Course: Mathematics III (6448)
Semester: Autumn, 2021

ASSIGNMENT No. 2

Q.1 a) Solve $\frac{dy}{dx} = 2x$ **such that** $y(1) = 4$**.**

Solution: $\frac{dy}{dx} = 2x$ ____(1)

This is an initial value problem.

$dy = 2xdx$

By integrating

$$\int dy = \int 2xdx$$

We have

$$y = 2\frac{x^2}{2}$$

So, $y = x^2 + c$, with arbitrary constant c, is the general solution of (1). Since $y(1) = 4$, we have: $4 = 1^2 + c \Rightarrow C = 3$

Thus $y = x^2 + 3$ is the solution of the initial value problem $\frac{dy}{dx} = 2x$.

b) Solve $\frac{d^2y}{dx^2} + y = 0$ **subject to the conditions** $y(1) = 3, y'(1) = -4$.

Solution: Here we observed that both the conditions relate to one value of x, this is an initial value problem. So the general solution of the given equation is:

$y = C_1 \sin x + C_2 \cos x$ _______ (1)

The initial conditions implies that:

$$3 = C_1 \sin(1) + C_2 \cos(1)$$

$$-4 = C_1 \cos(1) - C_2 \sin(1)$$

Solving for C_1 and C_2, we find,

$$C_1 = -4\cos(1) + 3\sin(1)$$

$$C_2 = 3\cos(1) + 4\sin(1)$$

With these values of C_1 and C_2, (1) is the required solution.

Q.2 a) Solve the linear differential equation: $\frac{dy}{dx} - 4y = 8$

Solution: This linear equation can be solved by separation of variables. Alternatively, since the equation is already in the standard form of $dy/dx+P(x)y=f(x)$.

The integrating factor is $e^{\int p(x)dx} = e^{\int(-4)dx} = e^{-4x}$.

Now multiply the integrating factor with the given equation we have

$$e^{-4x}\frac{dy}{dx} - 4ye^{-4x} = 8e^{-4x}$$

we have

$$\frac{d}{dx}\left[e^{-4x}y\right] = 8e^{-4x}$$

Integrating both sides gives from the above equation

$$\int d\left[e^{-4x}y\right] = \int 8e^{-4x}dx$$

Simplifying

$$e^{-4x}y = 8\frac{e^{-4x}}{-4} + c$$

Thus the solution of the differential equation is

$$y = -2 + ce^{4x}$$

In case of a_1, a_0, and g in (7.1) are constants, the differential equation is autonomous. In Example 7.1, you can verify from the normal form $dy/dx = 4(y + 2)$ that 2 is a critical point and that it is unstable and a repeller. Thus a solution curve with an initial point either above or below the graph of the equilibrium solution $y = -2$ pushes away from this horizontal line as x increases.

Notice in the general discussion and in Example 7.1 we disregarded a constant of integration in the evaluation of the indefinite integral in the exponent of $e^{\int p(x)dx}$.

If you think about the laws of exponents and the fact that the integrating factor multiplies both sides of the differential equation, you should be able to answer why writing $e^{p(x)+c}$ is pointless.

b) Solve the following exact differential equation:

$2xydx + (x^2 - 1)dy = 0$

Solution:

In above equation we have

$$M(x,y) = 2xy \qquad N(x,y) = x^2 - 1$$

we have the

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$

Thus the equation is exact and so there exist a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 - 1$$

From the first of these equations we obtain, after integrating,

$$f(x,y) = x^2y + g(y)$$

Taking the partial derivative of the last expression with respect to y and setting the result equal to $N(x,y)$ gives

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1$$

It follows that

$$g'(y) = -1 \quad \text{and} \quad g(y) = -y$$

Hence, $f(x,y) = x^2y - y$, and so the solution of the equation in implicit from is $x^2y - y = c$. The explicit form of the solution is easily seen to be $y = c/(x^2 - 1)$ and is defined on any interval not containing either $x = 1$ or $x = -1$.

Q.3 The given function family is the general differential equation solution at the defined interval. Find a family member who is a solution to the problem of initial value problem.

1. $y = c_1e^x + c_2e^{-x} \quad (-\infty, \infty) \quad y'' - y = 0, \quad y(0) = 0, \quad y'(0) = 1$
2. $y = c_1x + c_2x\ln x \quad (0, \infty) \quad x^2y'' - xy' + y = 0, \quad y(1) = 3, \quad y'(1) = -1$

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Verify that the indicated family of functions is a solution of the given differential equation. Assume an appropriate interval I of definition for each solution.

$$x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 12x^2$$
$$y = c_1x^{-1} + c_2x + c_3x\ln x + 4x^2$$

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Verify that the indicated family of functions is a solution of the given differential equation. Assume an appropriate interval I of definition for each solution.

$$\frac{dP}{dt} = P(1 - P); \quad P = \frac{c_1 e^t}{1 + c_1 e^t}$$

Q.4 a) **Given that $y_1 = e^x$ is solution of $y'' - y = 0$ on the interval $(-\infty, \infty)$, use reduction of order to find a second solution y_2.**

Solution

If $y = u(x)y_1(x) = u(x)e^x$, then the first two derivatives of y are obtained from the product rule

$$y' = ue^x + e^x u'$$
$$y'' = ue^x + 2e^x u' + e^x u''$$

By substituting and y into the original DE, it simplifies to

$$y'' - y = e^x\left(u'' + 2u'\right) = 0$$

Since $e^x \neq 0$, the last equation requires $u'' + 2u' = 0$. If we make the substitution $w = u'$, this linear second-order equation in u becomes $w' + 2w = 0$, which is a linear first-order equation in w. Using the integrating factor e^{2x}, we can write $d/dx[e^{2x}w] = 0$. After integrating we get

$$w = c_1 e^{-2x}$$

or

$$u_0 = c_1 e^{-2x}$$

Integrating again then yields

$$u = -\frac{1}{2}c_1 e^{-2x} + c_2$$

Hence

$$y = u(x)e^x = -\frac{1}{2}c_1 e^{-x} + c_2 e^x \quad (8.8)$$

By choosing $c_2 = 0$ and $c_1 = -2$ we obtain the desired second solution, $y_2 = e^{-x}$. Because $W(e^x, e^{-x})$ 6= 0 for every x, the solutions are linearly independent on $(-\infty, \infty)$.

b) Evaluate $\mathscr{L}\{\sin 2t\}$.

$$\mathscr{L}\{\sin 2t\} = \int_0^\infty e^{-st} \sin 2t \, dt$$

Using integration by parts we have

$$\mathscr{L}\{\sin 2t\} = \left.\frac{-e^{-st}\sin 2t}{s}\right|_0^\infty + \frac{2}{s}\int_0^\infty e^{-st}\cos 2t\ dt$$

$$\mathscr{L}\{\sin 2t\} = \frac{2}{s}\int_0^\infty e^{-st}\cos 2t\ dt$$

Again applying integration by parts

$$\mathscr{L}\{\sin 2t\} = \frac{2}{s}\left.\frac{-e^{-st}\cos 2t}{s}\right|_0^\infty - \frac{4}{s^2}\int_0^\infty e^{-st}\sin 2t\ dt$$

$$\mathscr{L}\{\sin 2t\} = \frac{2}{s^2} - \frac{4}{s^2}\int_0^\infty e^{-st}\sin 2t\ dt$$

Now as $\mathscr{L}\{\sin 2t\} = \int_0^\infty e^{-st}\sin 2t\ dt$, we have

$$\mathscr{L}\{\sin 2t\} = \frac{2}{s^2} - \frac{4}{s^2}\mathscr{L}\{\sin 2t\}$$

Simplifying we have

$$\mathscr{L}\{\sin 2t\} = \frac{2}{s^2+4},\ s > 0$$

Q.5 a) Evaluate $\mathscr{L}\{f(t)\}$ for $f(t) = \{0, \quad 0 \le t < 32, \quad t \ge 3$.

This piecewise-continuous function appears in figure. Since f is defined in two pieces, $\mathscr{L}\{f(t)\}$ is expressed as the sum of two integrals

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^3 e^{-st}(0)\, dt + \int_3^\infty e^{-st}(2)\, dt$$

$$\mathscr{L}\{f(t)\} = -\left.\frac{2e^{-st}}{s}\right|_3^\infty$$

$$\mathscr{L}\{f(t)\} = \frac{2e^{-3s}}{s},\ s > 0$$

b) Find the laplace transformation of the following functions.

f(t) = {–1, 0< t < 11, t ≥ 1

$$\begin{aligned}
F(s) &= \int_0^{\infty} e^{-ts} f(t)\, dt = \int_0^{\infty} e^{-ts}(1)\, dt \\
&= \lim_{b\to+\infty} \int_0^b e^{-ts}\, dt \\
&= \lim_{b\to+\infty} \left[\frac{e^{-ts}}{-s}\right]_0^b \quad \text{provided } s \neq 0. \\
&= \lim_{b\to+\infty} \left[\frac{e^{-bs}}{-s} - \frac{e^{0}}{-s}\right] \\
&= \lim_{b\to+\infty} \left[\frac{e^{-bs}}{-s} - \frac{1}{-s}\right]
\end{aligned}$$

At this stage we need to recall a limit from Cal 1:

$$e^{-x} \to \begin{cases} 0 & \text{if } x \to +\infty \\ +\infty & \text{if } x \to -\infty \end{cases}.$$

Hence,

$$\lim_{b\to+\infty} \frac{e^{-bs}}{-s} = \begin{cases} 0 & \text{if } s > 0 \\ +\infty & \text{if } s < 0 \end{cases}.$$

Thus,

$$F(s) = \frac{1}{s}, \; s > 0.$$