix with property $|R| = 1$, then for the eigen vector $\vec{x}$ having eigen value λ, the vector equation is:

$$R\vec{x} = \lambda\vec{x}$$

Equations (10.8.3) and (10.8.4), with orthogonal matrix R specify the physical motion of a rigid body with one point fixed always has the eigen value $\lambda = 1$.

Alternate Approach: Then the characteristic equation with $I_{3\times 3}$ unit matrix

$$|R - \lambda I| = 0$$

holds for $\lambda = 1$, *i.e.* we have to show that

$$|R - I| = 0$$

This can be shown as follows: Let R' is the transpose of R. Consider

$$R'(R - I) = R'R - R'$$

Since R is orthogonal matrix, then $R'R = I$, so we can write

$$\begin{aligned} R'(R - I) &= I - R' \\ &= -(R' - I) \\ &= -(R - I)' \end{aligned}$$

Taking determinant on both sides, we have

$$\left|R'(R - I)\right| = \left|-(R - I)'\right|$$

Since R is rotational matrix, so we have $|R| = |R'| = 1$
also $\left|(R - I)'\right| = |R - I|$, then above relation may be written as:

$$\begin{aligned} \left|R'\right| |(R - I)| &= (-1)^3 \left|(R - I)'\right| \\ |(R - I)| &= (-1) \left|(R - I)'\right| \\ |(R - I)| &= 0 \end{aligned}$$

Hence for eigen value $\lambda = 1$, the characteristic equation $|R - I| = 0$ is satisfied.

10.8.1 Position of the Body after Rotation

If we consider an arbitrary axis of rotation, say $\vec{X} = (x_1, x_2, x_3)^T$. Since on rotational axis, both axes has the same coordinates. This axis can be measured by well known vector equation:

$$A\vec{X} = \lambda\vec{X} \tag{10.8.4}$$

Using Euler's theorem, we have $\lambda = 1$, then Eq. (10.8.4) with $A = R$ becomes:

$$\vec{X} = R\vec{X}$$

so we have:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The above system is same as given by (10.5.4) to calculate the axis of rotation.

Example 10.8.1. *Determine whether the following matrix is rotational matrix. If yes, find angle of rotation and axis of rotation.*

$$R = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

Solution For a rotational matrix we have to show

a) $|R| = 1$

b) $RR^T = R^T R = I$ or $R^{-1} = R^T$

a) The determinant of R is

$$|R| = \begin{vmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{vmatrix}$$

Using determinant property, we can write

$$\begin{aligned}
|R| &= \frac{1}{3^3}\begin{vmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{vmatrix} \\
&= \frac{1}{27}[2(4+2)+(4-1)+2(4+2)] \\
&= 1
\end{aligned}$$

b) The transpose of R is

$$R^T = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Next RR^T is

$$\begin{aligned}
RR^T &= \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \\
&= \begin{pmatrix} \frac{4}{9}+\frac{4}{9}+\frac{1}{9} & -\frac{2}{9}+\frac{4}{9}-\frac{2}{9} & \frac{4}{9}-\frac{2}{9}-\frac{2}{9} \\ -\frac{2}{9}+\frac{4}{9}-\frac{2}{9} & \frac{1}{9}+\frac{4}{9}+\frac{4}{9} & -\frac{2}{9}-\frac{2}{9}+\frac{4}{9} \\ \frac{4}{9}-\frac{2}{9}-\frac{2}{9} & -\frac{2}{9}-\frac{2}{9}+\frac{4}{9} & \frac{4}{9}+\frac{1}{9}+\frac{4}{9} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= I
\end{aligned}$$

As both conditions are satisfied, the matrix R is rotational matrix.

Angle of Rotation: Let θ be the angle of rotation. Then trace of matrix R is:

$$Tr(R) = \frac{2}{3}+\frac{2}{3}+\frac{2}{3} = 2$$

using Eq. (10.5.1), we have

$$\begin{aligned} \theta &= \arccos\left(\frac{2-1}{2}\right) \\ &= \arccos\left(\frac{1}{2}\right) \\ &= \frac{\pi}{3} \ \ or \ \left(2n\pi + \frac{\pi}{3}\right) \end{aligned} \tag{10.8.5}$$

Axis of Rotation: Let $P = (x_1, x_2, x_3)$, be the position of the particle on the axis of rotation, Then its position vector is: $\vec{x} = (x_1, x_2, x_3)^T$ then the vector equation (10.8.3), becomes

$$\begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{aligned} x_1 &= \frac{2}{3}x_1 - \frac{1}{3}x_2 + \frac{2}{3}x_3 \\ x_2 &= \frac{2}{3}x_1 + \frac{2}{3}x_2 - \frac{1}{3}x_3 \\ x_3 &= -\frac{1}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 \end{aligned}$$

This system can be rearranged as:

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 0 \\ 2x_1 - x_2 - x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \end{aligned}$$

This system reduces to

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 0 \\ -x_2 + x_3 &= 0 \end{aligned}$$

The above system is in reduced Echelon form with x_3 as free variable. Set $x_3 = 1$, we have $x_2 = 1$ and $x_1 = 1$.

Hence the axis of rotation is $\overline{x} = (1,1,1)^T$ and the unit vector along the axis of rotation is

$$\begin{aligned}\hat{x} &= \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle \\ &= \langle \cos\alpha, \cos\beta, \cos\gamma \rangle\end{aligned}$$

Where α, β and γ are angles that the axis of rotation makes with coordinate axes. Hence

$$\cos\alpha = \frac{1}{\sqrt{3}}$$

and

$$\begin{aligned}\alpha &= \arccos\left(\frac{1}{\sqrt{3}}\right) = \beta = \gamma \\ &= 0.955 \quad \text{radian} = 54.7^\circ\end{aligned}$$

The axis of rotation makes an angle of 54.7° with all the coordinate axes.

10.9 Dynamics of Rotating Coordinate System

Consider two coordinate systems with common origan O. Let $OX_1X_2X_3$ be a fixed system and let $Ox_1x_2x_3$ is rotating with angular velocity ω about some arbitrary axis passing through origin. Initially both systems coincides with O as same origin. Let $\hat{i}', \hat{j}', \hat{k}'$ be unit vectors in fixed system while $\hat{i}, \hat{j}, \hat{k}$ are unit vectors in rotating system The position of the body is completely determined by giving the coordinates $P(X_1, X_2, X_3)$ with respect to origin O relative to the axis $OX_1X_2X_3$, while $P(x_1, x_2, x_3)$ with respect to origin O relative to the axis $Ox_1x_2x_3$. Let r be the position vector of point P then

$$\vec{r} = X_1\hat{i}' + X_2\hat{j}' + X_3\hat{k}' \tag{10.9.1}$$

and

$$\vec{r} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k} \tag{10.9.2}$$

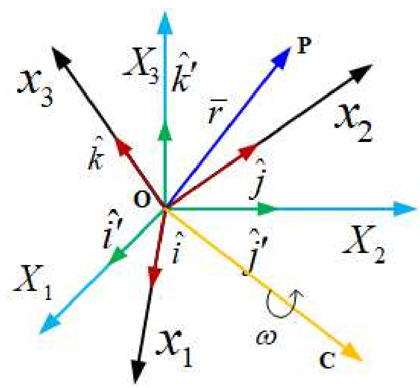


Figure 10.21: Rotation of coordinate system

The transformation equations from moving system to fixed system can be obtained by taking the dot product of (10.9.1) and (10.9.2) with $\hat{i}', \hat{j}', \hat{k}'$ respectively

$$\begin{aligned} \vec{r}\cdot\hat{i}' &= X_1 \\ \vec{r}\cdot\hat{j}' &= X_2 \\ \vec{r}\cdot\hat{k}' &= X_3 \end{aligned} \tag{10.9.3}$$

$$\begin{aligned} \vec{r}\cdot\hat{i}' &= x_1\hat{i}\cdot\hat{i}' + x_2\hat{j}\cdot\hat{i}' + x_3\hat{k}\cdot\hat{i}' \\ \vec{r}\cdot\hat{j}' &= x_1\hat{i}\cdot\hat{j}' + x_2\hat{j}\cdot\hat{j}' + x_3\hat{k}\cdot\hat{j}' \\ \vec{r}\cdot\hat{k}' &= x_1\hat{i}\cdot\hat{k}' + x_2\hat{j}\cdot\hat{k}' + x_3\hat{k}\cdot\hat{k}' \end{aligned} \tag{10.9.4}$$

From (10.9.3) and (10.9.4), we can write

$$\begin{aligned} X_1 &= \vec{r}\cdot\hat{i}' = x_1\hat{i}\cdot\hat{i}' + x_2\hat{j}\cdot\hat{i}' + x_3\hat{k}\cdot\hat{i}' \\ X_2 &= \vec{r}\cdot\hat{j}' = x_1\hat{i}\cdot\hat{j}' + x_2\hat{j}\cdot\hat{j}' + x_3\hat{k}\cdot\hat{j}' \\ X_3 &= \vec{r}\cdot\hat{k}' = x_1\hat{i}\cdot\hat{k}' + x_2\hat{j}\cdot\hat{k}' + x_3\hat{k}\cdot\hat{k}' \end{aligned} \tag{10.9.5}$$

(The equations for inverse transformation can be obtained by taking dot product of (10.9.1) and (10.9.2) with $\hat{i}, \hat{j}, \hat{k}$)

$$\begin{aligned} x_1 &= X_1\hat{i'}\cdot\hat{i} + X_2\hat{j'}\cdot\hat{i} + X_3\hat{k'}\cdot\hat{i} \\ x_2 &= X_1\hat{i'}\cdot\hat{j} + X_2\hat{j'}\cdot\hat{j} + X_3\hat{k'}\cdot\hat{j} \\ x_3 &= X_1\hat{i'}\cdot\hat{k} + X_2\hat{j'}\cdot\hat{k} + X_3\hat{k'}\cdot\hat{k} \end{aligned} \quad (10.9.6)$$

The dot products on right hand side of (10.9.5) are simply the cosines of the angles between the corresponding axes. If the rotation of angle θ is made about x_3 axis in anticlockwise direction, the system will transform as

$$\begin{aligned} X_1 &= x_1\cos\theta - x_2\sin\theta + (0)x_3 \\ X_2 &= x_1\sin\theta + x_2\cos\theta + (0)x_3 \\ X_3 &= (0)x_1 + (0)x_2 + x_3 \end{aligned} \quad (10.9.7)$$

The linear velocity $\vec{v}$ of a particle having position vector $\vec{r}$ and rotating with angular velocity $\vec{\omega}$

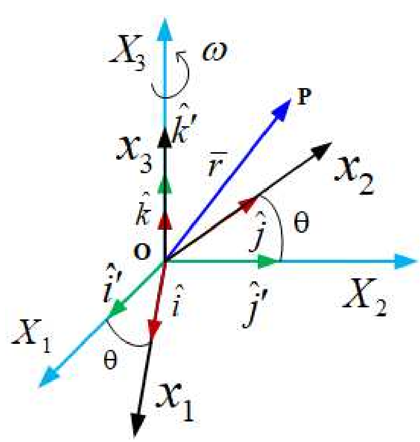


Figure 10.22: Rotation about x_3 axis

about the axis passing through the same origin is

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r} \quad (10.9.8)$$

The velocity with respect to fixed system can be written as

$$\left(\frac{d\vec{r}}{dt}\right)_{fix} = \left(\frac{d\vec{r}}{dt}\right)_{rot} \tag{10.9.9}$$

Here $\left(\frac{d\vec{r}}{dt}\right)_{fix}$ is the velocity of fixed system relative to fixed system and $\left(\frac{d\vec{r}}{dt}\right)_{rot}$ is velocity of rotating system relative to fixed system. The time derivative of $\vec{r}$, however will be different in two systems. In fixed system

$$\left(\frac{d\vec{r}}{dt}\right)_{fix} = \dot{X}_1\hat{i}' + \dot{X}_2\hat{j}' + \dot{X}_3\hat{k}' \tag{10.9.10}$$

However, in rotating system the unit vectors are changing in directions, hence their time derivatives will appear with respect to fixed axis.

$$\left(\frac{d\vec{r}}{dt}\right)_{fix} = \dot{x}_1\hat{i} + \dot{x}_2\hat{j} + \dot{x}_3\hat{k} + x_1\frac{d\hat{i}}{dt} + x_2\frac{d\hat{j}}{dt} + x_3\frac{d\hat{k}}{dt} \tag{10.9.11}$$

In rotating frame the unit vectors $\hat{i}, \hat{j}, \hat{k}$ are treated as constant unit vectors, then the velocity in the rotating system is

$$\left(\frac{d\vec{r}}{dt}\right)_{rot} = \dot{x}_1\hat{i} + \dot{x}_2\hat{j} + \dot{x}_3\hat{k} \tag{10.9.12}$$

Since $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors in a system rotating with angular velocity ω, then (10.9.8) can be applied as a special case as

$$\begin{aligned} \frac{d\hat{i}}{dt} &= \vec{\omega} \times \hat{i} \\ \frac{d\hat{j}}{dt} &= \vec{\omega} \times \hat{j} \\ \frac{d\hat{k}}{dt} &= \vec{\omega} \times \hat{k} \end{aligned} \tag{10.9.13}$$

Using (10.9.12) and (10.9.13), (10.9.11) can be write as

$$\left(\frac{d\vec{r}}{dt}\right)_{fix} = \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega} \times \vec{r} \tag{10.9.14}$$

(10.9.14) can be treated as an operator equation which gives the relation between the time derivative in the fixed and the rotating coordinate systems.

$$\left(\frac{d}{dt}\right)_{fix} = \left(\frac{d}{dt}\right)_{rot} + \vec{\omega}\times \tag{10.9.15}$$

The operator equation (10.9.15) can be operated on any vector. If we denote $\left(\frac{d\vec{r}}{dt}\right)_{rot} = \frac{\partial \vec{r}}{\partial t}$ (derivative relative to moving frame) in (10.9.14), then we have

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{\partial \vec{r}}{\partial t} + \vec{\omega} \times \vec{r} \tag{10.9.16}$$

And the operator equation (10.9.15) becomes

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{\omega}\times \tag{10.9.17}$$

If $F(x, y, z)$ be a vector, then

$$\frac{d\vec{F}}{dt} = \frac{\partial \vec{F}}{\partial t} + \vec{\omega} \times \vec{F} \tag{10.9.18}$$

is the rate of change of a vector.

10.9.1 Acceleration

We use operator equation (10.9.15) with velocity vector $\vec{v}$.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{\omega} \times \vec{v} \tag{10.9.19}$$

Using (10.9.16), we have

$$\begin{aligned}
\vec{a} &= \frac{d}{dt}\left(\frac{\partial \vec{r}}{\partial t} + \vec{\omega} \times \vec{r}\right) \\
&= \frac{\partial}{\partial t}\left(\frac{\partial \vec{r}}{\partial t} + \vec{\omega} \times \vec{r}\right) + \vec{\omega} \times \left(\frac{\partial \vec{r}}{\partial t} + \vec{\omega} \times \vec{r}\right) \\
&= \frac{\partial^2 \vec{r}}{\partial t^2} + \frac{\partial \vec{\omega}}{\partial t} \times \vec{r} + \vec{\omega} \times \frac{\partial \vec{r}}{\partial t} + \vec{\omega} \times \frac{\partial \vec{r}}{\partial t} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \\
&= \ddot{\vec{r}} + \dot{\vec{\omega}} \times \vec{r} + 2\vec{\omega} \times \dot{\vec{r}} + \vec{\omega}\,(\vec{\omega} \cdot \vec{r}) - \omega^2 \vec{r}
\end{aligned} \tag{10.9.20}$$

Corollary 10.9.1. *: The angular acceleration is the same in fixed and rotating systems.*

Proof: Consider the operator equation (10.9.15) with angular velocity vector $\vec{\omega}$.

$$\left(\frac{d\vec{\omega}}{dt}\right)_{fix} = \left(\frac{d\vec{\omega}}{dt}\right)_{rot} + \vec{\omega} \times \vec{\omega}$$

Since $\vec{\omega} \times \vec{\omega} = 0$, so we have

$$\left(\frac{d\vec{\omega}}{dt}\right)_{fix} = \left(\frac{d\vec{\omega}}{dt}\right)_{rot} \tag{10.9.21}$$

Hence the angular acceleration is the same in fixed and rotating systems.

Example 10.9.1. *The points* $(a, 2a, -a)$, $(-a, -a, a)$ *and* (a, a, a) *of a rigid body have instantaneous velocities* $\left\langle \frac{\sqrt{3}}{2}v, 0, \frac{\sqrt{3}}{2}v \right\rangle$, $\left\langle -\frac{1}{\sqrt{3}}v, 0, -\frac{1}{\sqrt{3}}v \right\rangle$ *and* $\left\langle 0, -\frac{1}{\sqrt{3}}v, \frac{1}{\sqrt{3}}v \right\rangle$ *. Show that the body has the line through the origin having direction cosines* $\left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$ *as instantaneous axis of rotation and that the magnitude of the angular velocity is* $\frac{1}{2a}v$.

Solution:

Set O as origin in the body. Let $A = (a, 2a, -a)$, $B = (-a, -a, a)$ and $C = (a, a, a)$ be the points of

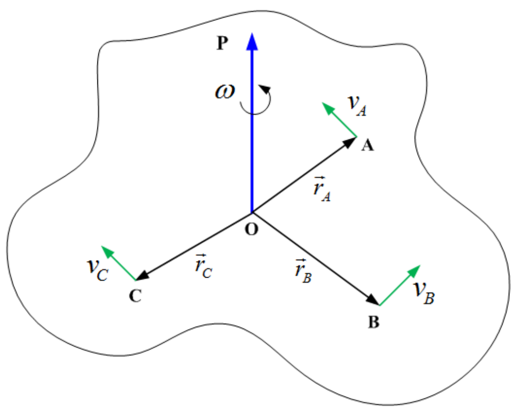


Figure 10.23: Rigid body with OP as axis of rotation

a rigid body, then their position vectors relative to O are

$$\begin{aligned}\vec{r}_A &= \langle a, 2a, -a\rangle \\ \vec{r}_B &= \langle -a, -a, a\rangle \\ \vec{r}_C &= \langle a, a, a\rangle\end{aligned}$$

Also let $\vec{v}_A = \left\langle \frac{\sqrt{3}}{2}v, 0, \frac{\sqrt{3}}{2}v \right\rangle$, $\vec{v}_B = \left\langle -\frac{1}{\sqrt{3}}v, 0, -\frac{1}{\sqrt{3}}v \right\rangle$ and $\vec{v}_C = \left\langle 0, -\frac{1}{\sqrt{3}}v, \frac{1}{\sqrt{3}}v \right\rangle$ be the linear velocities of A, B and C respectively. Let OP be the instantaneous axis of rotation with angular velocity ω, then

$$\vec{\omega} = \langle \omega_1, \omega_2, \omega_3\rangle$$

As the linear velocity of A relative to O is $\vec{v}_A$, can be given as

$$\vec{v}_A = \vec{\omega} \times \vec{r}_A$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ a & 2a & -a \end{vmatrix}$$

$$= \langle -a\omega_2 - 2a\omega_3, a\omega_1 + a\omega_3, 2a\omega_1 - a\omega_2 \rangle \tag{10.9.22}$$

But

$$v_A = \left\langle \frac{\sqrt{3}}{2}v, 0, \frac{\sqrt{3}}{2}v \right\rangle \tag{10.9.23}$$

From (10.9.22) and (10.9.23), we can write

$$-a\omega_2 - 2a\omega_3 = \frac{\sqrt{3}}{2}v \tag{10.9.24}$$

$$a\omega_1 + a\omega_3 = 0 \tag{10.9.25}$$

$$2a\omega_1 - a\omega_2 = \frac{\sqrt{3}}{2}v \tag{10.9.26}$$

From (10.9.25), we can write

$$\omega_2 = -\omega_1 \tag{10.9.27}$$

Similarly the velocity of B relative to O is

$$\vec{v}_B = \vec{\omega} \times \vec{r}_B$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ -a & -a & a \end{vmatrix}$$

$$= \langle a\omega_2 + a\omega_3, -a\omega_1 - a\omega_3, -a\omega_1 + a\omega_2 \rangle \tag{10.9.28}$$

But

$$v_B = \left\langle -\frac{1}{\sqrt{3}}v, 0, -\frac{1}{\sqrt{3}}v \right\rangle \tag{10.9.29}$$

From (10.9.28) and (10.9.29), we can write

$$a\omega_2 + a\omega_3 = -\frac{1}{\sqrt{3}}v \tag{10.9.30}$$

$$-a\omega_1 - a\omega_3 = 0 \tag{10.9.31}$$

$$-a\omega_1 + a\omega_2 = -\frac{1}{\sqrt{3}}v \tag{10.9.32}$$

From (10.9.31), we can write

$$\omega_3 = -\omega_1 \tag{10.9.33}$$

Adding (10.9.26) and (10.9.32),we have

$$\omega_1 = \frac{1}{2a\sqrt{3}}v = \frac{1}{\sqrt{3}}\frac{v}{2a} \tag{10.9.34}$$

From (10.9.27), we have

$$\omega_2 = -\frac{1}{\sqrt{3}}\frac{v}{2a} \tag{10.9.35}$$

and from (10.9.33), we have

$$\omega_3 = -\frac{1}{\sqrt{3}}\frac{v}{2a} \tag{10.9.36}$$

Hence

$$\vec{\omega} = \left\langle \frac{1}{\sqrt{3}}\frac{v}{2a}, -\frac{1}{\sqrt{3}}\frac{v}{2a}, -\frac{1}{\sqrt{3}}\frac{v}{2a} \right\rangle \tag{10.9.37}$$

$$= \frac{v}{2a}\left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle \tag{10.9.38}$$

Since

$$\left\| \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\| = 1$$

so we can write

$$\omega = \frac{v}{2a} \tag{10.9.39}$$

Since the points with velocity vector parallel to $\vec{\omega}$ lies on the line (axis of rotation). Hence the body has the line through O with direction cosines $\left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$

10.10 Cylindrical Coordinates

Polar coordinates can be extended to three dimensions in a very straightforward manner by adding the z coordinate. Every point in space is determined by the r and θ coordinates of its projection in the xy plane, and its z coordinates. Consider $Oxyz$ cartesian coordinate system. Let $P(x, y, z)$ be a point and $|PB|$ is perpendicular from P on xOy plane. Then

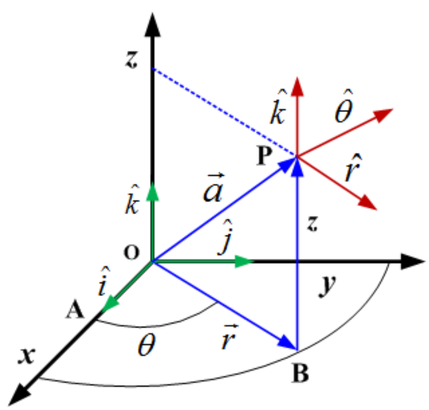


Figure 10.24: Cylindrical Coordinates

$$
\begin{aligned}
\angle AOB &= \theta \\
|\vec{OB}| &= r \\
|\vec{PB}| &= z
\end{aligned}
$$

Then (r, θ, z) are called the cylindrical polar coordinates of P. Let P varies with time, and

$$
\begin{aligned}
\vec{OP} = \vec{a} &= \vec{OB} + \vec{BP} \\
&= r\hat{r} + z\hat{z} \\
&= r\hat{r} + 0\hat{\theta} + z\hat{z} \qquad (10.10.1)
\end{aligned}
$$

Where $\hat{r}, \hat{\theta}, \hat{z}$ are unit vectors in the directions of r, θ, z. They constitute a right hand frame for which

$$
\begin{aligned}
\hat{r} \times \hat{\theta} &= \hat{z} \\
\hat{\theta} \times \hat{z} &= \hat{r} \\
\hat{z} \times \hat{r} &= \hat{\theta}
\end{aligned}
$$

Since P is in a frame specified by these unit vectors and moving with angular velocity ω along Oz axis

$$
\begin{aligned}
\omega &= \dot{\theta}\hat{z} \\
&= 0\hat{r} + 0\hat{\theta} + \dot{\theta}\hat{z}
\end{aligned} \tag{10.10.2}
$$

The velocity of P can be given by using operator equation (10.9.15) with position vector $\vec{a}$.

$$
\vec{v} = \frac{d\vec{a}}{dt} = \frac{\partial\vec{a}}{\partial t} + \vec{\omega} \times \vec{a} \tag{10.10.3}
$$

Using (10.10.1) and (10.10.2), (10.10.3) can be rewritten as

$$
\begin{aligned}
\vec{v} &= \frac{\partial}{\partial t}\left(r\hat{r} + 0\hat{\theta} + z\hat{z}\right) + \left(0\hat{r} + 0\hat{\theta} + \dot{\theta}\hat{z}\right) \times \left(r\hat{r} + 0\hat{\theta} + z\hat{z}\right) \\
&= \left(\dot{r}\hat{r} + 0\hat{\theta} + \dot{z}\hat{z}\right) + \left(0\hat{r} + r\dot{\theta}\hat{\theta} + 0\hat{z}\right) \\
&= \left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z}\right)
\end{aligned} \tag{10.10.4}
$$

The acceleration of P can be calculated by using operator equation (10.9.15) with velocity vector $\vec{v}$.

$$
\begin{aligned}
\vec{a} &= \frac{\partial}{\partial t}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z}\right) + \left(0\hat{r} + 0\hat{\theta} + \dot{\theta}\hat{z}\right) \times \left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z}\right) \\
&= \left[\ddot{r}\hat{r} + \left(r\ddot{\theta} + \dot{r}\dot{\theta}\right)\hat{\theta} + \ddot{z}\hat{z}\right] + \left(-r\dot{\theta}^2\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + 0\hat{z}\right) \\
&= \left[\left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta} + \ddot{z}\hat{z}\right]
\end{aligned} \tag{10.10.5}
$$

10.11 Spherical Coordinates

In spherical coordinates, we utilize two angles and a distance to specify the position of a particle. Consider $Oxyz$ cartesian coordinate system. Let $P(x, y, z)$ be a point. The circle in the POZ plane

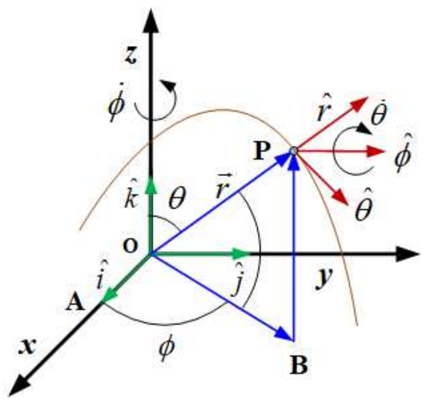


Figure 10.25: Spherical Coordinates

having center O and radius r meets the XOY plane in B. Then

$$\begin{aligned}\angle AOB &= \phi\\ \angle POZ &= \theta\\ OP &= r\end{aligned}$$

Then (r, θ, ϕ) are called the spherical polar coordinates of P. Let P varies with time, and

$$\begin{aligned}\vec{OP} &= \vec{r}\\ &= r\hat{r} + 0\hat{\theta} + 0\hat{z}\end{aligned} \tag{10.11.1}$$

Where $\hat{r}, \hat{\theta}, \hat{\phi}$ are unit vectors in the directions of r, θ, ϕ. They constitute a right hand frame for which

$$\begin{aligned}\hat{r} \times \hat{\theta} &= \hat{\phi}\\ \hat{\theta} \times \hat{\phi} &= \hat{r}\\ \hat{\phi} \times \hat{r} &= \hat{\theta}\end{aligned}$$

In this motion two angles are involved, hence the frame has angular velocity as

$$\omega = \dot{\phi}\hat{k} + \dot{\theta}\hat{\phi} \tag{10.11.2}$$

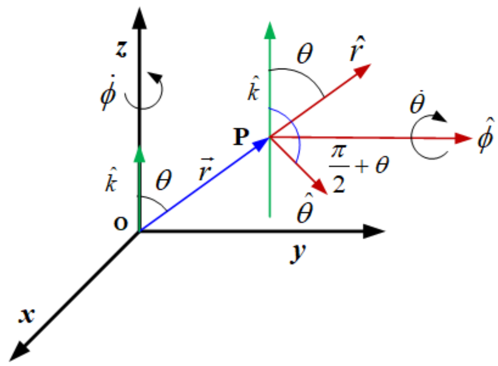


Figure 10.26: Spherical Coordinates

The units vectors $\hat{r}, \hat{\theta}$ and $\hat{k}$ lies in the same plane constituted by $\hat{r}$ and $\hat{k}$. The unit vectors $\hat{r}$ and $\hat{\theta}$ are at right angles and $\hat{k}$ makes angles θ with $\hat{r}$ and $\frac{\pi}{2}+\theta$ with $\hat{\theta}$ as shown in Fig. 10.27. Then $\hat{k}$ can be written as

$$\begin{aligned} \hat{k} &= \cos\theta \hat{r} + \cos\left(\frac{\pi}{2}+\theta\right)\hat{\theta} \\ &= \cos\theta \hat{r} - \sin\theta \hat{\theta} \end{aligned} \tag{10.11.3}$$

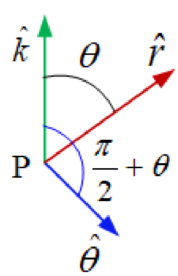


Figure 10.27: Spherical Coordinates

Using (10.11.3), (10.11.2) can be rewritten as

$$\omega = \dot{\phi}\cos\theta \hat{r} - \dot{\phi}\sin\theta\hat{\theta} + \dot{\theta}\hat{\phi} \tag{10.11.4}$$

The velocity of P can be given by using operator equation (10.9.15) with position vector $\vec{r}$ given by

(10.33).

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{\partial \vec{r}}{\partial t} + \vec{\omega} \times \vec{r} \tag{10.11.5}$$

Using (10.33) and (10.14.10), (10.14.11) can be rewritten as

$$\begin{aligned}
\vec{v} &= \frac{\partial}{\partial t}\left(r\hat{r} + 0\hat{\theta} + 0\hat{\phi}\right) + \left(\dot{\phi}\cos\theta\hat{r} - \dot{\phi}\sin\theta\hat{\theta} + \dot{\theta}\hat{\phi}\right) \times \left(r\hat{r} + 0\hat{\theta} + 0\hat{z}\right) \\
&= \left(\dot{r}\hat{r} + 0\hat{\theta} + 0\hat{\phi}\right) + \left(0\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi}\right) \\
&= \left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi}\right)
\end{aligned} \tag{10.11.6}$$

The acceleration of P can be calculated by using operator equation (10.9.15) with velocity vector $\vec{v}$.

$$\begin{aligned}
\vec{a} &= \frac{\partial}{\partial t}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi}\right) + \left(\dot{\phi}\cos\theta\hat{r} - \dot{\phi}\sin\theta\hat{\theta} + \dot{\theta}\hat{\phi}\right) \times \left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi}\right) \\
&= \left[\ddot{r}\hat{r} + \left(r\ddot{\theta} + \dot{r}\dot{\theta}\right)\hat{\theta} + \left(r\ddot{\phi}\sin\theta + \dot{r}\dot{\phi}\sin\theta + r\dot{\theta}\dot{\phi}\cos\theta\right)\hat{\phi}\right] \\
&+ \left[\left(-r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2\right)\hat{r} + \left(\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta\right)\hat{\theta} + \left(r\dot{\theta}\dot{\phi}\cos\theta + \dot{r}\dot{\phi}\sin\theta\right)\hat{\phi}\right] \\
&= \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta\right)\hat{\theta} \\
&+ \left(2r\dot{\theta}\dot{\phi}\cos\theta + r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta\right)\hat{\phi}
\end{aligned} \tag{10.11.7}$$

10.12 Screw Motion

A motion combination of rotation and translation motion is termed as screw motion as illustrated in Fig. 10.28.

Screw motion consist a of pairs of vectors, such as forces and moments and angular and linear velocity, that arise in the kinematics and dynamics of rigid bodies. The mathematical framework was developed by Sir Robert Stawell Ball in 1876 for application in kinematics and statics of mechanisms (rigid body mechanics).

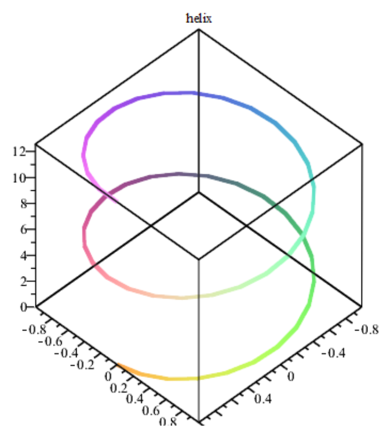


Figure 10.28: Screw motion

10.12.1 Screw Displacement

A spatial displacement of a rigid body can be defined by a rotation about a line and a translation along the same line, called a screw displacement. This is known as Chasles' theorem.

10.13 Chasle's Theorem

The most general rigid body displacement can be produced by a translation along a line followed (or preceded) by a rotation about that line.

To establish this result, discovered by Chasles in 1830, we note that the positions in space of any three non-collinear points of a rigid body determine the position of the rigid body. Let A, B, C be any three non-collinear points of the body which are displaced to other positions A_1, B_2, C_2. Thus the points A, B, C determine the initial position of the body and A_1, B_2, C_2 determine the final position. This screw displacement takes place in two steps. In first step, the motion is translation,

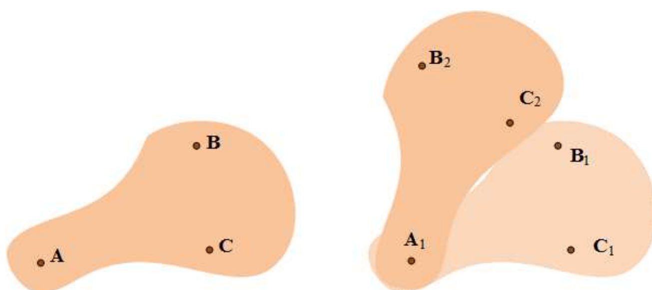


Figure 10.29: Screw displacement of Rigid body

in which A moves A_1, B moves B_1 and C moves C_1 as shown in Fig. 10.29. In second step, the motion is rotation about A_1 in which B_1 moves B_2 and C_1 moves C_2. The point A is called base point. We can choose any other point as base point. Then the translation will be altered but rotation will be independent of the choice of base point.

10.14 Fundamental Properties of Screw Motion

A screw is a six-dimensional vector constructed from a pair of three-dimensional vectors, such as forces and torques and linear and angular velocity, that arise in the study of spatial rigid body movement. The components of the screw define the Plcker coordinates of a line in space and the magnitudes of the vector along the line and moment about this line.

10.14.1 Vector Angular Velocity

Let a rigid body rotates about a fixed point O. Let ω be its angular velocity about the instantaneous axis OC at time t. Let P be a fixed point in the body such that

$$\vec{OP} = \vec{r} \tag{10.14.1}$$

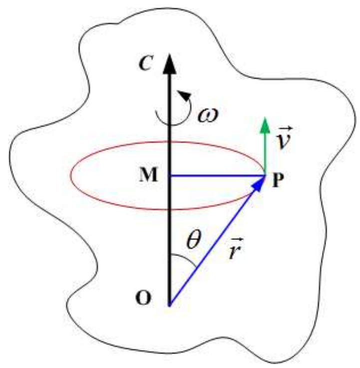


Figure 10.30: Rigid body with OC as axis of rotation

making an angle θ with OC axis. Draw a perpendicular PM from P on OC. Then in triangle OPM

$$PM = r\sin\theta \tag{10.14.2}$$

The linear speed in terms of angular speed is

$$v = \omega r \tag{10.14.3}$$

Where r is the radius of the circle. From Fig 10.30 we can see that the circular path has radius $r\sin\theta$, so the speed of P is

$$v = \omega r\sin\theta \tag{10.14.4}$$

(10.14.4) in vector form can be written as

$$\vec{v} = \vec{\omega}\times\vec{r} \tag{10.14.5}$$

If in addition, the body as a whole has a constant translation velocity $\vec{v}_0$, then the total linear velocity of the particle at P is

$$\vec{v} = \vec{v}_0 + \vec{\omega}\times\vec{r} \tag{10.14.6}$$

Corollary 10.14.1. *For a rotating rigid body about a fixed point, prove that*

$$curl(\vec{v}) = 2\vec{\omega}$$

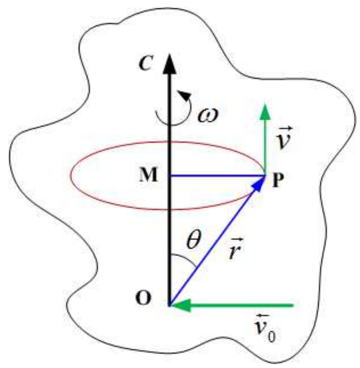


Figure 10.31: Rigid body with OC as axis of rotation having translation motion

Solution Consider

$$\vec{\omega} \quad = \quad \langle \omega_1, \omega_2, \omega_3 \rangle$$

and

$$\vec{r} \quad = \quad \langle x_1, x_2, x_3 \rangle$$

in cartesian coordinate system. Using (10.14.5), we can write

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x_1 & x_2 & x_3 \end{vmatrix}$$

$$= \quad \langle x_3\omega_2 - x_2\omega_3, x_1\omega_3 - x_3\omega_1, x_2\omega_1 - x_1\omega_2 \rangle \tag{10.14.7}$$

Next

$$curl(\vec{v}) \quad = \quad \nabla \times \vec{v} \tag{10.14.8}$$

Where

$$\nabla \quad = \quad \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right\rangle \tag{10.14.9}$$

is an operator. Using (10.14.7) and (10.14.9), (10.14.8) can be written as

$$curl\,(\vec{v}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ x_3\omega_2 - x_2\omega_3 & x_1\omega_3 - x_3\omega_1 & x_2\omega_1 - x_1\omega_2 \end{vmatrix}$$

$$\begin{aligned} &= \left(\frac{\partial}{\partial x_2}(x_2\omega_1 - x_1\omega_2) - \frac{\partial}{\partial x_3}(x_1\omega_3 - x_3\omega_1)\right)\hat{i} \\ &+ \left(\frac{\partial}{\partial x_3}(x_3\omega_2 - x_2\omega_1) - \frac{\partial}{\partial x_1}(x_2\omega_1 - x_1\omega_2)\right)\hat{j} \\ &+ \left(\frac{\partial}{\partial x_1}(x_1\omega_3 - x_3\omega_1) - \frac{\partial}{\partial x_2}(x_3\omega_2 - x_2\omega_3)\right)\hat{k} \end{aligned}$$

Since x_1, x_2, x_3 are linearly independent, then

$$\frac{\partial x_i}{\partial x_j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

so we have

$$\begin{aligned} curl\,(\vec{v}) &= (\omega_1 + \omega_1)\,\hat{i} + (\omega_2 + \omega_2)\,\hat{j} + (\omega_3 + \omega_3)\,\hat{k} \\ &= 2\,\langle \omega_1, \omega_2, \omega_3 \rangle \\ &= 2\vec{\omega} \end{aligned}$$

Hence the result.

10.14.2 General Rigid Body Motion as a Screw Motion

In screw motion, the axis of rotation lie in the direction of the translation motion *i.e.* the linear velocity v and angular velocity ω has the same direction. Consider a rigid body. Let O be a fixed point in the rigid body, having velocity v and ω be the instantaneous angular velocity of the body.

Let OC be the axis of rotation. Let P be another point of the body such that

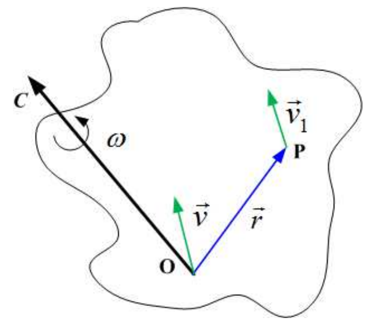


Figure 10.32: Rigid body with OC as axis of rotation

$$\vec{OP} = \vec{r}$$

and $\vec{v_1}$ be the velocity of P, then

$$\vec{v_1} = \vec{v} + \vec{\omega} \times \vec{r} \tag{10.14.10}$$

Now if we try to find a point P such that $\vec{v_1}$ and $\vec{\omega}$ are parallel, then

$$\vec{\omega} \times \vec{v_1} = 0 \tag{10.14.11}$$

Taking cross product of (10.14.10) with $\vec{\omega}$, next using (10.14.11), we have

$$\begin{aligned} 0 &= \vec{\omega} \times \vec{v} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \\ &= \vec{\omega} \times \vec{v} + (\vec{\omega} \cdot \vec{r})\,\vec{\omega} - \omega^2 \vec{r} \end{aligned}$$

or we can write

$$\omega^2 \vec{r} = \vec{\omega} \times \vec{v} + (\vec{\omega} \cdot \vec{r})\,\vec{\omega} \tag{10.14.12}$$

If $\omega \neq 0$, then (10.14.12) can be written as

$$\vec{r} = \frac{\vec{\omega} \times \vec{v}}{\omega^2} + \lambda \vec{\omega} \tag{10.14.13}$$

where $\lambda = \frac{(\vec{\omega} \cdot \vec{r})}{\omega^2}$ As λ varies, (10.14.13) represents the vector equation of straight line passing through the point represented by $\vec{r}_0 = \frac{\vec{\omega} \times \vec{v}}{\omega^2}$ and having the same direction as $\vec{\omega}$. At the particular

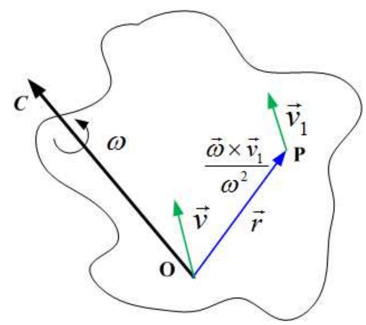


Figure 10.33: Screw motion

instant considered every point on the line (10.14.13), having the velocity parallel to $\vec{\omega}$. Hence the instantaneous motion of the body is a screw motion about the line. This line is called external axis or axis of screw. Every point on the axis moves along it and the body turns about the axis.

10.14.3 Pitch of Screw Motion

Since the velocity of P is $\vec{v_1}$ given by (10.14.10). Taking its dot product with $\vec{\omega}$

$$\begin{aligned}\vec{\omega}\cdot\vec{v_1} &= \vec{\omega}\cdot(\vec{v}+\vec{\omega}\times\vec{r})\\ &= \vec{\omega}\cdot\vec{v}+\vec{\omega}\cdot(\vec{\omega}\times\vec{r})\\ &= \vec{\omega}\cdot\vec{v}+0\end{aligned}$$

and we have

$$\vec{\omega}\cdot\vec{v_1} = \vec{\omega}\cdot\vec{v} \tag{10.14.14}$$

Next consider the quantity $\frac{\vec{\omega}\cdot\vec{v_1}}{\omega^2}$, with the velocity $\vec{v_1}$ is parallel to $\vec{\omega}$. Then

$$\begin{aligned}\frac{\vec{\omega}\cdot\vec{v_1}}{\omega^2} &= \frac{\omega v_1\cos(0)}{\omega^2}\\ &= \frac{v_1}{\omega}\\ &= \frac{\texttt{distance per second}}{\texttt{angle per second}}\end{aligned}$$

or

$$\frac{\vec{\omega}\cdot\vec{v_1}}{\omega^2} = \frac{\texttt{distance}}{\texttt{angle}} = \texttt{distance per unit angle} \tag{10.14.15}$$

The ratio (10.14.15) is called pitch of the screw, *i.e.* distance moved by the particle per unit angle.

10.14.4 Finite Rotations are not Commutative

The rotations and other rotational quantities derived from it may also be treated as vector quantities. The addition of such two vector quantities, should however commutative, but the reality is not so. Consider two regular trihedral systems with common origin O, one fixed and the other is rotatable. Let $OX_1X_2X_3$, fixed in space and $Ox_1x_2x_3$ is rotatable. First we made a rotation of $\theta_1 = \frac{\pi}{2}$ about $X_3 - axis$. In this rotation, x_3 will remain unchanged, while x_1 and x_2 will interchange their positions . Considering the new positions, we made an other rotation of $\theta_2 = \frac{\pi}{2}$ about $X_2 - axis$. In this rotation, x_1 (along $X_2 - axis$) will remain unchanged, while x_2 and x_3 will interchange their positions (see Fig. 10.34 left side). In this way we complete the operation $\theta_1 + \theta_2$. Next we reverse the order. First we made a rotation of $\theta_2 = \frac{\pi}{2}$ about $X_2 - axis$. In this rotation, x_2 will remain unchanged, while x_1 and x_3 will interchange their positions. Considering the new positions, we made an other rotation of $\theta_1 = \frac{\pi}{2}$ about $X_3 - axis$. In this rotation, x_1 (along $X_3 - axis$) will remain unchanged, while x_2 and x_3 will interchange their positions (see Fig. 10.34 right side). In this way we complete the operation $\theta_2 + \theta_1$. Above Fig. 10.34, shows that addition of two finite rotations $\theta_2 = \frac{\pi}{2}$ & $\theta_2 = \frac{\pi}{2}$ is not commutative. *i.e.*, $\theta_1 + \theta_2$ does not yield the position of the body as by $\theta_2 + \theta_1$

The body which is subjected to these two finite rotations is not found to be in the same final state. Hence finite rotations can not be regarded as vector quantities.

10.14.5 Infinitesimal Rotations are Commutative

Consider two regular trihedral systems with common origin O, one fixed and the other is rotatable. Let $OX_1X_2X_3$, fixed in space and $Ox_1x_2x_3$ is rotatable. We made a rotation of a very small angle about $X_3 - axis$. The axes $X_1 - axis$ and $X_2 - axis$ moves but maintaining the same direction. Here $\theta \approx 0$ so $\sin\theta \approx \theta$. Then we shall see that, for very small rotations $\Delta\theta_1 \& \Delta\theta_2$, the addition will be commutative. To represent infinitesimal rotation as a vector, we draw a straight line along the axis of rotation with the properties.

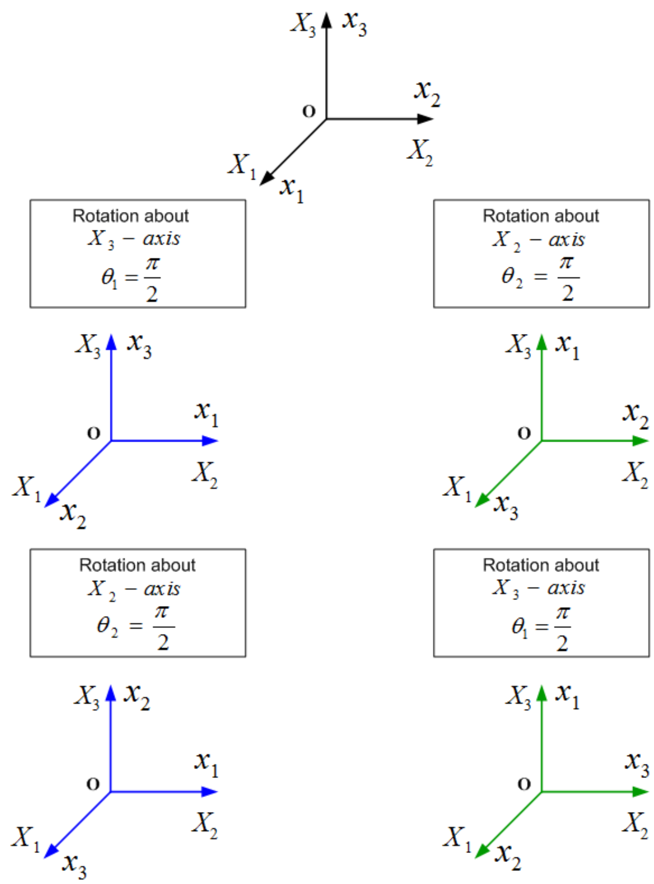


Figure 10.34: Finite rotations are not commutative

i) the length of which is proportional to the magnitude of the angle of rotation.

ii) the arrowhead points in the direction of advancement of the tip of the righthand screw.

Let O and P are points fixed in a rigid body and

$$\vec{OP} = \vec{r} \tag{10.14.16}$$

Let the rigid body turns through a small angle $\Delta\theta_1$ in the positive sense about an axis through O specified by unit vector $\hat{a}_1$, then

$$\vec{\omega} = \Delta\theta_1 \hat{a}_1 \tag{10.14.17}$$

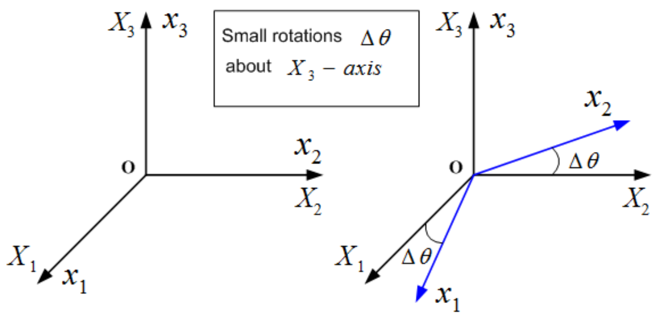


Figure 10.35: Infinitesimal rotations

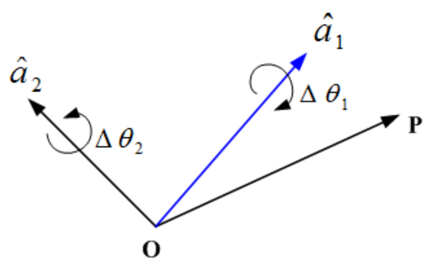


Figure 10.36: Infinitesimal rotations

and the movement of P is given by using (10.14.5) with ω in (10.14.17)

$$\vec{v} = \Delta\theta_1 \hat{a}_1 \times \vec{r} \tag{10.14.18}$$

Hence the new position of P relative to O

$$\vec{r}_1 = \vec{r} + \Delta\theta_1 \hat{a}_1 \times \vec{r} \tag{10.14.19}$$

Now suppose the body rotates through another small angle $\Delta\theta_2$ in the opposite sense about an axis through O specified by a unit vector $\hat{a}_2$. and the movement of P is

$$\vec{v} = \Delta\theta_2 \hat{a}_2 \times \vec{r}_1 \tag{10.14.20}$$

$$\vec{r}_{12} = \vec{r}_1 + \Delta\theta_2 \hat{a}_2 \times \vec{r}_1 \tag{10.14.21}$$

$$= \vec{r} + \Delta\theta_1 \hat{a}_1 \times \vec{r} + \Delta\theta_2 \hat{a}_2 \times (\vec{r} + \Delta\theta_1 \hat{a}_1 \times \vec{r}) \tag{10.14.22}$$

Since $\Delta\theta_1$ and $\Delta\theta_2$ are very small, consider only first order term, we can write

$$= \quad \vec{r} + (\Delta\theta_1 \hat{a}_1 + \Delta\theta_2 \hat{a}_2) \times \vec{r} \tag{10.14.23}$$

If the operations were performed in the reverse order, the position of P relative to O, considering only first order term,is

$$\vec{r}_{21} = \vec{r} + (\Delta\theta_2 \hat{a}_2 + \Delta\theta_1 \hat{a}_1) \times \vec{r} \tag{10.14.24}$$

Thus to first order

$$\vec{r}_{12} = \vec{r}_{21} \tag{10.14.25}$$

Hence for infinitesimal rotations, commutative law holds.

10.14.6 Composition of Angular velocity

If Δt be the time interval in which these operations take place, the velocity of P relative to O is

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{\vec{r}_{12} - \vec{r}}{\Delta t} &= \lim_{\Delta t \to 0} \left(\frac{\Delta\theta_1}{\Delta t} \hat{a}_1 + \frac{\Delta\theta_2}{\Delta t} \hat{a}_2 \right) \times \vec{r} \\ &= \left(\dot{\theta}_1 \hat{a}_1 + \dot{\theta}_2 \hat{a}_2 \right) \times \vec{r} \\ &= (\vec{\omega}_1 + \vec{\omega}_2) \times \vec{r} \end{aligned}$$

where $\vec{\omega}_1 = \dot{\theta}_1 \hat{a}_1$ vector angular velocity about $\hat{a}_1$
and $\vec{\omega}_2 = \dot{\theta}_2 \hat{a}_2$ vector angular velocity about $\hat{a}_2$
This result confirms that vector quantities about a point of a rigid body are commutative with respect to addition, as the other quantities are.

Exercises

1. consider the point $(x, y) = (1, 1)$ in a frame rotated by 45^o counter-clockwise. Find its new position.

2. Let an $xy-$coordinate system be obtained by rotating an $\acute{x}\acute{y}-$coordinate system through an angle of $\theta = 60°$.

   (a) Find the $xy-$coordinates of the point whose $\acute{x}\acute{y}-$coordinates are $(-2, 6)$.

   (b) Find an equation of the curve $\sqrt{3}\acute{x}\acute{y} + \acute{y}^2 = 6$ in $xy-$coordinates.

3. Let an $xy-$coordinate system be obtained by rotating an $\acute{x}\acute{y}-$coordinate system through an angle of $\theta = 30°$.

   (a) Find the $xy-$coordinates of the point whose $\acute{x}\acute{y}-$coordinates are $\left(1, -\sqrt{3}\right)$.

   (b) Find an equation of the curve $2\acute{x}^2 + 2\sqrt{3}\acute{x}\acute{y} = 3$ in $xy-$coordinates.

4. Find the angle and axis of rotation corresponding to the following rotational matrices.

   (a) $$\begin{pmatrix} \frac{5}{2} & -\frac{1}{2} & \frac{3}{4} \\ \frac{2}{3} & \frac{1}{2} & -\frac{3}{4} \\ -\frac{2}{3} & \frac{4}{5} & -\frac{5}{2} \end{pmatrix}$$

   (b) $$\begin{pmatrix} \frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} & -\frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} & -\frac{4}{5} \end{pmatrix}$$

(c) $\begin{pmatrix} \frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{3}{4} \end{pmatrix}$

5. The points $(a, 0, 0)$, $\left(0, \frac{a}{\sqrt{3}}, 0\right)$ and $(0, 0, 2a)$ of a rigid body have instantaneous velocities $\langle u, 0, 0\rangle$, $\langle u, 0, v\rangle$ and $\left\langle u + v, \sqrt{3}v, \frac{v}{2}\right\rangle$ respectively, referred to a regular trihedral system. Find the magnitude and direction of spin of the body and the point at which the certain axis cuts the xz-plane.

Bibliography

[1] Q.K.Ghori, Introduction to Mechanics, West Pak Publishing Co. Ltd., Lahore 1971.

[2] D. Halliday, R. Resnick and J. Walker, Fundamentals of Physics,John Wiley & Sons, lnc., 8th ed. extended, 2008.

[3] D. G. Zill and M. R. Cullen, Differential Equations with Boundary-Value Problems, 7th Edition, 2009.

[4] Richard Bronson and G. B. Costa, Schaum's Outline of Differential Equations, McGraw-Hill Companies, Inc., 3rd Edition, 2006.

[5] H. Anton, I. Bivens and S. Davis, Calculus, John Wiley & Sons, lnc., Early Transcendentals 10th edition, 2012.

[6] L. D. Landau, E. M. Lifshitz, Mecanics, Pergamon Press, Oxford: butterworth, 3rd edition, 15 2005.

[7] B. Ahmad, S. Borisenok, Control of effective potential minima for Kapitza oscillator by periodical kicking pulses, Physics Letters A, 373, 701-707, 2009.

[8] B. Ahmad, Stabilization of Kapitza Oscillator by Periodic Symmetric Pulses, Nonlinear Dynamics, 62, 499-506, 2010.

[9] K. Singh, Mechanics, Department of Mathematics, Directorate of Distance Education, Guru Jambheshwar University of Science & Technology Hisar-125001.

[10] S.T. Thornton and J.B. Marion, Classical Dynamics of Particles and Systems, Cengage Learning India Private Ltd. Third Indian Reprint, 2009.

[11] S.S. Bhavikatti, A textbook of Classical Mechanics, New Age International Publishers, Second Edition Reprint, 2009.

[12] R.G. Takwale and P.S. Puranik, Introduction to Classical Mechanics, Tata McGraw-Hill Publishing Company Ltd. 39th reprint 2009.

[13] Lecture notes of Mechanics MIT, USA.

[14] Wikipedia.

Index

Lecture #	Topics
1	Velocity And Acceleration in Cartesian coordinate system (1,2 &3 dimensional)
2	Motion Of A Free Particle Along The Vertical Line
3	Air resistance motion along vertical line
4	Rectilinear motion
5	Rectilinear motion
6	Tangential And Normal Components Of Velocity And Acceleration
7	Radial And Transverse Components Of Velocity And Acceleration
8	Simple Harmonic Motion
9	Forced Harmonic Motion
10	Damped Harmonic Motion
11	Forced and damped Harmonic Motion
12	Projectile Motion Parabola Of Safety Time Period Maximum Height
13	Range On The Inclined Plane Maximum Range On Horizontal And Inclined Plane
14	Projectile Motion With Air Resistance
15	Motion under central force Elliptic Orbit Under Central Force Planetary Orbits
16	Motion under central force Kepler’s Law
17	Motion under central force Apse And Apsidal Distance
18	Rotation in 2-space, Rotation of coordinate axis, rotational matrix
19	Rotation in 3-space, rotational matrix, Eulerian angles
20	Euler’s Theorem, Angle of rotation and axis of rotation
21	Rotating coordinate systems
22	Chasle’s theorem, screw motion
23	Rotation about arbitrary axis
24	Position vector after rotation, rotational matrix, Euler's Dynamical Equations
25	Rotation of coordinate axis
26	Kinematics in cylindrical coordinates
27	Kinematics in spherical coordinates
28	Angular Momentum, Rotational kinetic energy (Mechanics I)
29	Equi-momental Systems, Momental Ellipsoid (Mechanics I)
30	Principle of Gyroscopic Compass (Mechanics I)